Broken Continuous Symmetry

- In general the order parameter $\vec{M}$ can be a vector with components (or other things like tensor, complex number etc.)
- if $n \geq 2$ → novel phenomenon arises ⇒

Landau functional for 2-component order parameter

$$\mathcal{L}[\vec{M}] = \int d\vec{r} \left[\frac{1}{2} (\vec{\nabla} \vec{M})^2 + \frac{w_0}{4} \left(\vec{M}^2 + \frac{r_0}{w_0} \right)^2 \right]$$

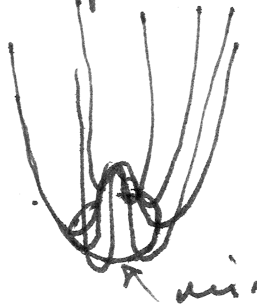
$$\mathcal{L}[\vec{M}] = \int d\vec{r} \left[\frac{1}{2} (\vec{\nabla} \cdot M_1)(\vec{\nabla} M_1) + \frac{1}{2} (\vec{\nabla} M_2) \cdot (\vec{\nabla} M_2) + \frac{w_0}{4} \left(M_1^2 + M_2^2 + \frac{r_0}{w_0} \right)^2 \right]$$

- the "potential term" ⇒ $\frac{w_0}{4} \left((M_1^2 + M_2^2) + \frac{r_0}{w_0} \right)^2$

has a minimum at $M_1 = M_2 = 0$ if $r_0 > 0$

has a minimum at $M_1^2 + M_2^2 = \frac{|r_0|}{w_0}$

"Mexican hat" potential



minima are form a circle

⇒ physically ⇒ the magnetization may lie in any orientation (finite but direction can be anything)

②

- can choose orientation by turning on a magnetic field → $-\mu \vec{B} \cdot \int \vec{M}(\vec{r}) d\vec{r}$

~~choose~~ choose direction 1:

$$-\mu B_1 \int M_1(\vec{r}) d\vec{r}$$

⇒ in this case ordered state does have a direction ⇒ if $B \to 0$ $M_1 = \frac{|r_0|}{w_0}$ $M_2 = 0$

- calculate correlation function

$$\left.\frac{\delta^2 \mathcal{L}}{\delta M_1(\vec{r}) \delta M_1(\vec{r})}\right|_{M_1 = \frac{|r_0|}{w_0}, M_2 = 0} \Longrightarrow \tilde{G}_{11}(q) = \frac{1}{q^2 + 2|r_0|}$$

$$\frac{\delta^2 \mathcal{L}}{\delta M_2 \, \delta M_2} \longrightarrow \tilde{G}_{22}(q) = \frac{1}{q^2}$$

$$\frac{\delta^2 \mathcal{L}}{\delta M_1 \delta M_2} = \frac{\delta^2 \mathcal{L}}{\delta M_2 \delta M_1} \to \tilde{G}_{12}(q) = 0 = \tilde{G}_{21}(q)$$

- in the 1-direction: correlation length $\xi \sim |T - T_c|^{-1/2}$ (same as before)
- in the 2-direction: correlation length is $\xi \sim \infty$

⇓

<u>Goldstone boson</u> (in the direction parallel to B field ⇒ need energy to move; in the direction perpendicular ⇒ no energy needed)

Ginzburg-Landau Hamiltonian

- use Ginzburg-Landau Hamiltonian to derive a theory based on the Boltzmann-weight, partition function associated with GL Hamiltonian

Hamiltonian:

$$H[\varphi(\vec r)] = \int d\vec r \left[\frac{\vec\nabla\varphi\cdot\vec\nabla\varphi}{2} + \frac{r_0}{2}\varphi^2 + \frac{u_0}{4!}\varphi^4 \right]$$

Partition function!!

$$Z[B(\vec r)] = \int D[\varphi(\vec r)] \exp\left[-H[\varphi(\vec r)] - \int d\vec r\, B(\vec r)\varphi(\vec r)\right]$$

$$= \int D[\varphi(\vec r)] \exp\left(-H_1[\varphi(\vec r); B(\vec r)]\right)$$

$\int D[\varphi(\vec r)]$ - denotes a functional integral over $\varphi(\vec r)$

also $\Rightarrow$ β, μ parameters have been ~~swallowed~~ absorbed in notation (discrete analog: $\int \prod_i d\varphi(\vec r_i)$)

- role of lattice $\Rightarrow$ shortest length scale $\underline{a}$ (lattice constant) $\Rightarrow$ in evaluating the functional integral apply the "ultraviolet cutoff" $\Rightarrow$ $q_{max} = \frac{2\pi}{a}$

largest wave-vector $q_{max} \approx \frac{2\pi}{a}$

- this Hamiltonian is not easy to derive
  - as we have seen $\Rightarrow$ coarse graining on Ising model gives something similar

Landau's theory can be recovered via saddle-point approximation (4)

- maximum of integrand: $\left.\dfrac{\delta H[\varphi(\vec r)]}{\delta \varphi(\vec r)}\right|_{\varphi(\vec r)=\varphi_0(\vec r)} = B(\vec r)$

$\varphi_0(\vec r)$ is a functional of $B(\vec r)$

$\Downarrow$

$$Z[B] \simeq \exp\left[-H[\varphi_0] + \int d\vec r\, B(\vec r)\varphi_0(\vec r)\right]$$

(saddle point approximation)

$$\ln Z = -H[\varphi_0] + \int d\vec r\, B(\vec r)\varphi_0(\vec r)$$

$$M(\vec r) = \frac{\delta \ln Z}{\delta B(\vec r)} = -\int \left.\frac{\delta H[\varphi(\vec r)]}{\delta \varphi(\vec r')}\right|_{\varphi_0} \frac{\delta \varphi(\vec r')\, d\vec r'}{\delta B(\vec r)}$$

$$\cancel{+B(\vec r)} + \varphi_0(\vec r) + \int B(\vec r')\frac{\delta \varphi_0(\vec r')}{\delta B(\vec r)}\, d\vec r'$$

$$\boxed{M(\vec r) = \varphi_0(\vec r)}$$

Legendre transform of $\ln Z \rightarrow$ Gibbs potential Γ

$$\Gamma[M(\vec r)] = \int d\vec r\, M(\vec r) B(\vec r) - \ln Z$$

$$= \int d\vec r \left(\frac{1}{2}\left[\vec\nabla M\right]^2 + \frac{r_0}{2} M^2(\vec r) + \frac{u_0}{4!} M^4(\vec r)\right)$$

$M(\vec r) = \varphi_0(\vec r) \longrightarrow$ only true within saddle-pt. approximation

$\Gamma[M(\vec r)] = H[M(\vec r)] \rightarrow$ only true within saddle-pt. approximation

Beyond Landau's theory

- perform expansion around saddle-point approximation

$$Z[B(\vec r)] = \int \mathcal{D}[\varphi(\vec r)] \exp\Big[\underbrace{-H[\varphi(\vec r)] + \int d\vec r\, B(\vec r)\varphi(\vec r)}_{\tilde H}\Big]$$

$\varphi_0(\vec r)$ - saddle pt. solution

$\psi(\vec r) = \varphi(\vec r) - \varphi_0(\vec r) \rightarrow$ fluctuations around saddle-pt. solution

$$\tilde H \cong H[\varphi_0(\vec r)] - \int d\vec r\, B(\vec r)\varphi_0(\vec r) + \frac{1}{2}\int d\vec r\, d\vec r\,' \psi(\vec r)\mathcal{H}(\vec r, \vec r\,')\psi(\vec r\,')$$

$$\mathcal{H}(\vec r, \vec r\,') = \frac{\delta \tilde H}{\delta\varphi(\vec r)\delta\varphi(\vec r\,')} = \left[-\nabla^2 + r_0 + \frac{u_0}{2}\varphi^2(\vec r)\right]\delta(\vec r - \vec r\,')$$

$$Z[B(\vec r)] = \exp\left[-H[\varphi_0(\vec r)] + \int d\vec r\, B(\vec r)\varphi_0(\vec r)\right]$$
$$\underbrace{\int \mathcal{D}[\psi(\vec r)] \exp\left[-\frac{1}{2}\int d\vec r\, d\vec r\,' \psi(\vec r) \mathcal{H}(\vec r, \vec r\,')\psi(\vec r\,')\right]}_{\text{functional analog of a Gaussian integral}}$$

$$\int du_1 \ldots du_N \exp\left[-\frac{1}{2}\sum_{ij} u_i A_{ij} u_j\right] \propto \left[\det \hat A\right]^{-1/2} = \exp\left[-\frac{1}{2}\operatorname{Tr}\ln \hat A\right]$$

Similarly

$$\int \mathcal{D}[\psi(\vec r)] \exp\left[-\frac{1}{2}\int d\vec r\, d\vec r\,' \psi(\vec r)\mathcal{H}(\vec r, \vec r\,')\psi(\vec r\,')\right] = \exp\left[-\frac{1}{2}\operatorname{Tr}\ln \mathcal{H}\right]$$

$H(\vec{r},\vec{r}')$ → not diagonal in position space ⑥

→ diagonal in resiprocal space

$$\mathrm{Tr}\, f = \int d\vec{r}\, d\vec{r}'\, \delta(\vec{r}-\vec{r}')\, f(\vec{r},\vec{r}') = \int d\vec{r}\, d\vec{r}' \int d\vec{q}\, \frac{e^{i\vec{q}(\vec{r}-\vec{r}')}}{(2\pi)^D} f(\vec{r}-\vec{r}')$$

$$= V \int \frac{d\vec{q}}{(2\pi)^D}\, \tilde{f}(\vec{q})$$

$$\mathrm{Tr} \ln \mathcal{H}(\vec{r},\vec{r}') = V \int \frac{d\vec{q}}{(2\pi)^D} \ln\left(q^2 + r_0(T) + \frac{u_0}{2} M^2\right)$$

approximate free energy:

$$\ln Z = H_1[\varphi_0] - \frac{1}{2} \mathrm{Tr} \ln \mathcal{H}$$

$$M = \frac{\delta \ln Z}{\delta B(\vec{r})} = \varphi(\vec{r}) + \frac{\delta\, \frac{1}{2} \mathrm{Tr} \ln \mathcal{H}}{\delta B(\vec{r})}$$

Gibbs potential:

$$\Gamma[M] = H[M] + \frac{V}{2} \int \frac{d\vec{q}}{(2\pi)^D} \ln\left(q^2 + r_0 + \frac{u_0}{2} M^2\right)$$

(up to second order!)

Ginzburg Criterion

consider $\Gamma[M] = H[M] + \frac{V}{2}\int \frac{d\vec{q}}{(2\pi)^D} \ln\left(q^2 + r_0 + \frac{u_0}{2}M^2\right)$

$$\frac{1}{V}\frac{\partial \Gamma}{\partial M} = B = r_0 M + \frac{u_0}{6}M^3 + \frac{u_0}{2} M \int \frac{d\vec{q}}{(2\pi)^D} \frac{1}{\left(q^2 + r_0 + \frac{u_0}{2}M^2\right)}$$

case $T > T_c$ $\quad B = M = 0$

compute susceptibility

$$\frac{\partial B}{\partial M} = \frac{1}{\chi} = r_0 + \frac{u_0}{2}\int \frac{d\vec{q}}{(2\pi)^D}\frac{1}{(q^2 + r_0)}$$

$$\frac{1}{\chi} = r_0 + \frac{u_0}{2}\int \frac{d\vec{q}}{(2\pi)^D}\frac{1}{(q^2 + r_0)}$$

dimensional analysis

$r_0 \sim \ell^{-2}$ $\qquad (q^2 + r_0)$

$u_0 \sim$ $\qquad u_0 \, \ell^{-D} \, \ell^{2} \rightarrow$

$\Rightarrow u_0 \sim \ell^{D-4}$

$r_0 \sim \ell^{-2}$

$u_0 \sim \ell^{D-4}$

- consider an expansion in $\underline{u_0}$: $g = c_0 + c_1 u_0 + c_2 u_0^2 + \dots$

- $g = 0$ at T_c $\Rightarrow$ $\quad r_{0c} + \frac{u_0}{2}\int \frac{d\vec{q}}{(2\pi)^D}\frac{1}{q^2 + r_{0c}} = 0$

$\Rightarrow T$

- the higher order terms in the expansion for ξ originate from higher-order fluctuation terms in the expansion $H = H[u_0] + \frac{\delta H}{\delta u \delta u} + \dots$ (8)

- in principle possible to carry out expansion to arbitrary accuracy $\Rightarrow$ it turns out that convergence depends on dimensionality

dimensions: $[u] = ?$ $r_0 \int d\vec{r}\, u^2 \rightarrow$ dimensionless

$$\ell^{D-2}$$

$$[u] = \frac{2-D}{2} \quad (\text{length})$$

$$[r_0] = -2 \quad (\text{length})$$

$$[u_0] = D-4 \quad u_0 \int d\vec{r}\, u^4 \rightarrow \text{dimensionless}$$

(length)

$$u_0\, \ell^D\, \ell^{4-2D} = u_0\, \ell^{4-D}$$

$$\xi = r_0 + \frac{u_0}{2} \int \frac{d\vec{q}}{(2\pi)^D} \frac{1}{q^2 + r_0}$$

$$0 = r_{0c} + \frac{u_0}{2} \int \frac{d\vec{q}}{(2\pi)^D} \frac{1}{q^2 + r_{0c}}$$

$\Downarrow$

$$\xi = (r_0 - r_{0c}) + \frac{u_0}{2} \int \frac{d\vec{q}}{(2\pi)^D} \frac{(r_{0c} - r_0)}{(q^2 + r_0)(q^2 + r_{0c})}$$

$$= (r_0 - r_{0c}) \Big[1 - \frac{u_0}{2} \int \frac{d\vec{q}}{(2\pi)^D} \frac{1}{(q^2 + r_0)(q^2 + r_{0c})}$$

replace r_0 with δ $\Rightarrow$ lowest order approximations
r_{0c} with 0 from expansion (9)

$$\delta = (r_0 - r_{0c})\left[1 - \frac{u_0}{2}\int \frac{d\vec{q}}{(2\pi)^D} \frac{1}{q^2(q^2+\delta)}\right]$$

if $D > 0$ integral is convergent even at $q=0$

for $\delta = 0$

define $q' = q/\sqrt{\delta}$

$$\delta = (r_0 - r_{0c})\left[1 - \frac{u_0}{2}\delta^{\frac{D-4}{2}}\int \frac{d\vec{q}'}{(2\pi)^D} \frac{1}{q'^2(q'^2+1)}\right]$$

$u_0 \delta^{\frac{D-4}{2}}$ $\rightarrow$ dimensionless parameter

$\rightarrow$ expansion parameter

$D > 4$ $\quad$ $D < 4$ $\Rightarrow$ fundamentally different

$D < 4$ $\Rightarrow$ convergence is not expected