# Ginzburg-Landau theory



- relates order parameter to the underlying symmetries of the system
- $1^{st}$ order phase transitions: may or may not involve symmetry breaking
    - solid-liquid; solid-vapour => there is symmetry breaking
    - liquid-gas => no symmetry breaking
    - solid to solid => change in symmetry
- continuous phase transition => free energy changes continuously → symmetry is always broken
    - order parameter => sensitive to symmetry breaking → ~~less~~ becomes finite in the less symmetric phase
    - examples of symmetries which break at a continuous phase transition
        * Curie point: rotational symmetry
        * normal to superfluid transition in $^4He$: gauge symmetries
- order parameter can be a vector/scalar or a complex field

* SL theory does not describe all features of a phase transition correctly, and is only valid near the critical point (2)

* but it describes the role of symmetry in the nature of phase transitions well → and is a good starting point

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let $\eta$ denote the order parameter

let $\Phi$ denote the free energy

$$\Phi(T, Y, f) = \Phi_0(T, Y) + \alpha_2(T, Y)\eta^2 + \alpha_4(T, Y)\eta^4 + \dots - f\eta$$

$f$ - conjugate to the order parameter

$\alpha_i(T, Y)$ - coefficients which depend on details of phase transition

- no ~~fist~~ first order term in $\eta$ (if $f = 0$) ⇒ this way $\eta = 0$ will correspond to one of the ordered states

- $\Phi_0(T, Y)$ - part of free energy not involved in the phase transition

$\eta$ realised in nature → corresponds to $\frac{\partial \Phi}{\partial \eta} = 0$

~~min~~ minimum

-continuous phase transition: occurs if there is no cubic term ($\alpha_3(T,Y)=0$)

$\Downarrow$ $\Downarrow$

$$\Phi(T,Y) = \Phi_0(T,Y) + \alpha_2(T,Y)\eta^2 + \alpha_4(T,Y)\eta^4$$

-global stability requires that $\alpha_4(T,Y)>0$

$$\frac{\partial \Phi(T,Y)}{\partial \eta} = 0 \Rightarrow 2\alpha_2(T,Y)\eta + 4\alpha_4(T,Y)\eta^3 = 0$$

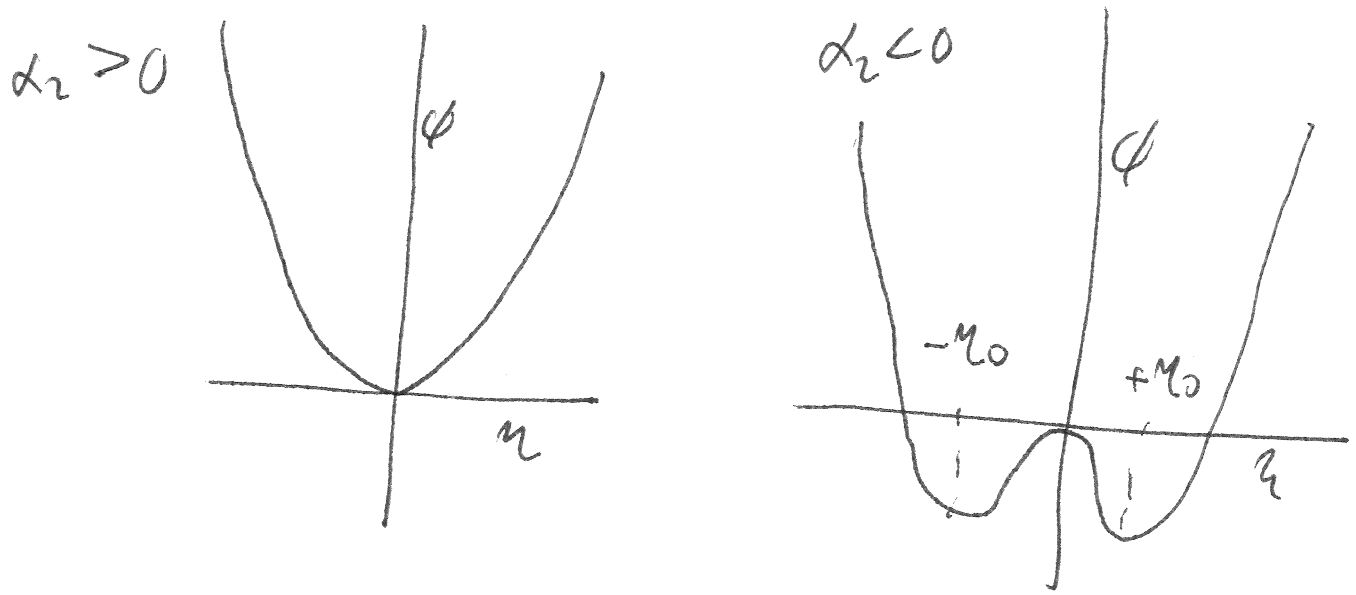
solutions: $\eta = 0, \pm\sqrt{-\frac{\alpha_2(T,Y)}{2\alpha_4(T,Y)}}$

either $\alpha_2(T,Y) > 0$ and the only solution is $\eta = 0$

or $\alpha_2(T,Y) < 0$ and there are three solutions $\eta = 0, \pm\eta_0$ $\quad \eta_0 = \sqrt{\frac{-\alpha_2}{2\alpha_4}}$

* free energies $\to +\infty$ if $\eta \to \pm\infty$ (global stability)

therefore we must have



- we can take $\alpha_2 = \tilde{\alpha}(T - T_c)$ (this can be derived from mean-field theory $\xi$ → see later) (4)

→ $T < T_c$ minima are $\pm \eta_0$; $T > T_c$ minimum is $\eta = 0$

critical point $T = T_c \Rightarrow \alpha_2 = 0$

critical exponents:

* $\eta_0 = \pm\sqrt{\dfrac{\alpha(T-T_c)}{\alpha_4}} \Rightarrow \eta_0 \sim |T-T_c|^{\beta} \Rightarrow \boxed{\beta = 1/2}$

* free energy

$$\varphi(T,y) = \varphi_0(T,y) + \alpha_2\left(\frac{-\alpha_2}{2\alpha_4}\right) + \alpha_4\left(\frac{-\alpha_2}{2\alpha_4}\right)^4$$

$$= \varphi_0(T,y) - \frac{\alpha_2^2}{4\alpha_4} = \varphi_0 - \frac{\tilde{\alpha}^2(T-T_c)^2}{4\alpha_4}$$

specific heat:

$$C = -T\frac{\partial^2\varphi}{\partial T^2} = -T\frac{\partial}{\partial T}\left[-\frac{\tilde{\alpha}^2(T-T_c)}{2\alpha_4}\right] = \frac{\tilde{\alpha}^2}{2\alpha_4}$$

$$C \sim |T-T_c|^{-\alpha} \qquad \boxed{\alpha = 0}$$

⇓

$C$ → has a finite jump at critical point

$$C = \begin{cases} 0 & T > T_c \\ \dfrac{\tilde{\alpha}^2}{2\alpha_4} & T < T_c \end{cases}$$

→ consider finite field $\mathcal{J}$

$$\phi(T,\eta) = \phi_0(T,\eta) + \alpha_2 \eta^2 + \alpha_4 \eta^4 - \mathcal{J}\eta$$

$$\frac{\partial \phi}{\partial \eta} = 0 \Rightarrow 2\alpha_2 \eta + 4\alpha_4 \eta^3 - \mathcal{J} = 0$$

at $T_c$ $\quad \alpha_2 = 0 \Rightarrow \quad \eta_s^3 \sim \mathcal{J}$

$$\eta \sim \mathcal{J}^{1/\delta} \qquad \boxed{\delta = 3}$$

can also obtain susceptibility

$$\frac{\partial}{\partial \mathcal{J}} \to \frac{\partial \eta}{\partial \mathcal{J}}\left[2\alpha_2 + 12\alpha_4\eta^2\right] - 1 = 0$$

$$\chi = \frac{\partial \eta}{\partial \mathcal{J}} \Rightarrow \frac{\partial \eta}{\partial \mathcal{J}} = \frac{1}{2\alpha_2 + 12\alpha_4 \eta^2}$$

since $\eta^2 = -\frac{\alpha_2}{2\alpha_4} \quad T < T_c$

$$\chi = \frac{\partial \eta}{\partial \mathcal{J}} = \frac{1}{4\alpha_2} = \frac{1}{4\tilde{\alpha}(T-T_c)}$$

~~$\chi \sim |T-T_c|^{-\nu}$~~

$$\chi = \begin{cases} \dfrac{1}{2\tilde{\alpha}|(T-T_c)|} & T > T_c \\[2ex] \dfrac{1}{4\tilde{\alpha}|(T-T_c)|} & T < T_c \end{cases}$$

$$\chi \sim |T-T_c|^{-\nu} \Rightarrow \boxed{\nu = 1}$$

* Ginzburg-Landau theory can also be derived from mean-field theory through a coarse graining procedure
* this leads to the Ginzburg-Landau functional
* can also be used to derive the remaining critical exponents

## Landau functional

- derive entropy expression → let $P_i^+, P_i^-$ denote ~~average spin on a site: $M_i =$~~ probability of spin (+) on site i and spin (−) on site i

$$M_i = P_i^{(+)} - P_i^{(-)} \qquad \Rightarrow P_i^{(+)} = \frac{1+M_i}{2}$$
$$1 = P_i^{(+)} + P_i^{(-)} \qquad P_i^{(-)} = \frac{1-M_i}{2}$$

- entropy $\Rightarrow S = -k \sum_i \left[ \left(\frac{1+M_i}{2}\right) \ln\left(\frac{1+M_i}{2}\right) + \left(\frac{1-M_i}{2}\right) \ln\left(\frac{1-M_i}{2}\right) \right]$

- expand up to 4th order

$$\Rightarrow \left(\frac{1+M_i}{2}\right) \ln\left(\frac{1+M_i}{2}\right) + \left(\frac{1-M_i}{2}\right) \ln\left(\frac{1-M_i}{2}\right)$$
$$= \frac{M_i^2}{2} + \frac{M_i^4}{12} + \dots$$

$$\Psi = -\frac{1}{2}\sum_{ij} J_{ij} M_i M_j - \mu \sum_i B_i M_i + \frac{1}{\beta}\sum_i \left(\frac{M_i^2}{2} + \frac{M_i^4}{12}\right)$$

( from the original free energy expression

$$\Psi = -\frac{1}{2}\sum_{ij} J_{ij} M_i M_j - \mu \sum_i B_i M_i - kT \sum_i \left[ \left(\frac{1+M_i}{2}\right) \ln\left(\frac{1+M_i}{2}\right) + \left(\frac{1-M_i}{2}\right) \ln\left(\frac{1-M_i}{2}\right)\right]$$

it is possible to derive the mean-field equations $\frac{\partial \Psi}{\partial M_i} = 0 \Rightarrow \frac{1}{\beta} \tanh^{-1} M_i = \mu B_i + \sum_j J_{ij} M_j$ )

- can also rewrite the interaction term

$M_i = M(\vec{r}_i)$ $\quad$ $\vec{r}_i \to$ position vector

$M_{i+\mu} = M(\vec{r}_i + a\vec{e}_\mu)$ $\quad$ $a$ - lattice constant

$$-\frac{1}{2}\sum_{ij} J_{ij} M_i M_j = -J \sum_i \sum_\mu M_i M_{i+\mu}$$

$$J_{ij} = \begin{cases} J & \text{for nearest neighbours} \\ 0 & \text{otherwise} \end{cases}$$

$\sum_\mu \to$ sum over nearest neighbours

$$-J \sum_i \sum_\mu M_i M_{i+\mu} = +\frac{J}{2}\sum_{i\mu}\left\{\left[M_i^2 - 2M_i M_{i+\mu} + M_{i+\mu}^2\right] - M_i^2 - M_{i+\mu}^2\right\}$$

$$= \frac{J}{2}\sum_{i,\mu}\left((M_i - M_{i+\mu})^2 - \frac{J}{2}\sum_{i\mu}(M_i^2 + M_{i+\mu}^2)\right)$$

$$= \frac{J}{2}\sum_{i,\mu}(M_i - M_{i+\mu})^2 - \cancel{\beta\sum_i}\, 2JD\sum_i M_i^2$$ ⑧

$$\Downarrow \qquad\qquad \Downarrow$$

$$\Phi = \frac{J}{2}\sum_{i\mu}(M_i - M_{i+\mu})^2 - \mu\sum_i B_i M_i + \left(\frac{1}{\beta} - 2JD\right)\sum_i M_i^2$$

$$+ \frac{1}{12\beta}\sum_i M_i^4$$

Note: * $\sum_i M_i^2$ is multiplied by $\left(\frac{1}{\beta} - 2JD\right)$

mean-field definition of $T_c \Rightarrow T_c = \frac{2JD}{k}$

* also $M_i = M$ for homogeneous system

$(B_i = B)$

$$\Downarrow \qquad \Downarrow$$

in this case we recover the Ginzburg-Landau theory described before $[\Phi_0 + \alpha_2 M^2 + \alpha_4 M^4 - BM]$

- coarse graining:

$$a^D \sum_i M_i^2 \to \cancel{a} \int d\vec{r}\, M^2(\vec{r})$$

$$a^D \sum_i M_i^4 \to \int d\vec{r}\, M^4(\vec{r})$$

$$a^D \sum_i \mu M_i B_i \to \mu\int d\vec{r}\, B(\vec{r}) M(\vec{r})$$

- last term $\Rightarrow$ $a^D \sum_i (M_i - M_{i+\mu})^2$

$$= a^{D-2}\sum_i\left(\frac{M_i - M_{i+\mu}}{a}\right)^2 \to a^2\int d\vec{r}\,(\nabla M \cdot \nabla M)$$

perform rescaling of magnetization: $m \to \sqrt{Ja^2}\, M$

$$\Rightarrow \mathcal{L}[M(\vec{r})] = \int d\vec{r} \left[ (\vec{\nabla} M \cdot \vec{\nabla} M) + \frac{r_0}{2} M^2(\vec{r}) + \frac{u_0}{4!} M^4(\vec{r}) - \mu B(\vec{r}) M(\vec{r}) \right]$$ 9

in summary:

$$\Phi = -\frac{1}{2} \sum_{ij} J_{ij} M_i M_j - \mu \sum_i B_i M_i + \frac{1}{\beta} \sum_i \left[ \left(\frac{1+M_i}{2}\right) \ln\left(\frac{1+M_i}{2}\right) + \left(\frac{1-M_i}{2}\right) \ln\left(\frac{1-M_i}{2}\right) \right]$$

$\Downarrow$ coarse graining

$$\mathcal{L}[M(\vec{r})] = \int d\vec{r} \left[ (\vec{\nabla} M) \cdot (\vec{\nabla} M) + \frac{r_0}{2} M^2(\vec{r}) + \frac{u_0}{4!} M^4(\vec{r}) - \mu B(\vec{r}) M(\vec{r}) \right]$$

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$\mathcal{L}[M]$ is a functional $\Rightarrow$ value of $\mathcal{L}$ depends on the whole ~~functio~~ function $M(\vec{r})$ (the value of $M(\vec{r})$ at every point $\vec{r}$)

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before continuing let us review functional differentiation

- suppose $I[f]$ is a functional ($I$ depends on function $f(x)$)
- if we want to find $\frac{\delta I}{\delta f(x)}$. (how $I$ varies with a change in $f(x)$ at point $x$)

we need to do the following:

1.) $f(x') \rightarrow f(x') + \epsilon\,\delta(x-x')$

2.) $I \rightarrow I[f(x') + \epsilon\,\delta(x-x')]$

3.) take the simple derivative $\frac{\partial I}{\partial \epsilon}$

4.) set $\epsilon = 0$

5.) integrate over $\delta(x-x')$

<u>examples:</u>

$$I[f] = \int dx'\, f^3(x')$$

~~1.) $f(x') \rightarrow$~~

find $\frac{\delta I}{\delta f(x)}$

1.) $f(x') \rightarrow f(x') + \epsilon\,\delta(x-x')$

2.) $I[f] \rightarrow I[f+\epsilon\delta]$

$\Rightarrow \int dx' \left[f(x') + \epsilon\,\delta(x-x')\right]^3$

3.) $\frac{\partial I}{\partial \epsilon} \Rightarrow 3\int dx' \left[f(x') + \epsilon\,\delta(x-x')\right]^2 \delta(x-x')$

4.) $\epsilon \Rightarrow 0 \Rightarrow 3\int dx' \left[f(x')\right]^2 \delta(x-x')$

5.) integrate $\Rightarrow \frac{\delta I}{\delta f(x)} = 3 f^2(x)$

if derivatives are also involved → trickier
but usually one can use integration by parts

example: $I = \int dx' f(x') \frac{\partial^2 f(x')}{\partial x'^2}$

find $\frac{\delta I}{\delta f(x)} = ?$

1.) $f(x') \to f(x') + \epsilon\, \delta(x-x')$

2.) $I \to \int dx' \left(f(x') + \epsilon\, \delta(x-x')\right) \frac{\partial^2}{\partial x'^2} \left(f(x') + \epsilon\, \delta(x-x')\right)$

$= \int dx' \left(f(x') + \epsilon\, \delta(x-x')\right) \frac{\partial^2}{\partial x'^2} f(x')$

$+ \epsilon \int dx' \left(f(x') + \epsilon\, \delta(x-x')\right) \frac{\partial^2}{\partial x'^2} \delta(x-x')$

in second term $\frac{\partial^2}{\partial x'^2} \delta(x-x')$ is problematic because the derivative of $\delta$-fn. is undefined

⇒ integrate by parts

(for simplicity neglect boundary contributions)

$I \to \int dx' f(x') \frac{\partial^2}{\partial x'^2} f(x') + \epsilon \int dx'\, \delta(x-x') \frac{\partial^2 f(x')}{\partial x'^2}$

$+ \epsilon \int dx' f(x') \frac{\partial^2}{\partial x'^2} \delta(x-x') + \epsilon^2 \int dx'\, \delta(x-x') \frac{\partial^2}{\partial x'^2} \delta(x-x')$

$\Rightarrow$ using these rules

$$\frac{\delta \mathcal{L}[M]}{\delta M(\vec{r})} = -\nabla^2 M(\vec{r}) + r_0 M(\vec{r}) + \frac{u_0}{3!} M^3(\vec{r}) - B(\vec{r}) = 0$$

$$\cancel{\frac{\delta^2 \mathcal{L}}{\delta M(\vec{r})\,\delta B(\vec{r}')} =}$$

if $M(\vec{r}) = M$ (constant)

$\Downarrow$

$$\boxed{r_0 M + \frac{u_0}{3!} M^3 - B = 0}$$

similar to meanfield equations

correlation function: $G(\vec{r}, \vec{r}') = \dfrac{\delta M(\vec{r})}{\delta B(\vec{r}')}$

$$\cancel{= \frac{\delta^2 \mathcal{L}}{\delta}}$$

$$\left(-\nabla^2 + r_0 + \frac{u_0}{2!} M^2(\vec{r})\right) \frac{\delta M(\vec{r})}{\delta B(\vec{r}')} = \delta(\vec{r} - \vec{r}')$$

Fourier transform and let $B(\vec{r}) \to B$ (constant)

$$\Rightarrow \cancel{\frac{\delta M(\vec{r})}{\delta B(\vec{r}')}} = G(q) = \frac{1}{q^2 + r_0 + \frac{u_0 M^2}{2}}$$

$$G(q) = \frac{1}{q^2 \left[1 + \frac{r_0 + \frac{u_0 M^2}{2}}{q^2}\right]}$$

$T > T_c$ $\quad G(q) = \dfrac{1}{q^2 \left[1 + \frac{\bar{r}_0 (T - T_c)}{q^2}\right]} = \dfrac{g(q\xi)}{q^{2-\eta}}$

$\Rightarrow \boxed{\eta = 0} \quad \Rightarrow \quad \xi \sim |T - T_c|^{-1/2} \quad \boxed{\nu = 1/2}$