Critical Exponents

- at critical point: order parameter of ordered phase starts to grow from zero (continuously)
- above T_c: clusters of finite order parameter form but on average order parameter is zero
- below T_c: order parameter is finite
- as critical pt. is approached various thermodynamic functions go to zero or diverge

→ introduce $\varepsilon = \frac{T - T_c}{T_c}$ → expansion parameter

→ thermodynamic functions

$$f(\varepsilon) = A\,\varepsilon^{\lambda}\,(1 + B\,\varepsilon^{\gamma} + \dots)$$

$$\lambda = \lim_{\varepsilon \to 0} \frac{\ln f(\varepsilon)}{\ln \varepsilon}$$

$\lambda > 0$ → $f(\varepsilon) \longrightarrow 0$ at T_c

$\lambda < 0$ → $f(\varepsilon)$ diverges at T_c

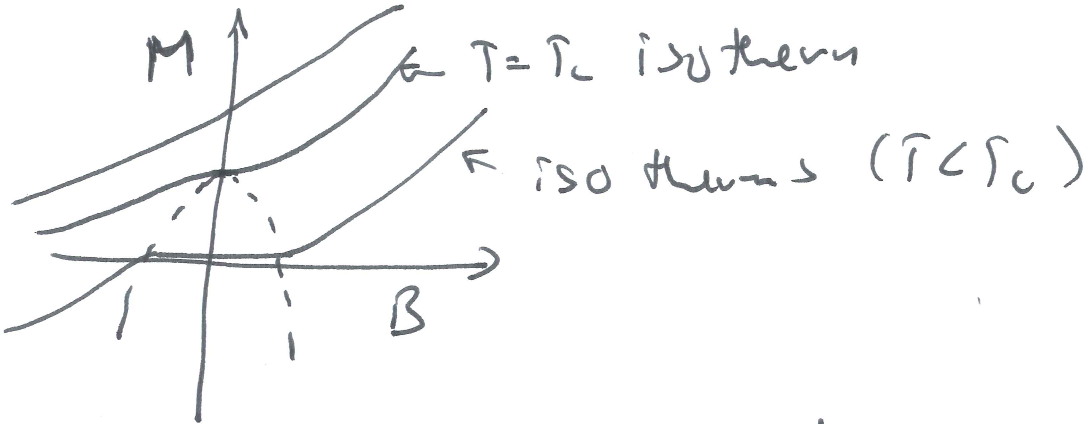
$\lambda = 0$ → can be logarithmic

$$f(\varepsilon) = A \ln \varepsilon + B$$

or $f(\varepsilon) = A + B\,\varepsilon^{1/2}$

Critical exponents around Curie point

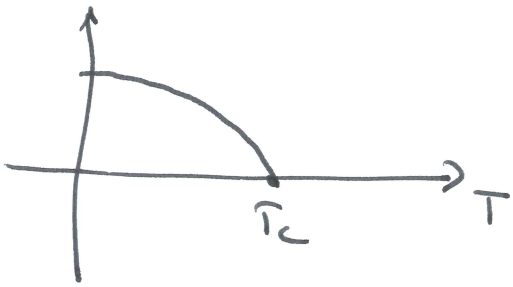
$\boxed{\delta:}$ $M \sim B^{1/\delta}$ at $T = T_c$



δ describes the behaviour of the magnetization M as a function of B at T_c

$\boxed{\beta:}$ $M(\varepsilon) \sim B^{\beta}$ $\quad \varepsilon = \frac{T-T_c}{T_c}$

from below



- how magnetization behaves around critical temperature T_c as T_c is approached from below

(3)

$\boxed{\alpha:}$ heat capacity

$$C \sim \varepsilon^{-\alpha} \qquad \cancel{T}_c \quad \varepsilon \to 0^+ \quad B=0$$

$$(-\varepsilon)^{-\alpha'} \qquad \varepsilon \to 0^- \quad B=0$$

B-field is zero

~~α~~ describes the behaviour of the specific heat as the critical temperature is approached either from above or below

$\boxed{\gamma:}$

$$\chi \sim \varepsilon^{-\gamma} \qquad \varepsilon \to 0^+ \qquad B=0$$

$$\varepsilon^{-\gamma'} \qquad \varepsilon \to 0^- \qquad B=0$$

describes behaviour of the ~~specific~~ susceptibility ~~heat~~ as the critical temperature is approached either from above or below

there are also two more exponents which are related to the correlation function

$$G(\vec{r}) \to \tilde{G}(\vec{q}) = \int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}}\, G(\vec{r})$$

according to experimental results

$\boxed{\eta:}$

$$\tilde{G}(\vec{q}) = \frac{f(q\xi)}{q^{2-\eta}} \qquad \text{(valid for } qa \ll 1)$$

$\boxed{\nu:}$

$$\xi \sim \varepsilon^{-\nu}$$

* ~~these~~ exponents are experimentally measurable
* they are also <u>universal</u>: for ~~a~~ large classes of models they are identical

critical exponents can also be related to each other $\Rightarrow$ <u>scaling laws</u>

* ~~at~~ $T = T_c$

$$\tilde{G}(\vec{q}) = \frac{K'}{q^{2-\eta}}$$

$$\tilde{G}(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^D} e^{-i\vec{q}\cdot\vec{r}} \frac{1}{q^{2-\eta}} f(q\xi)$$

$$\Downarrow$$

$$G(\vec{r}) = \frac{1}{r^{D+\eta-2}} g\left(\frac{r}{\xi}\right)$$

since $d\vec{q} \Rightarrow$ dimensions of $\frac{1}{r^D}$

$q^{2-\eta} \to$ dimensions of $r^{2-\eta}$

(valid for $r \gg a$)

for $T \neq T_c$ $\tilde{G}(0)$ is <u>finite</u>

$$\tilde{G}(0) = \frac{f(q\xi)}{q^{2-\eta}} \Rightarrow f(q\xi) = (q\xi)^{2-\eta}$$

$$\tilde{G}(0) \sim \xi^{2-\eta} \sim \varepsilon^{-\nu(2-\eta)}$$

$\tilde{G}(0)$ is the integrated correlation fn.

$\to$ proportional to χ

$$\chi \sim \varepsilon^{-\gamma} \sim \varepsilon^{-\nu(2-\eta)} \Rightarrow \boxed{\text{SCALING LAW: } \gamma = \nu(2-\eta)}$$

Chart of critical exponents

	Experiment	Mean-field (Ising)	Exact (Ising)
α	0-0.2	0	0.12
β	0.3-0.4	1/2	0.31
δ	4-5	3	5.2
γ	1.2-1.4	1	1.25
ν	0.6-0.7	1/2	0.64
η	0.1	0	0.056

Mean-field theory

- in most cases the analytical solution is too complicated $\Rightarrow$ we must resort to approximate solutions of models
- one such approach is the mean-field theory

- consider the Ising model on a square lattice

$$H = -K \sum_{\langle i,j \rangle} S_i S_j - B \sum_i S_i \qquad K > 0 \text{ (ferromagnetic)}$$

- ~~for simp~~

- expect that S_i consists of two terms

$$S_i = \delta S_i + \bar{S}_i$$

$\bar{S}_i$ = average $\rightarrow$ same for all sites $\bar{S}$

δS_i = fluctuation around average

- can write Hamiltonian in terms of $S_i = \delta S_i + \bar{S}$

$$H = -K \sum_{\langle i,j \rangle} (\delta S_i + \bar{S})(\delta S_j + \bar{S}) - B \sum_i S_i$$

$$= -K \sum_{\langle i,j \rangle} [\underbrace{\delta S_i \, \delta S_j}_{} + \delta S_i \bar{S} + \delta S_j \bar{S} + \bar{S}^2] - B \sum_i S_i$$

this term is second order in fluctuations $\Rightarrow$ <u>neglect!</u>

approximate Hamiltonian (mean-field) 7

$$H_{MF} = -K \sum_{\langle i,j \rangle} (\delta S_i \bar{S} + \delta S_j \bar{S} + \bar{S}^2) - B \sum_i S_i$$

$$= -K \sum_{\langle i,j \rangle} ((S_i - \bar{S})\bar{S} + (S_j - \bar{S})\bar{S} + \bar{S}^2)$$

$$- B \sum_i S_i$$

$$= -K \sum_{\langle i,j \rangle} (S_i \bar{S} + S_j \bar{S} - \bar{S}^2) - B \sum_i S_i$$

$$= -K\nu \sum_i S_i \bar{S} + \frac{K\nu}{2} N \bar{S}^2 - B \sum_i S_i$$

$$= -(K\nu\bar{S} + B) \sum_i S_i + \frac{K\nu N}{2} \bar{S}^2$$

ν - coordination number (number of nearest neighbours) → for square lattice $\nu = 4$

based on this Hamiltonian we can calculate the partition function, free energy, etc.

$$Q = \sum_{S_1} \cdots \sum_{S_N} e^{\beta(K\nu\bar{S} + B)\sum_i S_i - \beta\frac{K\nu N}{2}\bar{S}^2}$$

$$= [2\cosh[\beta(K\nu\bar{S} + B)]]^N e^{-\beta\frac{K\nu N}{2}\bar{S}^2}$$

$$F = -\frac{1}{\beta} \ln Q = \frac{K\nu N \bar{S}^2}{2} - \frac{N}{\beta} \ln\left(2\cosh[\beta(K\nu\bar{S} + B)]\right)$$

F depends on $\bar{s}$: find $\bar{s}$ by minimizing

$$F \Rightarrow \frac{\partial F}{\partial \bar{s}} = 0$$

$$K\nu N\bar{s} - \frac{N}{\beta}\,\frac{1 \times \beta K\nu}{2\cosh[\beta(K\nu\bar{s}+B)]} \times 2\sinh[\beta(K\nu\bar{s}+B)]$$

$$= 0$$

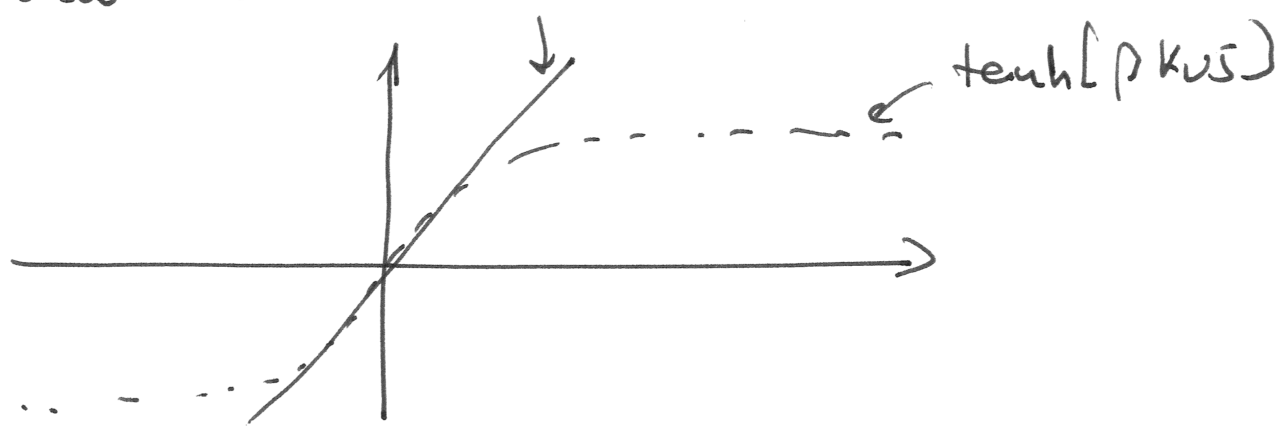
$$\Rightarrow \boxed{\bar{s} = \tanh[\beta(K\nu\bar{s}+B)]}$$

mean-field equation

let us consider $B = 0$

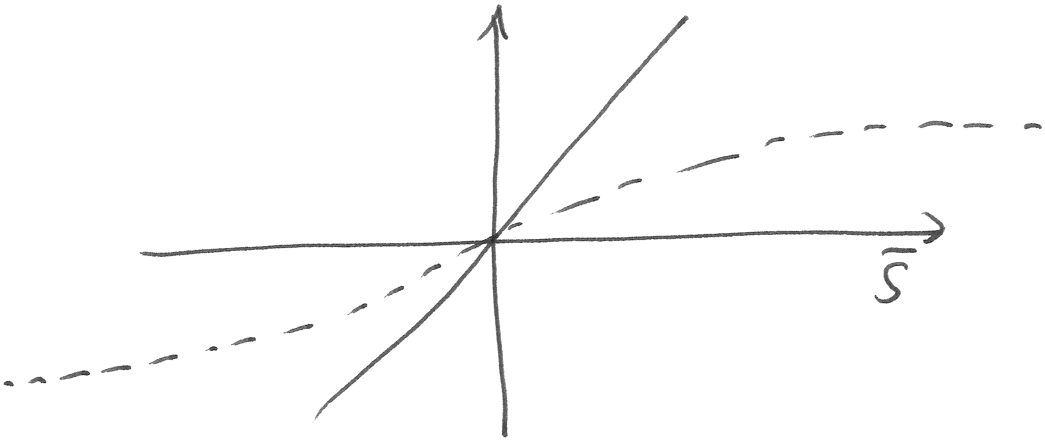
$$\bar{s} = \tanh[\beta K\nu\bar{s}]$$

graphical solution: $\bar{s}$



depending on $\beta K\nu$ there can be different numbers of solutions of $\bar{s} = \tanh(\beta K\nu\bar{s})$

(recall $\frac{\partial F}{\partial \bar{s}} = 0 \Rightarrow$ can be either minimum or maximum for small $\beta K \nu$ we have

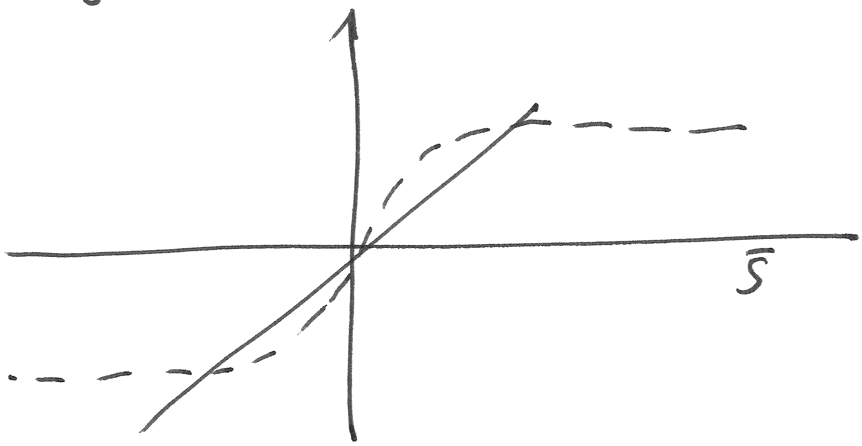


<u>one solution</u> is it a maximum or a minimum?

as $\bar{s} \to \infty$ $F \to \infty$ (see formula on page 7)

must be a minimum $\Rightarrow$ $\bar{s} = 0$ (disordered state)

for large $\beta K \nu$



<u>three solutions</u>

since as $\bar{s} \to \infty$ $F(\bar{s}) \to \infty$

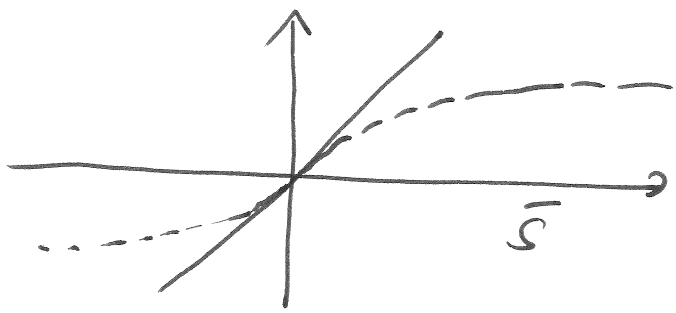
$\Downarrow$

must be two minima and one maximum

for the two minima $\bar{s}_1 = -\bar{s}_2$ (ordered) 10

(plus magnetization, minus magnetization)

if $\beta K \nu = 1$ we have crossing point between having <u>one</u> minimum and <u>two</u> minima



$\Downarrow$ critical point

$$\boxed{\beta_c K \nu = 1} \Rightarrow \boxed{T_c = \frac{K\nu}{k_B}}$$