Problem 1: Renormalization 1D Ising

Consider the 1D Ising model. Apply the following renormalization scheme: form blocks of two lattice sites, and take the block spin to be the spin of the left spin in each block. Use this scheme to show that in the Ising model in 1D there is not phase transition at finite temperature.

$$Q = \sum_{\sigma_1} \dots \sum_{\sigma_N} e^{J(\sigma_1\sigma_2 + \dots \sigma_N \sigma_1)}$$

$$= \sum_{S_1} \dots \sum_{S_{N/2}} e^{J'(S_1 S_2 + \dots S_{N/2} S_1)}$$

transfer matrix

$$\begin{pmatrix} e^{J'} & e^{-J'} \\ e^{-J'} & e^{J'} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2\cosh 2J & 1 \\ 1 & 2\cosh 2J \end{pmatrix}$$

one unknown J', 2 distinct eq'ns.

need another unknown $\Rightarrow$ introduce fn. $f(J)$

$$\begin{pmatrix} e^{J'} & e^{-J'} \\ e^{-J'} & e^{J'} \end{pmatrix} = f(J) \begin{pmatrix} 2\cosh 2J & 2 \\ 2 & 2\cosh 2J \end{pmatrix}$$

$$\Rightarrow \quad e^{J'} = (2\cosh 2J) f(J)$$

$$e^{-J'} = 2 f(J) \rightarrow f(J) = \frac{e^{-J'}}{2}$$

$$\Rightarrow e^{2J'} = \cancel{4}\cosh 2J$$

only solutions $\boxed{J' = J = 0}$

(fixed point) ~~(trit)~~

$$\boxed{J = J' = \infty}$$

(fixed point)

Problem 2: Landau theory of phase transitions

Taking the free energy to be of the form $\psi = A_2\eta^2 + A_3\eta^3 + A_4\eta^4$. The coefficients in general can depend on temperature and other thermodynamic variables. What restriction does stability impose on the coefficients? Show that this expression for the free energy corresponds to a first order phase transition. Calculate the value of the order parameter in the ordered and disordered phases. Sketch the free energy for the three regions $T<T_c$, $T_c<T<T^*$, and $T^*<T$, where T_c indicates the phase transition temperature, T^* indicates the highest temperature at which the ordered state can be metastable. Calculate the order parameter in terms of the coefficients just below T^*. $\Rightarrow \eta = 0$

stability: $A_4 > 0$

roots of $\Psi \Rightarrow$ 3 roots $\Rightarrow$ free

$$2A_2\eta + 3A_3\eta^2 + 4A_4\eta^3 = 0 \Rightarrow$$

$$\underline{\eta_1 = 0} \qquad \eta = \frac{-3A_3 \pm \sqrt{9A_3^2 - 4(2A_2)(4A_4)}}{8A_4}$$

$$\underline{\eta_{2,3}} = \frac{-3A_3 \pm \sqrt{9A_3^2 - 32A_2A_4}}{8A_4}$$

roots η_2 and η_3 can "coalesce" at pt.

$$9A_3 - 32A_1A_2 = 0$$

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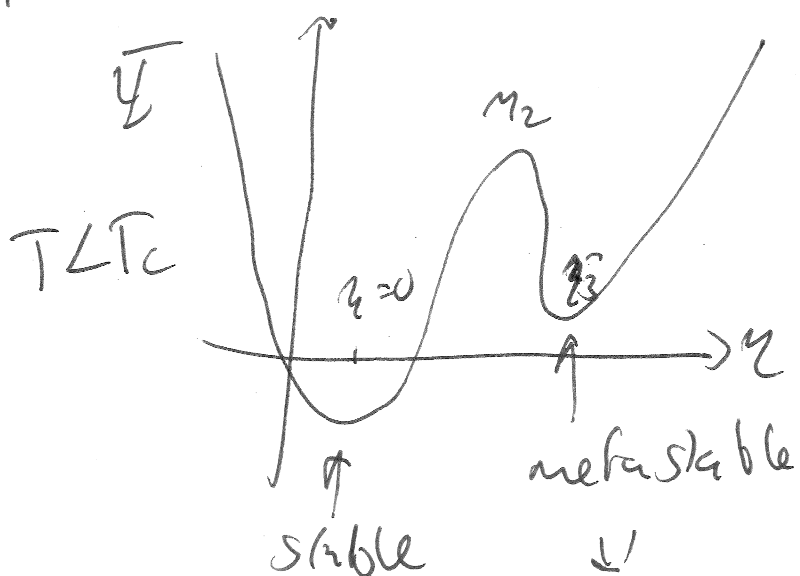
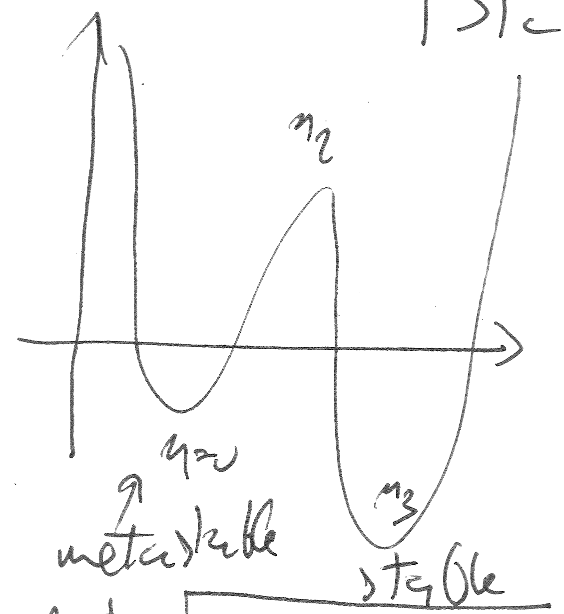
for three roots: possible free energies are





1st order p.t.

Problem 3: Mean-field theory

Consider an Ising model on a 2D square lattice in the presence of a magnetic field *B*, with spins which can take four possible values: -3/2,-1/2,1/2,3/2. Derive the mean-field equations for this model. (Optional: Try also to identify the phase transition, by graphical/numerical means.)

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i$$

$$= -J \sum_{\langle i,j \rangle} (\bar\sigma + \delta\sigma_i)(\bar\sigma + \delta\sigma_j) - B \sum_i \sigma_i$$

$$= -J \sum_{\langle i,j \rangle} \bar\sigma^2 - J \sum_{\langle i,j \rangle} \bar\sigma (\delta\sigma_i + \delta\sigma_j) - B \sum_i \sigma_i$$

$$\sigma_i = \bar\sigma + \delta\sigma_i$$

$$= -J \sum_{\langle i,j \rangle} \bar\sigma (\sigma_i + \sigma_j) - B \sum_i \sigma_i + J \sum_{\langle i,j \rangle} \bar\sigma^2$$

let ν = number of nearest neighbors

$$H_{MF} = -J\nu\bar\sigma \sum_i \sigma_i - B \sum_i \sigma_i + \frac{JN\nu\bar\sigma^2}{2}$$

$$F = -kT \ln Q \qquad Q = \sum_{\sigma_1} \cdots \sum_{\sigma_N} e^{-\frac{\beta J N \nu \bar\sigma^2}{2}} e^{\beta(J\nu\bar\sigma + B)\sum_i \sigma_i}$$

$$Q = e^{-\frac{\beta J N \nu \bar\sigma^2}{2}} \left[2\cosh \frac{(\beta J \nu \bar\sigma + B)}{2} + 2\cosh \left(\frac{3(\beta J \nu \bar\sigma + B)}{2} \right) \right]$$

mean-field eq'ns.: $\frac{\partial F}{\partial \bar\sigma} = 0$

$$F = \frac{JN\nu\bar\sigma^2}{2} - kT \ln \left[2\cosh \left(\frac{\beta J \nu \bar\sigma + B}{2} \right) + 2\cosh \left(\frac{3(\beta J \nu \bar\sigma + B)}{2} \right) \right]$$

Worksheet:

$$\frac{\partial F}{\partial \bar{\sigma}} = JN\nu\bar{\sigma} - \frac{kT\left(\sinh\left(\frac{\beta J\nu\bar{\sigma}+B}{2}\right) + \sinh\left(\frac{3(\beta J\nu\bar{\sigma}+B)}{2}\right)\right)}{\left[\cosh\left(\frac{\beta J\nu\bar{\sigma}+B}{2}\right) + \cosh\left(\frac{3(\beta J\nu\bar{\sigma}+B)}{2}\right)\right]}$$

$$= 0$$

M.F. equations:

$$JN\nu\bar{\sigma} = \frac{kT\left[\sinh\left(\frac{\beta J\nu\bar{\sigma}+B}{2}\right) + \sinh\left(\frac{3(\beta J\nu\bar{\sigma}+B)}{2}\right)\right]}{\left[\cosh\left(\frac{\beta J\nu\bar{\sigma}+B}{2}\right) + \cosh\left(\frac{3}{2}(\beta J\nu\bar{\sigma}+B)\right)\right]}$$