Problem 1:

Consider a *classical* composite and possibly correlated system, for which the joint probability of a given state being occupied is equal to $P_{mm'}$. $P_{mm'}$ satisfies $\sum_{mm'} P_{mm'} = 1$, and $P_{mm'} \geq 0$. The probability of one of the states of a particular subsystem being occupied is given by $Q_m = \sum_{m'} P_{mm'}$, and $R_m = \sum_{m'} P_{m'm}$. Show that

$$S^{(QR)} \leq S^{(Q)} + S^{(R)},$$

where $S^{(QR)} = -\sum_{mm'} P_{mm'} \ln P_{mm'}$, $S^{(Q)} = -\sum_m Q_m \ln Q_m$, and $S^{(R)} = -\sum_m R_m \ln R_m$.

* $\ln P_m Q_{m'} = \ln P_m + \ln Q_{m'}$ (1^{st} identity in problem 7.7.5)

* $\sum_{mm'} P_{mm'} \ln P_m = \sum_m P_m \ln P_m$ (2^{nd} identity in 7.7.5)

* introduce $\tilde{P}_{mm'} = P_m Q_{m'}$

* use inequality $-\sum_{mm'} P_{mm'} \ln P_{mm'} \leq -\sum_{mm'} P_{mm'} \ln \tilde{P}_{mm'}$ (classical version of Eq. (2.56))

$$\underline{\underline{S^{(QR)} \leq S^{(Q)} + S^{(R)}}}$$

Problem 2:

Calculate the phase space volume for N non-interacting relativistic particles in one dimension. The energy momentum relation is given by $\epsilon = |p|c$..

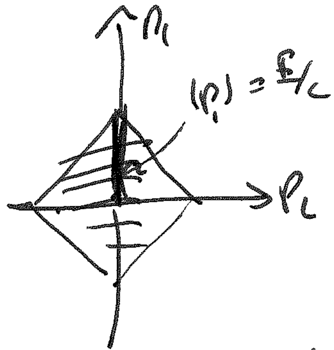
Total energy: $E = c\left(|p_1| + |p_2| + \dots + |p_N|\right)$

phase space volume:

$$\Omega(E) = \frac{L^N}{h^N N!} \underbrace{\int_{|p_1|+|p_2|+\dots+|p_N| \le \frac{E}{c}} dp_1 \cdots dp_N}_{\text{volume of an } N\text{-dimensional hypercube with side } \frac{\sqrt{2}E}{c}}$$

* consider 2-particle system:

$$|p_1| + |p_2| \le \frac{E}{c} \Rightarrow$$



square with side $\sqrt{2}\,E/c$

$$\Omega(E) = \frac{L^N}{h^N N!}\left(\frac{\sqrt{2}E}{c}\right)^N$$

phase space volume $\Phi(E) = \frac{\partial \Omega(E)}{\partial E}$

$$\boxed{\Phi(E) = \frac{L^N}{h^N N!}\left(\frac{\sqrt{2}}{c}\right)^N N E^{N-1}}$$

Problem 3: (ideal gas!)

Given a container thermally isolated from the surroundings, divided by a fixed, diathermic, impermeable wall into two compartments. Initially the left(right) compartment has $N(N')$ particles, and volume $V(V')$. The particles on each side are the same type (mass m). Consider the following process:

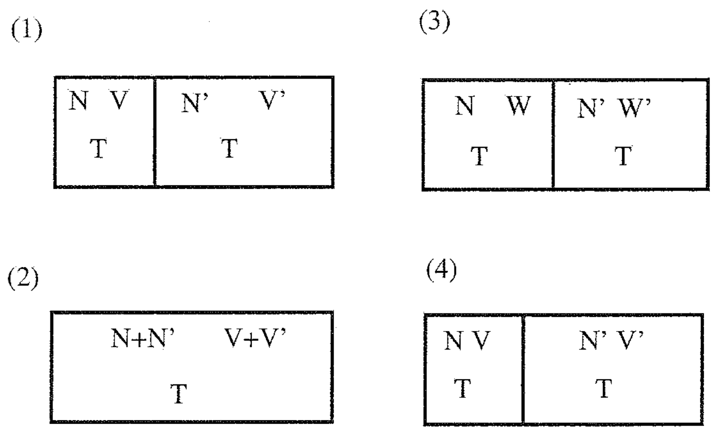
(1)->(2) The diathermic impermeable wall is removed and the system is allowed to equilibrate.

(2)->(3) The diathermic, impermeable wall is replaced so that the number of particles in the left(right) compartment is $N(N')$, but the volumes are different from those in state (1).

(3)->(4) The wall is quasi-statically pushed back to the original position in state (1).

Calculate the entropy change for each process and the entire process, and comment on it from the point of view of reversibility. Is the entropy in state (1) the same as in state (4)? Why or why not?

Calculate also the work done in each subprocess and the entire process.



Start: canonical ensemble, ideal gas

$$Z_N = \frac{V^N}{h^{3N} N!} \left(\frac{2\pi m}{\beta} \right)^{3N/2}$$

$$\rho = \frac{e^{-\beta H}}{Z}$$

entropy?

$$S = -k \, \mathrm{Tr}\, \rho \ln \rho$$

$$= -k \, \mathrm{Tr}\, \rho \left[-\beta H - \ln Z \right]$$

$$\frac{S}{k} = \beta E + \ln Z$$

$$\bar{E} = -\frac{\partial \ln Z}{\partial \beta} = \frac{3NkT}{2}$$

Worksheet:

$$S = \frac{3N}{2} + N\ln V - N\ln N + N + \frac{3N}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right)$$

$$= N\left[\ln\frac{V}{N} + \frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right) + \frac{5}{2}\right]$$

state (1):

$$S_1 = N\left[\ln\frac{V}{N} + \frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right) + \frac{5}{2}\right]$$
$$+ N'\left[\ln\frac{V'}{N'} + \frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right) + \frac{5}{2}\right]$$

state (2):

$$S_2 = (N+N')\left[\ln\frac{V+V'}{N+N'} + \frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right) + \frac{5}{2}\right]$$

state (3)

$\cancel{S_3 = N\Big[}$

$$W = \frac{N}{N+N'}(V+V')$$

$$W' = \frac{N'}{N+N'}(V+V')$$

$$S_3 = N\left[\ln\frac{N}{N+N'}(V+V') + \frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right) + \frac{5}{2}\right]$$
$$+ N'\left[\ln\frac{N'}{(N+N')}(V+V') + \frac{3}{2}\ln\left(\frac{2\pi m}{h^2\beta}\right) + \frac{5}{2}\right]$$

state (4)

same as state (1)

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work done is always $T\Delta S$ since energy is constant (so for 1→2 $T\Delta S_{12}$ etc.)

for last process 3→4: process is not spontaneous ⇒ work supplied by external agent !!!