Physics 552: Midterm 1

Name:
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SOLUTIONS

Problem 1: Probability

Consider a random walk for which the probability of taking a step of length $x + \mathrm{d}x$ is given by $P(x) = \frac{1}{\pi}\frac{a}{x^2+a^2}\mathrm{d}x$.

1.) Find the average displacement after N steps. (10 pts.) 0 (even distribution $P(x)$)

2.) Given that Y_N is the *average* distance traveled after N steps, find the probability density of Y_N in the limit of large N. (10 pts.)

probability density of Y_N: $Y = \frac{1}{N}\sum_i Y_i$

$$P_{Y_N}(y) = \int dx_1 \ldots dx_N \; P_x(x_1) \ldots P_x(x_N)\, \delta\left(y - \frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

characteristic fn.

$$\tilde{P}(k) = \int dy\, e^{iky} P_{Y_N}(y)$$

$$= \left[\int dx\, e^{i\frac{k}{N}x} P_x(x)\right]^N$$

For large N

$$\tilde{P}(k) = \int dx\left[1 + \cancel{\frac{ik}{N}}^{\,0} x + \frac{i^2k^2}{2N^2}x^2 + \ldots\right] P_x(x)$$

$$= \int dx\left[1 - \frac{k^2}{2N^2}\sigma^2 + \ldots\right] P_x(x)$$

σ diverges, $\int dx \frac{x^2}{x^2+a^2} \to \infty \Rightarrow$ ~~limit of large~~ central limit theorem may not hold

Problem 2: Entropy and Complexity

A one-dimensional system consists of L lattice sites. There are $N_\uparrow$ up-spin electrons, and $N_\downarrow$ down-spin electrons. When an up-spin electron and a down-spin electron occupy the same site the energy cost is U. Calculate the energy, entropy, temperature, and chemical potential, and heat capacity of this system. (5 pts. each)

$E = UD$ $\quad$ D – number of sites w/ up-spin and down-spin particle simultaneously

number of configurations: $\dfrac{L!}{D!\,(N_\uparrow - D)!\,(N_\downarrow - D)!\,(L + D - N_\uparrow - N_\downarrow)!}$

entropy: $S = k_B \ln \dfrac{L!}{D!\,(N_\uparrow - D)!\,(N_\downarrow - D)!\,(L + D - N_\uparrow - N_\downarrow)!}$

$$S = k_B \Big[L \ln L - D \ln D - (N_\uparrow - D) \ln (N_\uparrow - D) + (N_\downarrow - D) \ln (N_\downarrow - D) - (L + D - N_\uparrow - N_\downarrow) \ln (L + D - N_\uparrow - N_\downarrow) \Big]$$

entropy as a fn. of E

$$S(E) = \Big[L \ln L - \frac{E}{U} \ln \frac{E}{U} - \left(N_\uparrow - \frac{E}{U}\right) \ln \left(N_\uparrow - \frac{E}{U}\right) \cancel{+}\left(N_\downarrow - \frac{E}{U}\right) \ln \left(N_\downarrow - \frac{E}{U}\right) - \left(L + \frac{E}{U} - N_\uparrow - N_\downarrow\right) \ln \left(L + \frac{E}{U} - N_\uparrow - N_\downarrow\right) \Big]$$

$$\frac{1}{k_B T} = \cancel{k_B} \frac{\partial S}{\partial E} = -\frac{1}{U} \ln \frac{E}{U} - \cancel{\frac{1}{U}} + \frac{1}{U} \ln \left(N_\uparrow - \frac{E}{U}\right) \cancel{+ \frac{1}{U}} + \frac{1}{U} \ln \left(N_\downarrow - \frac{E}{U}\right) \cancel{+ \frac{1}{U}} - \frac{1}{U} \ln \left(L + \frac{E}{U} - N_\uparrow - N_\downarrow\right) - \frac{1}{U}$$

$$\frac{U}{k_B T} = \ln \frac{\left(N_\uparrow - \frac{E}{U}\right)\left(N_\downarrow - \frac{E}{U}\right)}{\frac{E}{U}\left(L + \frac{E}{U} - N_\uparrow - N_\downarrow\right)} = \ln \frac{(N_\uparrow U - E)(N_\downarrow U - E)}{E(LU + E - N_\uparrow U - N_\downarrow U)}$$

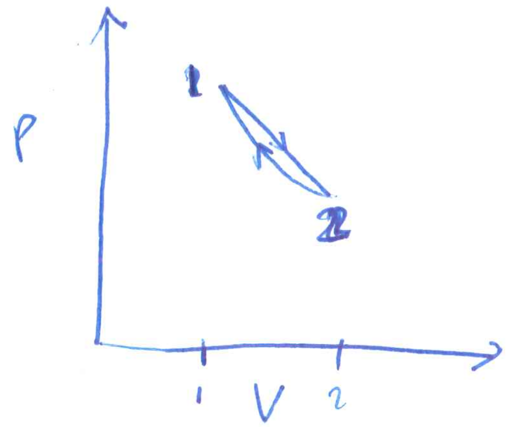
(con't on worksheet)

Problem 3: Thermodynamics

Consider a heat engine based on a two-step cycle. Step one is adiabatic compression from V_2 to V_1, with $V_1 < V_2$. Step two is expansion along a straight line along the $P - V$ plane.

1.) When integrated along the straight line segment, the heat supplied must, since energy is conserved during the cycle, equal the net work done during the cycle. Why does this not give rise to an efficiency of $\eta = \frac{\Delta W}{\Delta Q}$ and violate the second law of thermodynamics? (10 pts.)

2.) Calculate the efficiency η if the working substance is an ideal monatomic gas, and if $V_2 = 2V_1$. (10 pts.)



1.) efficiency:

1.) (2 → 1) $\Delta Q_{21} = 0$ (adiabatic)

$\Delta U = \Delta Q^a_{21} - \Delta W^a_{21}$

$\Delta U = -\Delta W^a_{21}$

2.) (1 → 2) $\Delta U_{12} = \Delta Q_{12} - \Delta W_{12}$

$\Delta Q_{12} - \Delta W_{12} - \Delta W^a_{21} = 0$

efficiency: $\eta = \dfrac{\Delta W_{12} - \Delta W^a_{21}}{\Delta Q_{12} - \Delta W^a_{21}}$

efficiency is not simply heat absorbed in this case ⇒ adiabatic step requires work input

(2.) (con't on worksheet)

Problem 4: Phase transitions

1.) For a binary mixture of particles of type 1 and type 2 the Gibbs free energy is $G = n_1\mu_1 + n_2\mu_2$. The combined first and second laws are $\mathrm{d}G = -S\mathrm{d}T + V\mathrm{d}P + \mu_1\mathrm{d}n_1 + \mu_2\mathrm{d}n_2$.

1. Show that $s\mathrm{d}T - v\mathrm{d}P + \sum_{i=1,2} x_i\mathrm{d}\mu_i = 0$. (5 pts.)
2. Show that $\sum_{i=1,2} x_i(\mathrm{d}\mu_i + s_i\mathrm{d}T - v_i\mathrm{d}P) = 0$. (5 pts.)

2.) The Berthelot equation of state is the following: $\left(P + \frac{a}{Tv^2}\right)(v-b) = RT$. Does this equation satisfy the law of corresponding states? (10 pts.)

1.) Gibbs-Duhem: $SdT - VdP + n_1 d\mu_1 + n_2 d\mu_2 = 0$

$$\Downarrow$$

$$sdT - vdP + \sum_{i=1,2} x_i d\mu_i = 0$$

$$s = x_1 s_1 + x_2 s_2$$
$$v = x_1 v_1 + v_2 v_2$$
$$\Rightarrow \sum_i x_i (s_i dT - v_i dP + d\mu_i) = 0$$

2.) Berthelot: find T_c, P_c, V_c

$$P = \frac{RT}{v-b} - \frac{a}{Tv^2}$$

$$\left.\frac{\partial P}{\partial v}\right)_T = 0 \Rightarrow = -\frac{RT}{(v-b)^2} + \frac{2a}{Tv^3} = 0$$

$$\left.\frac{\partial^2 P}{\partial v^2}\right)_T = 0 \Rightarrow = \frac{2RT}{(v-b)^3} - \frac{6a}{Tv^4}$$

$$\rightarrow \frac{2\cancel{a}}{(v-b)\cancel{T}\cancel{v^3}} - \frac{\cancel{6}^{3}\cancel{a}}{\cancel{T}v^{\cancel{4}}} = 0$$

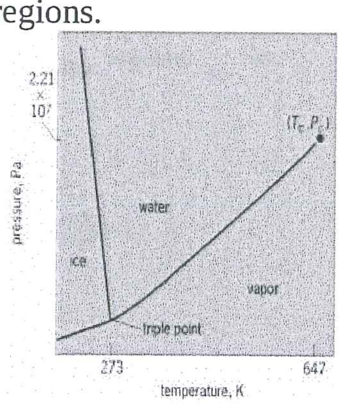
$$2v = 3v - 3b$$
$$\Rightarrow \boxed{v_c = 3b}$$

$$\Rightarrow -\frac{RT}{4b^2} + \frac{2a}{T\,27b^3} = 0$$

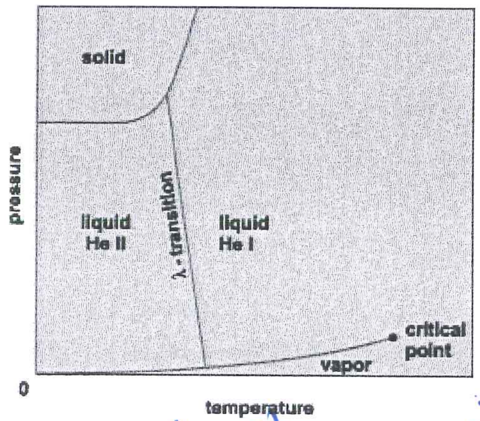
$$T^2 = \frac{8a}{R\,27b} \Rightarrow T_c = \sqrt{\frac{8a}{27bR}}$$

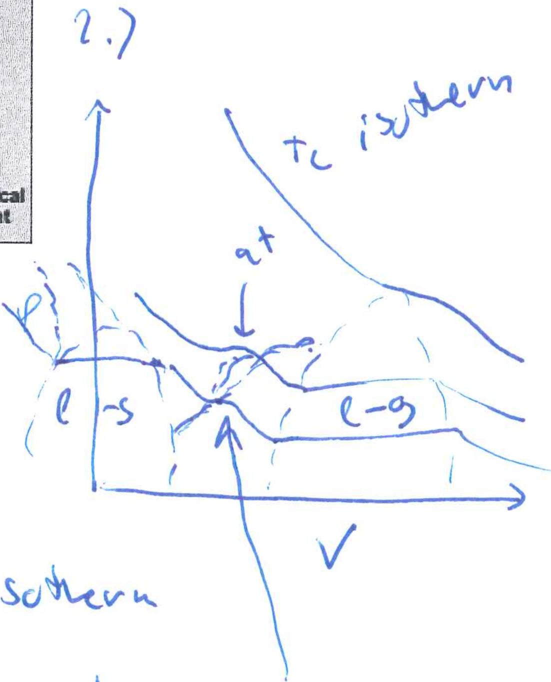
Problem 5: Phase Diagrams

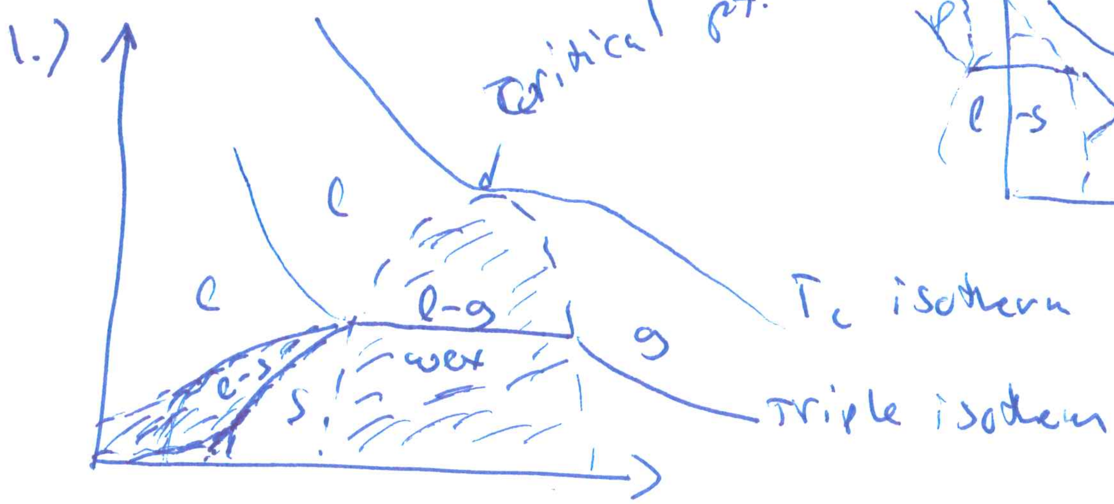
1.) Based on the $P-T$ phase diagram of a one-component system, and assuming that for the solid-liquid coexistence curve it holds that $\frac{\mathrm{d}P}{\mathrm{d}T}_{\mathrm{coex}} < 0$, sketch the $P-V$ phase diagram including coexistence curves. Indicate the single phase regions as well as the coexistence regions, the critical point, and the triple point. Sketch isotherms for all distinct regions.



2.) Based on the $P-T$ phase diagram for bosonic helium, sketch the $P-V$ phase diagram,including coexistence curves. Indicate the single phase regions as well as the coexistence regions, the critical point and the λ-line. Sketch isotherms for all distinct regions.







λ-line
derivatives go to zero

Worksheet 1:

→ Problem 2

$$e^{U/k_BT} = \frac{(N_\uparrow U - E)(N_\downarrow U - E)}{E(LU + E - N_\uparrow U - N_\downarrow U)}$$

$$E(LU + E - N_\uparrow U - N_\downarrow U)e^{U/k_BT} = (N_\uparrow U - E)(N_\downarrow U - E)$$

$$N_\uparrow N_\downarrow U^2 - E(N_\uparrow U + N_\downarrow U)(1 - e^{U/k_BT}) - ELUe^{U/k_BT}$$
$$+ E^2(1 - e^{U/k_BT}) = 0$$

$$AE^2 + BE + C = 0$$

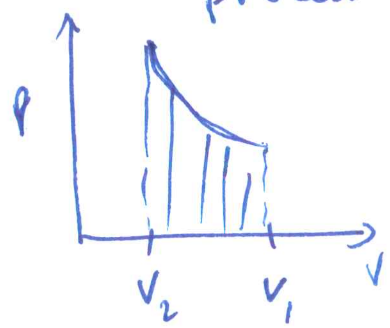
$$A = (1 - e^{U/k_BT})$$
$$B = -(N_\uparrow U + N_\downarrow U)(1 - e^{U/k_BT}) - LUe^{U/k_BT}$$
$$C = N_\uparrow N_\downarrow U^2$$

$$E = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$E = \frac{\left[(N_\uparrow U + N_\downarrow U)(1 - e^{U/k_BT}) - LUe^{U/k_BT}\right] \pm \sqrt{(N_\uparrow U - N_\downarrow U)(1 - e^{U/k_BT})^2 + LUe^{U/k_BT}(1 - e^{U/k_BT}) \cdot 2(N_\uparrow U + N_\downarrow U) + L^2U^2e^{2U/k_BT}}}{2(1 + e^{U/k_BT})}$$

Worksheet 2:

3.) process 1



$\Delta U = \Delta Q - \Delta W \Rightarrow \Delta Q = 0$ (adiabatic)

$dU = \frac{3}{2} nR\,dT = -\frac{nRT}{V} dV$

$\frac{3}{2} \ln T = -\ln V$

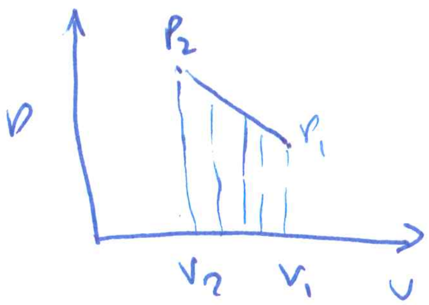
$\left(\frac{T_f}{T_i}\right)^{\frac{3}{2}} = \frac{V_i}{V_f}$ $\quad T_f^{3/2} V_f = T_i^{3/2} V_i$

$V_f = \frac{1}{2} V_i \Rightarrow$ $\quad T_f^{3/2} \cancel{V_i} = 2 T_i^{3/2} \cancel{V_i}$

$T_f = 2^{2/3} T_i$

$\Delta U = nRT_i (2^{2/3} - 1)$

process 2



work done (total)

$\left(\frac{P_1 + P_2}{2}\right)(V_1 - V_2)$

$= -\frac{P_1 V_2 - P_1 V_1 + P_2 V_2 - P_2 V_1}{2}$

$= -\frac{\frac{P_1 V_1}{2} - P_1 V_1 + \frac{P_2 V_1}{2} - P_2 V_1}{2}$

$= \frac{P_1 V_1 + P_2 V_1}{4} = \frac{nRT_i + 2nRT_f}{4}$

$\frac{nRT_i}{4}(1 + 2 \cdot 2^{2/3})$

net work $= \Delta W_{net} = nRT_i \left(-\frac{3}{2} + 2^{2/3}\right)$

efficiency: $\eta = \frac{\Delta W_{net}}{\Delta W_1 + \Delta W_{net}} = \frac{\left(-\frac{3}{2} + 2^{2/3}\right)}{2 \cdot 2^{2/3} - 5/2}$

Worksheet 3:

4.) Berthelot:

$$P_c = \frac{\cancel{R}}{2b}\sqrt{\frac{8aR}{27b\cancel{R}}} - \frac{a}{9b^2}\sqrt{\frac{27bRa}{8ab}}$$

$$P_c = \frac{2}{2b\cdot 3}\sqrt{\frac{2aR}{3b}} - \frac{\cancel{81}}{\cancel{9}b\cdot 2}\sqrt{\frac{3aR}{2b}}$$

$$= \frac{1}{6b}\sqrt{\frac{8}{3}}\left(\frac{aR}{b}\right)^{1/2} - \frac{1}{6b}\left(\frac{3}{2}\right)^{1/2}\left(\frac{aR}{b}\right)^{1/2}$$

$$= \frac{1}{6b}\left(\frac{aR}{b}\right)^{1/2}\left[\left(\frac{8}{3}\right)^{1/2} - \left(\frac{3}{2}\right)^{1/2}\right] = \left(\frac{aR}{216b^3}\right)^{1/2}$$

$$\frac{\sqrt{16}-\sqrt{9}}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

$$P_c = \left(\frac{aR}{216b^3}\right)^{1/2}$$

reduced variables

$$\bar{P} = \frac{P}{P_c} \quad \bar{T} = \frac{T}{T_c} \quad \bar{V} = \frac{V}{V_c}$$

$$\bar{P}\left(\frac{aR}{216b^3}\right)^{1/2} = \frac{R\left(\frac{8a}{27bR}\right)^{1/2}\bar{T}}{b(3\bar{V}-1)} - \frac{a}{\bar{T}\left(\frac{8a}{27bR}\right)^{1/2}\bar{V}^2 9b^2}$$

$\Rightarrow$ a, b cancel $\Rightarrow$ law of corresponding states satisfied