

Widom Scaling Hypothesis

assume:

$$f(T, B) = t^{1/4} \psi(B/t^{x/4})$$

$$t = \left| \frac{T - T_c}{T_c} \right|$$

$$m|_{B=0} = - \left(\frac{\partial f}{\partial B} \right)_T = - (t^{1/4} / t^{x/4}) \psi'(0)$$

$$\beta = \frac{1-x}{\gamma}$$

$$\chi = \frac{\partial m}{\partial B} = - \left(\frac{\partial^2 f}{\partial B^2} \right)_T = t^{\frac{1-2x}{4}} \psi''(0)$$

$$\gamma = \frac{2x-1}{\gamma}$$

$$C_B = -T \left(\frac{\partial^2 f}{\partial T^2} \right)_{B=0}$$

$$\frac{\partial f}{\partial T}$$

$$= - \frac{T_c}{T_c^2} \frac{\partial^2 f}{\partial T^2}$$

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial t} \frac{\partial t}{\partial T} = \frac{1}{T_c} \frac{\partial}{\partial t}$$

$$\frac{\partial^2}{\partial T^2} = \frac{1}{T_c^2} \frac{\partial^2}{\partial t^2}$$

$$\frac{\partial f}{\partial T} \Big|_{T=0} = \frac{1}{4} t^{1/4-1} \psi(B/t^{x/4}) = \frac{1}{4} t^{1/4-1} \psi(0)$$

$$\frac{\partial^2 f}{\partial T^2} = \frac{1}{4} \left(\frac{1}{4} - 1 \right) t^{1/4-2} \psi(0)$$

$$= - \frac{\psi(0)}{T_c} \frac{1}{4} \left(\frac{1}{4} - 1 \right) t^{\frac{1}{4}-2}$$

$$\alpha = 2 - \frac{1}{\gamma}$$

can also write

$$\mathcal{H}(T, B) = B^{1/\gamma} \bar{\psi}(B/t^{x/\gamma})$$

$$u|_{T=T_c} = - \left. \frac{\partial \mathcal{H}}{\partial B} \right|_T = \frac{\bar{\psi}(\infty)}{x} B^{1/\gamma - 1}$$

$$\delta = \frac{x}{1-x}$$

$\Downarrow$

$$\rho = \frac{1-x}{\gamma}$$

$$\delta = \frac{2x-1}{\gamma}$$

$$\alpha = 2 - \frac{1}{\gamma}$$

$$\delta = \frac{x}{1-x}$$

$$\alpha + 2\rho + \delta = 2$$

$$\alpha + \beta(\delta + 1) = 2$$

based on Kadanoff one can also show that

$$(2-\eta)v = \delta$$

$$vd = 2 - \alpha$$

Real Space Renormalization

①

- method to calculate critical exponents based on Kadanoff scaling idea
- ingredients/steps:
 - lattice must have discrete scaling symmetry
 - lattice is defined by a Hamiltonian
 - lattice sites are blocked $\Rightarrow$ new Hamiltonian obtained from block variables $\mathcal{H}^{(0)} \rightarrow \mathcal{H}^{(1)} \rightarrow \mathcal{H}^{(n)}$
 - parameters of the Hamiltonian trace out a trajectory: $\vec{H}_0 \rightarrow \vec{H}_1 \rightarrow \vec{H}_n$
 - trajectories tend towards fixed points: $\vec{H} = \vec{P}(\vec{H})$
 - fixed points can be attractive, repulsive, or mixed
 - ~~consider~~ a high-T fixed point / low-T fixed point are separated by a critical surface
- $\rightarrow$ - on critical surface, trajectories tend toward one fixed point $\Rightarrow \vec{H}^*$ (at least in our case)
 - $\downarrow$
critical point
- to obtain critical exponents one considers trajectories on the critical surface and close to the critical surface
- one considers how these quantities renormalize (how they change along the process)

- for example: $\xi^{(n)} = \frac{g^{(0)}}{b^n}$

- for trajectories starting close to critical surface
at some point trajectory will turn away
and go off to the high or low temperature
fixed points

- turning pt. found by condition $\frac{\xi^{(n)}}{a} \sim 1$

$$\frac{\xi^{(0)}}{ab^n} \sim 1$$

$$\log \xi - n \log a - n \log b = 0$$

$$n \log b = \log \xi - n \log a$$

$$n = \frac{\log \xi}{\log b} - \frac{n \log a}{\log b}$$

- other condition

$$\vec{H} = \vec{H}^* + x_A \lambda_A^n \vec{H}_A + x_R \lambda_R^n \vec{H}_R$$

- repulsive component small initially

- if we are sufficiently close to the
critical surface $\rightarrow$ ~~trajectory will~~
~~stay close, but goes off eventually~~

$$x_R \sim \gamma_R (T - T_c)$$

$$\vec{H} = \vec{H}^* + x_A \lambda_A^n \vec{H}_A + \underbrace{\gamma_R (T - T_c) \lambda_R^n}_{\sim 1} \vec{H}_R$$

turning away when ~ 1

$$\log \gamma_R + \log (T - T_c) + n \log \lambda_R = 0$$

$$n = -\frac{\log \gamma_R}{\log \lambda_R} - \frac{\log (T - T_c)}{\log \lambda_R}$$

$\Downarrow$

$$\xi \sim |T - T_c|^D$$

$\Downarrow$

$$D = \frac{\log b}{\log \lambda_R}$$

Renormalization of B, M, K, G_c

- for linear blocking transformation

$$\sigma_n^{(n+1)} = \frac{1}{b^d} \sum_{i \in S_n} \sigma_i^{(n)}$$

$$w \rightarrow \frac{1}{2} \quad \text{for high-} T$$

$$w \rightarrow 1 \quad \text{for low } T$$

in general not known

renormalizing B

$$B^{(n+1)} \sum_i \sigma_i^{(n+1)} = \frac{B^{(n)}}{b^d} \sum_i \sigma_i^{(n)} = B^{(n)} \sum_i \sigma_i^{(n)}$$
$$B^{(n+1)} = b^d B^{(n)} \Rightarrow \lambda_B = b^d$$

renormalizing M

$$S^{(n+1)} = \frac{S^{(n)}}{b}$$

$$m^{(n)} = \langle \sigma_n^{(n)} \rangle$$

$$m^{(n+1)} = \langle \sigma_n^{(n+1)} \rangle = \frac{1}{b^d} \sum_{i \in S_n} \langle \sigma_i^{(n)} \rangle$$
$$= b^{d(1-w)} m^{(n)}$$

renormalizing G_c

$$G_c^{(n+1)}(x) = b^{2d(1-w)} G_c^{(n)}(bx)$$

$\psi^{(n)}$

renormalization of $\psi^{(n)}$

$$\psi^{(n+1)} = b^{d(1-2\alpha)} \psi^{(n)}$$

to obtain critical exponents first show that renormalization does not alter the relations which define critical exponents:

example: $m \sim B^{1/\sigma}$ as $B \rightarrow 0$ $T \rightarrow T_c$

at stage 0: $m^{(0)} \sim B^{(0)1/\sigma}$

$$\begin{aligned} m^{(n)} &= m^{(0)} \prod_{i=1}^n b^{d(1-w)} \\ B^{(n)} &= B^{(0)} \prod_{i=1}^n b^{dw} \end{aligned}$$

$$\frac{m^{(n)}}{\prod_{i=1}^n b^{d(1-w)}} \approx \frac{B^{(n)1/\sigma}}{[\prod_{i=1}^n b^{dw}]^{1/\sigma}} \Rightarrow m^{(n)} \sim B^{(n)1/\sigma}$$

critical exponents for $T = T_c$

$$G_c(x) \approx x^{-(d-2+\eta)} \quad x \rightarrow \infty \quad T = T_c$$

$$\Downarrow$$

$$\frac{G_c(x_1)}{G_c(x_2)} = \left(\frac{x_1}{x_2} \right)^{-(d-2+\eta)}$$

at T_c (as $n \rightarrow \infty$) $G_c^{(n+1)}(x)$, $G_c^{(n)}(x)$ apply to the same system $\Rightarrow$ indices (n) , $(n+1)$ are irrelevant

it holds that $G_c^{(n+1)}(x) = b^{2d(1-w)} G_c^{(n)}(bx)$

$$\Downarrow$$

$$G_c(x) = b^{2d(1-w)} G_c(bx)$$

$$\frac{G_c(x)}{G_c(bx)} = \left(\frac{1}{b} \right)^{-(d-2+\eta)} = b^{2d(1-w)}$$

$$\Rightarrow \eta = d+2-2wd$$

Exponent δ

$$m \sim \beta^{1/\delta} \quad \beta \rightarrow 0 \quad T = T_c$$

$$\frac{m^{(n+1)}}{m^{(n)}} = \left(\frac{\beta^{(n+1)}}{\beta^{(n)}} \right)^{1/\delta} \Rightarrow b^{d(1-w_c)} = b^{dw_c/\delta}$$

$$\Downarrow \quad \eta = d + 2 - 2dw_c$$

Relation between
critical exponents

$$\delta = \frac{w_c}{1-w_c} = \frac{d+2-\eta}{d-2+\eta}$$

Remaining critical exponents are $T \neq T_c$

- study model at region of 'turning point'

- as renormalising proceeds $\Rightarrow$ T will tend towards T_c
until turning point

- near turning point: $\vec{H}^{(n)} \rightarrow \vec{H}^{(n+1)} \rightarrow \vec{H}^{(n+2)} \rightarrow \dots$

Hamiltonian will move slowly, since it is
almost a fixed point

- we can even control how close we are to critical
point, by trying initial temperatures to
start trajectory which are closer to T_c

$\Downarrow$

$\Downarrow$

$\Downarrow$

in this region: renormalised Hamiltonians will
look very similar $\Rightarrow$ they will display same
critical behaviour

HOWEVER: expectation values will differ greatly,
even T_c will in general change

$$m \sim k(T - T_c)^{\beta}$$

$m, T, T_c \Rightarrow$ change

$k, \beta \Rightarrow$ remain the same

exponent β

$$m \sim |T - T_c|^\beta \quad \beta = 0 \quad T \rightarrow T_c \quad \text{from below}$$

can write in terms of correlation length

$$\xi \sim |T - T_c|^{-\nu}$$

$$m \stackrel{\Downarrow}{\sim} \xi^{-\beta/\nu} \quad \xi \rightarrow \infty$$

$$\frac{m_1}{m_2} \sim \left(\frac{\xi_1}{\xi_2} \right)^{-\beta/\nu}$$

$$b^{d(1-w)} \sim \left(\frac{1}{b} \right)^{-\beta/\nu}$$

$$\frac{\beta}{\nu} = d(1-w) = \beta = \nu d(1-w)$$

$$\boxed{\beta = \frac{\nu}{2}(d + m - 2)}$$

exponent γ

$$\chi \sim |T - T_c|^{-\gamma}$$

$$\chi \sim \xi^{\gamma/\nu}$$

$$\frac{\chi_1}{\chi_2} \sim \left(\frac{\xi_1}{\xi_2} \right)^{\gamma/\nu}$$

$$b^{d(1-2w)} \sim b^{-\gamma/\nu}$$

$$\gamma = \nu d(1-2w)$$

$$\gamma = \nu(2 - m)$$

exponent α

- expresses relation of specific heat (divergence) as a function of temperature

$\Downarrow$

- not related in any simple way to block variables

- need to express free energy in terms of renormalization transformation

at any step n :

$$\hat{f}^{(n)} = \frac{\beta F^{(n)}}{N^{(n)}} = -\frac{1}{N^{(n)}} \log Z^{(n)}$$

(dimensionless free energy)

- consider renormalization transformation

$$P^{(n+1)}(\{\sigma_i^{(n+1)}\}) = \exp(-\mathcal{H}^{(n+1)}(\{\sigma_i^{(n+1)}\}))$$

$\mathcal{H}^{(n+1)}(\{\sigma_i^{(n+1)}\})$ - undetermined up to a constant

For a linear transformation
can fix constant so that

$$\mathcal{H}^{(n+1)}(\{0\}) = 0$$

$$P^{(n+1)}(\{0\}) \stackrel{!}{=} \frac{1}{Z^{(n+1)}}$$

$$\hat{f}^{(n)} \stackrel{!}{=} \frac{1}{N^{(n)}} \log [P^{(n)}(\{0\})]$$

$$P^{(n+1)}(\{0\}) = \sum_{\sigma_i^{(n+1)}=0} P^{(n)}(\{\sigma_i^{(n)}\})$$

$$= \frac{1}{Z^{(n)}} \sum_{\sigma_i^{(n+1)}=0} \exp(-\mathcal{H}^{(n)}(\{\sigma_i^{(n)}\}))$$

$$\frac{1}{Z^{(n+1)}} = \frac{1}{Z^{(n)}} \sum_{\sigma_i^{(n+1)}=0} \exp(-\mathcal{H}^{(n)}(\{\sigma_i^{(n)}\}))$$

$$\frac{1}{Z^{(n+1)}} = \frac{\sum_{\sigma_i^{(n+1)} = 0} \exp[-\mathcal{H}(\{\sigma_i^{(n)}\} s)]}{Z^{(n)}}$$

$$-\log Z^{(n+1)} = -\log Z^{(n)} + \log \sum_{\sigma_i^{(n+1)} = 0} \exp[-\mathcal{H}(\{\sigma_i^{(n)}\} s)]$$

$$f^{(n+1)} = f^{(n)} - g^{(n)}$$

$$\frac{f^{(n+1)}}{N^{(n+1)}} = b^d \frac{f^{(n)}}{N^{(n)}} - \frac{g^{(n)}}{N^{(n)}}$$

$$f^{(n+1)} = b^d (\hat{f}^{(n)} - g^{(n)})$$

$\hat{f}, g$ - all intensive properties

free energy: $f = k_B T \hat{f}^{(0)}$

$$f^{(n)} = k_B T \sum_{n=0}^{\infty} b^{-nd} g^{(n)}$$

- obtain by inverse iteration (assume that $\lim_{n \rightarrow \infty} b^{-nd} \hat{f}^{(n)} \rightarrow 0$)

$$f^{(n+1)} = b^d (\hat{f}^{(n)} - g^{(n)}) \rightarrow \frac{f^{(n+1)}}{b^d} + g^{(n)} = \hat{f}^{(n)}$$

$$f^{(n)} = b^d (\hat{f}^{(n-1)} - g^{(n-1)}) \rightarrow \frac{f^{(n)}}{b^d} + g^{(n-1)} = \hat{f}^{(n-1)}$$

$$\frac{f^{(n+1)}}{b^{2d}} + \frac{g^{(n)}}{b^d} + g^{(n-1)} = \hat{f}^{(n-1)}$$

- can write f in terms of correlation length

$$\xi^{(n)} = \frac{\xi}{b^n}$$

g - dimensionless

$$\text{define: } x_n = a / \xi^{(n)} = \frac{a b^n}{\xi}$$

$$f = k_B T \sum_{n=0}^{\infty} \left(\frac{\xi x_n}{a} \right)^{-d} g(x_n)$$

$$= k_B T \left(\frac{\xi}{a} \right)^{-d} \sum_{n=0}^{\infty} x_n^{-d} g(x_n)$$

$$f = k_B T \left(\frac{\xi}{a} \right)^{-d} \sum_{n=0}^{\infty} r_n^{-d} g(r_n)$$

convert sum to integral

$$r_n = \frac{a b^n}{\xi} \longrightarrow n=0 \Rightarrow r_0 = \frac{a b^0}{\xi}$$

$$\frac{dx}{dn} = \frac{a b^n}{\xi} \log b = x \log b$$

$$\frac{dn}{dx} dx = \frac{dx}{x \log b}$$

1)

$$f = k_B T \left(\frac{\xi}{a} \right)^{-d} \frac{1}{\log b} \int_{a/\xi}^{\infty} x^{-d} g(x) \frac{dx}{x}$$

as $\xi \rightarrow \infty$ f depends on ξ as $f = k_B T \left(\frac{\xi}{a} \right)^{-d} \tilde{A}$

$\tilde{A}$ - constant

Since $\xi \sim (T - T_c)^{-\nu}$

we have $f \sim (T - T_c)^{\nu d}$

specific heat: $C = T \frac{d^2 f}{dT^2} \sim |T - T_c|^{\nu d - 2}$

$$\alpha = 2 - \nu d = \alpha'$$

Scaling laws

$$\begin{aligned}\alpha &= 2 - \nu d \\ \beta &= \frac{1}{2} \nu (d + n - 2) \\ \gamma &= \nu (2 - \eta) \\ \delta &= \frac{d + 2 - \eta}{d - 2 + \eta}\end{aligned}$$

can obtain scaling relations

$$2\beta + \gamma = 2 - \alpha \quad (\text{Rushbrooke})$$

$$2\beta\delta - \gamma = 2 - \alpha \quad (\text{Griffiths})$$

$$\gamma = \nu(2 - n) \quad (\text{Fisher})$$

$$\nu d = 2 - \alpha \quad (\text{Josephson})$$

- in deriving scaling laws, a particular form for the block transformation: can be shown in general

$$e^{-H'(O_i')} = e^{-\beta G} \sum_{\substack{\{O_j\} \rightarrow O_i' \\ \text{constant w.r.t } O_i'}} e^{-H(\{O_j\})}$$

$$\sum_i e^{-H'(O_i')} = e^{-\beta G} \sum_i \sum_{\{O_j\} \rightarrow O_i'} e^{-H(\{O_j\})}$$

↓

$$F' = F - G$$

$$N' \phi(\{H_i'\}) = N \phi(\{H_i\}) - N g(\{H_i\})$$

$$\phi(\{H_i'\}) = b^d (\phi(\{H_i\}) - g)$$

$$f(\{H_i'\}) = b^q (f(\{H_i\}) - g)$$

close to critical point

$$\vec{H}' = \vec{H} + \vec{P}(\vec{H}' - \vec{H})$$

$\Downarrow$

$$f(\lambda_1 x_1, \lambda_2 x_2, \dots) = b^d (f(x_1, x_2, \dots) - g(x_1, x_2, \dots))$$

assumption: singularities in f originate from f not g
(not always true, and often hard to prove, but
in linear transformations it is the case)

$\downarrow$

regular transformation

$$f_{\text{sing}}(\lambda_1 x_1, \lambda_2 x_2, \dots) = b^d f(x_1, x_2, \dots)$$

- one eigenvalue $\lambda_1 > 1 \rightarrow$ near critical point
 $x_1 \sim T - T_c$

- approx: free energy (singular part)
varies as $x_1^{a_1}$

if all $x_i \ll 0$ except x_1

$\Downarrow$

$$(\lambda_1 x_1)^{a_1} = b^d x_1^{a_1}$$

$$\lambda_1 = b^{d/a_1}$$

- one can also make this approximation for
magnetic eigenvalue

in this case $T = T_c$ $B \neq 0$

~~$$\lambda_2 \sim b^{d/a_2}$$~~

$$f_{\text{sing}} \sim x_2^{a_2}$$

$\Downarrow$

$$\lambda_2^{a_2} x_2^{a_2} = b^d x_2^{a_2}$$

$$\lambda_2 = b^{d/a_2}$$

we can thus write:

$$f_{\text{sing}}(b^{y_1} x_1, b^{y_2} x_2, 0, 0) = b^d f_{\text{sing}}(x_1, x_2, 0, 0)$$

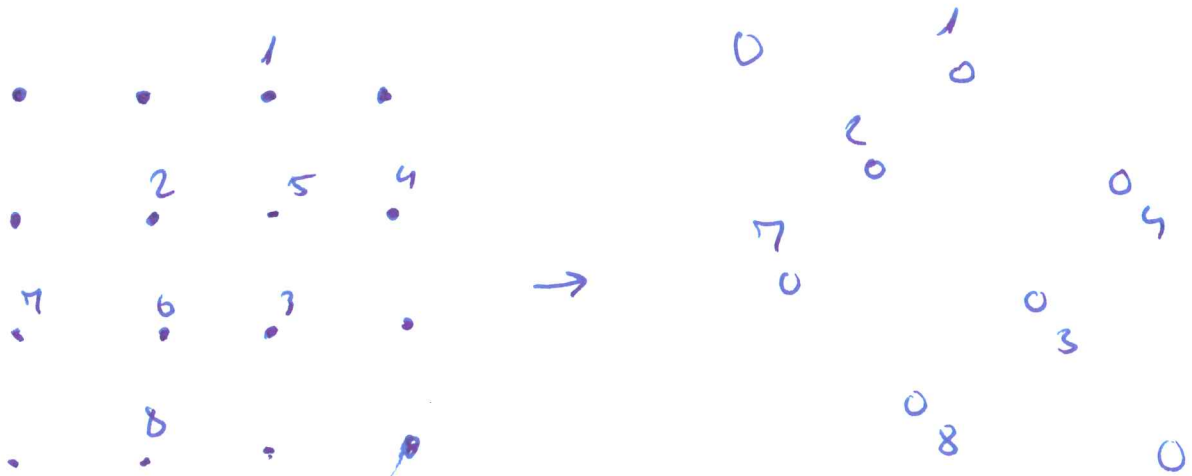
$$b = x_1^{-1/y_1}$$

$$x_1^{d/y_1} f_{\text{sing}}(1, x_1^{-y_2/y_1} x_2, 0, 0) = f_{\text{sing}}(x_1, x_2)$$

$$x_1^{d/y_1} f_{\text{sing}}(1, \frac{x_2}{x_1^{y_2/y_1}}, 0, 0) = f_{\text{sing}}(x_1, x_2)$$

assumption of Widom scaling hypothesis

RG 2D Ising



$$Q = \sum_{\sigma_1 \dots \sigma_N} \dots \exp [K \sigma_5 (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] \\ \times \exp [K \sigma_6 (\sigma_2 + \sigma_3 + \sigma_7 + \sigma_8)]$$

$$= \sum_{\sigma_1} \dots \sum_{\sigma_N} \left[\exp [K (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] \right. \\ \left. + \exp [-K (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] \right] \dots$$

can write:

$$\exp [K (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] + \exp [-K (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] \\ \stackrel{f(K)}{=} f(K) \exp [K' (\sigma_1 \sigma_2 + \sigma_1 \sigma_4 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4)]$$

has to be made equal for all possibilities:

$$\left. \begin{aligned} \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 &= \pm 1 \\ \sigma_1 = \sigma_2 = \sigma_3 = -\sigma_4 &= \pm 1 \\ \sigma_1 = \sigma_4 = -\sigma_3 = -\sigma_2 &= \pm 1 \\ \sigma_1 = -\sigma_2 = \sigma_3 = -\sigma_4 &= \pm 1 \end{aligned} \right\} \begin{array}{l} \text{too many equations} \\ \text{for two} \\ \text{unknowns!} \end{array}$$

One way to go!

$$e^{K(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)} + e^{-K(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)}$$

$$= f(K) \exp \left(K_1 (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \sigma_4 \sigma_1) \right) \\ \exp \left(K_2 (\sigma_1 \sigma_3 + \sigma_2 \sigma_4) \right) \\ \exp \left(K_3 \sigma_1 \sigma_2 \sigma_3 \sigma_4 \right)$$

$\Downarrow$

$\Downarrow$

$$K_1 = \frac{1}{4} \ln \cosh(4K)$$

$$K_2 = \frac{1}{8} \ln \cosh(4K)$$

$$K_3 = \frac{1}{8} \ln \cosh(4K) - \frac{1}{2} \ln \cosh(2K)$$