

Real Space Renormalization

①

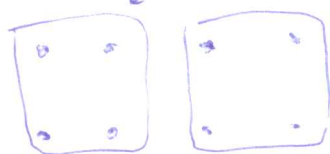
- Kadanoff scaling: great physical intuition
but lack of mathematical precision

Renormalizing the Lattice

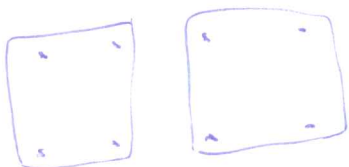
- lattice to be renormalized must have discrete scaling symmetry

- discrete scaling symmetry: when blocking lattice, one should obtain back the same lattice

example:
block of spins



represent each block
by one spin
→ → → →



2D cubic lattice

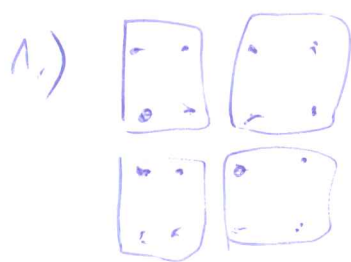


2D cubic lattice

renormalized lattice will have less spins by a factor
 $p = b^d$

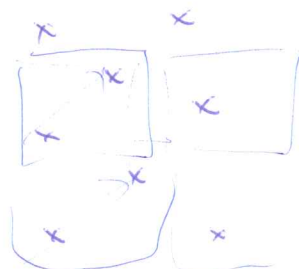
- possible routes to renormalize square lattice

③



$$p = 4 = b^d$$

$$d=1 \quad b=2$$

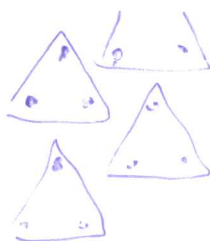


in this

$$p = 2 = b^d$$

$$b = \sqrt{2} \quad d=2$$

- renormalizing the triangular lattice



Block variables

- refer to variables as spins

- original lattice $\sigma_k^{(0)}$ - spins

- new lattice - k th block $\Rightarrow \sigma_k^{(1)}$ - spin associated with block

$$\sigma_k^{(1)} = f(\{\sigma_i^{(0)}\}) \quad \sigma_i^{(0)} \quad i \in \text{block } k$$

- subsequently we shrink our lattice down to size of original

- real-space renormalization:

successive applications of

$$\text{blocking} \Rightarrow \sigma_k^{(n+1)} = f(\{\sigma_i^{(n)}\})$$

$i \in \text{block } k$

- definition of $\pm$

- one has to choose it

- examples: 1.) $\sigma_n^{(n+1)} = A^{(n)} \sum_{i \in S_n} \sigma_i^{(n)}$

↑
rescaling to keep magnitude fixed

$$\sigma_n^{(n+1)} \leftarrow \frac{\sigma_n^{(n+1)}}{\sum_{i \in S_n} \sigma_i^{(n)}}$$

2.) majority rule (for Ising model)
- applicable to odd-membered blocks

3.) choose spin of one particular place
in block (lower-left corner, etc.)

- constraint on $\pm \Rightarrow$ set of states available to $\sigma_i^{(n)}$
should be the same as that of $\sigma_i^{(n+1)}$

(for example: $\sigma_n^{(n+1)} = A^{(n)} \sum_{i \in S_n} \sigma_i^{(n)}$ would
not work for the Ising model)

Renormalization of the Hamiltonian

Boltzmann distribution:

$$P(\{\sigma_i\}) = \frac{1}{Q} \exp[-\mathcal{H}(\{\sigma_i\})]$$

- we can obtain the renormalized probability
distribution

$$P^{(0)}(\{\sigma_i^{(0)}\}) = \frac{1}{Q^{(0)}} \exp[-\mathcal{H}^{(0)}(\{\sigma_i^{(0)}\})]$$

$$P^{(1)}(\{\sigma_i^{(1)}\}) = \frac{1}{Q^{(1)}} \exp[-\mathcal{H}^{(1)}(\{\sigma_i^{(1)}\})]$$

$$= \text{Tr}_{\{\sigma_i^{(0)}\}} P^{(0)}(\{\sigma_i^{(0)}\})$$

- to obtain effective Hamiltonian at stage $n+1$ from stage n

$\Rightarrow$ sum over all configurations in n consistent with a particular configuration at level $n+1$

- expectation values have to yield the same value when evaluated in each set of spins

$$\begin{aligned}\langle X \rangle &= \frac{1}{\sum_{\{\sigma_i^{(n+1)}\}} X(\{\sigma_i^{(n+1)}\}) \exp[-H^{(n+1)}(\{\sigma_i^{(n+1)}\})]} \\ &= \frac{1}{\sum_{\{\sigma_i^{(n+1)}\}} \exp[-H^{(n+1)}(\{\sigma_i^{(n+1)}\})]} \\ &= \frac{1}{\sum_{\{\sigma_i^{(n)}\}} \sum_{\{\sigma_i^{(n+1)}\}} X(\{\sigma_i^{(n+1)}\}) \exp[-H^{(n)}(\{\sigma_i^{(n)}\})]}\end{aligned}$$

- impose one more condition

Form of the Hamiltonian at each iteration is the same, the constants entering the Hamiltonian change

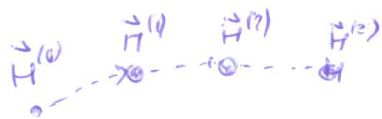
$\Downarrow$ $\Downarrow$
normalisation is a mapping of the Hamiltonian at iteration (n) to another Hamiltonian at iteration $(n+1)$

$$\Downarrow \\ H^{(n+1)} = Q(H^{(n)})$$

- one can also write $\vec{H}^{(n+1)}$ - vector of parameters of Hamiltonian

$$\vec{H}^{(n+1)} = \vec{R}(\vec{H}^{(n)})$$

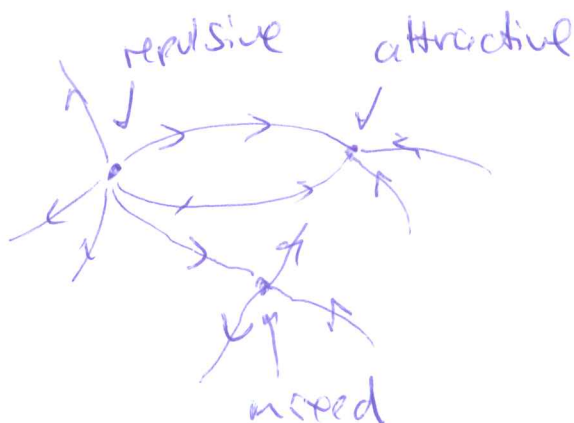
- thus renormalization caused a vector to move in discrete steps around a space (space of parameters)



- a path in parameter space is traced out

Fixed points

- in general there can be points where $\vec{H} = \vec{R}(\vec{H})$
 - i.e. where the renormalization maps Hamiltonian onto itself
- types of fixed points
 - attractive fixed point: in its neighborhood $\vec{H}$ -vectors tend in its direction
 - repulsive fixed point: in its neighborhood $\vec{H}$ -vectors tend away from it
 - mixed fixed point: in some directions towards in some others away



to determine type of fixed point, consider the neighborhood of fixed point

$$\vec{H}_0 + \delta \vec{H}' = \vec{R}(\vec{H}_0 + \delta \vec{H}) = \vec{R}(\vec{H}_0) + \vec{M} \delta \vec{H}$$

$$M_{ij} = \left. \frac{\partial R_i(\vec{H})}{\partial H_j} \right|_{\vec{H} = \vec{H}_0}$$

for stationary point

$$\delta \vec{H}' = \vec{M} \cdot \delta \vec{H}$$

diagonalize $\vec{M} \Rightarrow$ for ~~positive~~ eigenvalues $\lambda_e > 1$

direction is repulsive

$\Rightarrow$ for eigenvalues $\lambda_A < 1$

direction is attractive

- for systems undergoing a phase transition, identities of ~~the~~ two of the fixedpoints can be understood as follows

1) high-temperature fixed point ($T \rightarrow \infty$)

- all our variables assume random values and become uncorrelated

- effective Hamiltonian of system at $T \rightarrow \infty$

- why is it a fixed point?

- if we block an uncorrelated system of spins $\Rightarrow$ resulting system will also be uncorrelated

2.) low-temperature fixed point ($T \rightarrow 0$)

- effective Hamiltonian for system at $T \rightarrow 0$

- if a system is maximally correlated (ordered completely) $\Rightarrow$ renormalized system will also be completely ordered

low-T and high-T fixed points are attractive

- consider state above $T_c \Rightarrow$ at each iteration correlation length shrinks by factor b

$$s^l = \frac{s}{b}$$

$$\text{as } n \rightarrow \infty \quad s^l \rightarrow 0$$

- tends to high-T fixed point

- below T_c

- state is ordered, but not completely

- as we perform blocking iterations, disorder (or incomplete order) will vanish

$n \rightarrow \infty$ tends toward low-T fixed point

② $T_c \Rightarrow$ critical surface separates those fixed points which are ~~separated from~~ low T_c / high T_c

- possible modes of behaviour on the critical surface

* as $n \rightarrow \infty \quad \vec{r}_i \rightarrow \vec{r}_i^*$ (finite limit)

* as $n \rightarrow \infty \quad \vec{r}_i \rightarrow \infty$

* as $n \rightarrow \infty \quad \vec{r}_i$ goes in a closed path

* as $n \rightarrow \infty \quad \vec{r}_i$ moves chaotically on critical surface

* as $n \rightarrow \infty$

- for us: Only first two options important

(last two fall in field of 'critical dynamics')

- finite fixed point $\Rightarrow$ attractive on critical surface
- infinite critical fixed point $\Rightarrow$ can often be mapped onto finite critical fixed point

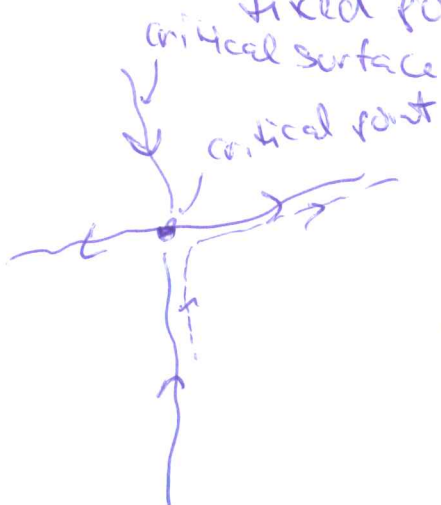
$\Downarrow \quad \Downarrow \quad \Downarrow$

at critical fixed point matrix $\tilde{M}$ has one eigenvalue $\lambda_1 < 1$

others $\lambda_i > 1$

Calculation of ν

- start with Hamiltonian just a little above ~~start~~ T_c and field zero ($B=0$)
- ~~iteration will lead us in direction of critical point~~
- iterating will ~~lead~~ generate a trajectory
 - trajectory will stay close to critical surface at first, but it will turn away from it and head off to infinity (attractive fixed point at $T \rightarrow \infty$)



Question: when does trajectory turn away?

- to answer: consider correlation length ξ near T_c ξ diverges
as $T \rightarrow \infty$ $\xi \rightarrow 0$

$\Downarrow$
crossover: correlation length and lattice parameter are comparable

- correlation length as a function of iterations

$$\xi^{(n)} = \frac{\xi}{b^n}$$

$\Rightarrow$ crossover at $\frac{\xi}{ab^n} = u \quad u \sim 1$

- one can also answer this question differently

- in vicinity of critical point $\vec{H}^*$ one can

write

$$\vec{H} = \vec{H}^* + \lambda_A x_A \vec{H}_A + \lambda_R x_R \vec{H}_R$$

x_A, x_R - numbers

$\vec{H}_R, \vec{H}_A$ - eigenvectors of matrix $\vec{H}$

λ_A, λ_R - eigenvalues of matrix $\vec{H}$

after m - iterations

$$\vec{H} = \vec{H}^* + \lambda_A^m x_A \vec{H}_A + \lambda_R^m x_R \vec{H}_R$$

if we let $T \rightarrow T_c \quad x_R \rightarrow 0$

in this case $\vec{H} \rightarrow \vec{H}^*$ as $n \rightarrow \infty$

- let's expand $x_R \approx \gamma_R (T - T_c)$

$$\vec{H} = \vec{H}^* + \lambda_A^m x_A \vec{H}_A + \lambda_R^m \gamma_R (T - T_c) \vec{H}_R$$

- turning away from critical point expected to

occur when

$$\lambda_R^m \gamma_R (T - T_c) = v \quad v \sim 1$$

$v \sim$ same order of magnitude as u

$$\Rightarrow u \sim v \Rightarrow \xi = ab^n \lambda_R^m \gamma_R$$

- assume $m \rightarrow n$ | eliminate n from two definitions

$$\frac{\xi}{a b^n} = 4 \quad \lambda_n^n \gamma_n (T - T_c) = 0$$

$$\log \xi - \log a - n \log b = 0$$

$$n \log \lambda_n + \log \gamma_n + \log (T - T_c) = 0$$

$$\frac{\log \xi - \log a}{\log b} = - \frac{\log \gamma_n}{\log \lambda_n} - \frac{\log (T - T_c)}{\log \lambda_n}$$

$$\Downarrow$$

$$\nu = \frac{\log b}{\log \lambda_n} \Rightarrow \xi \sim (T - T_c)^{-\nu}$$

both b and λ_n are available $\Rightarrow$ we can calculate ν

Renormalization of B , H , K and G_c

- for these quantities renormalization procedure has to be specified specifically

~~σ_n~~ $\Rightarrow$ 1

- here we study case

$$\sigma_n^{(n+1)} = \frac{1}{b^{dw}} \sum_{i \in S_n} \sigma_i^{(n)}$$

[this is just $\sigma_n^{(n+1)} = A^{(n)} \leq \sigma_1^{(n)}$ with $A^{(n)} = \frac{1}{b^{dw}}$]

d -dimensionality

w - some scaling parameter

- choice must guarantee that spins in renormalization

remain finite / construction: mean-square

value constrained to remain finite

$\Downarrow$

$\Downarrow$

w chosen so that mean-square of $\langle \sigma_i^{(n)} \rangle^2$ is a constant of transformation

Calculations

- w depends on temperature
- in general not easy

- one can do it at high-T fixed point and low-T fixed point

- high-T $\Rightarrow$ spins become uncorrelated
 $\Rightarrow$ can write their distributions as

$$p^{(n)}(\sigma_i^{(n)}) = \frac{1}{b^n \Delta} \exp\left(-\frac{\sigma_i^{(n)2}}{2\Delta}\right)$$

- mean-square value of sum $\sum_{i \in S_n} \sigma_i^{(n)} \rightarrow b^d \Delta$

- mean-square value of sum $\sum_{i \in S_n}^{(n+1)} \rightarrow b^{d(1-2w_1)} \Delta$

to keep it constant: $\boxed{w_n = \frac{1}{2}}$

- low-T fixed point:

ordered state: all spins lined up $\uparrow\uparrow\uparrow\uparrow\ldots$

mean square value remains the same $\boxed{\text{if } w_2 = 1}$

at T_c : no general result

can be calculated in some cases

Non-zero external field

$$B^{(n+1)} \leq \sigma_i^{(n+1)} = \frac{B^{(n+1)}}{b^{dw}} \leq \sigma_i^{(n)} \quad (\text{from foregoing explanation})$$

$$\Downarrow$$

$$b^{dw} B^{(n)} = B^{(n+1)}$$

- for nonzero B - increase dimension by one (or parameter space) and add magnetic eigenvalue

$$\lambda_B = b^{dw}$$

$$B^{(n+1)} = b^{dw} B^{(n)}$$

$\Rightarrow$ does not "mix" with other eigenvalues

Show later that $\eta = d+2 - 2d\omega_c$

$$\log \lambda_B = b^{d\omega_c}$$

$$\log \lambda_B = d\omega_c \log b$$

$$\eta = d+2 - 2 \frac{\log \lambda_B}{\log b}$$

short-coming of linear renormalization transformations
 $\Rightarrow$ cannot calculate ω

Renormalization of μ, ν, G_c

$$\xi^{(n+1)} = \frac{\xi^{(n)}}{b}$$

$$m^{(n)} = \langle \sigma_n^{(n)} \rangle \Rightarrow m^{(n+1)} = \langle \sigma_n^{(n+1)} \rangle = \frac{1}{b^{d\omega_c}} \sum_{i \in S_n} \langle \sigma_i^{(n)} \rangle$$

$\underbrace{\hspace{10em}}_{b^d \langle \sigma_i^{(n)} \rangle}$

$$\Downarrow$$

$$m^{(n+1)} = b^{d(1-\omega)} m^{(n)}$$

for $G_c(i,j)$

$$G_c^{(n)}(i,j) = \langle \sigma_i^{(n)} \sigma_j^{(n)} \rangle_c$$

$$G_c^{(n+1)}(i,j) = \frac{1}{b^{d\omega_c}} \sum_{\substack{h \in S_i \\ l \in S_j}} \langle \sigma_h^{(n)} \sigma_l^{(n)} \rangle_c$$

$\Downarrow$

$$G_c^{(n+1)}(i,j) = b^{2d(1-\omega)} G_c^{(n)}(i,j)$$

$\Downarrow$

since distances rescale in unrenormalized system

$$G_c^{(n+1)}(x) = b^{2d(1-\omega)} G_c^{(n)}(bx)$$

$$\underline{\underline{\nu^{(n+1)} = b}}$$

How to obtain critical exponents?

$\xi^{(n)}, m^{(n)}, \chi^{(n)}, G_c^{(n)} \rightarrow$ values of particular physical quantities at iteration (n)

$\Rightarrow$ problem: are they a reasonable representation of the real system, i.e. system at iteration (0) ?

the answer is no, but ~~it~~ they can be used to obtain critical exponents!

consider:
 $m \sim B^{1/5} \quad B > 0 \quad T = T_c \quad (\text{iteration } (0))$

at n th stage of renormalization it should hold that

$$m^{(n)} \sim B^{(n)1/5}$$

but! since

$$\begin{aligned} B^{(n+1)} &= b^{dw} B^{(n)} \\ m^{(n+1)} &= b^{d(1-w)} m^{(n)} \end{aligned}$$

$$\Downarrow$$
$$B^{(n)} = B^{(0)} \prod_{i=1}^n b^{dw^{(i)}}$$

$$m^{(n)} = m^{(0)} \prod_{i=1}^n b^{d(1-w^{(i)})}$$

$$\Downarrow$$
$$m^{(0)} \prod_{i=1}^n b^{d(1-w^{(i)})} = B^{(0)1/5} \left(\prod_{i=1}^n b^{dw^{(i)}} \right)^{1/5}$$

point is that the proportion between $m^{(n)}$ and $B^{(n)}$ is independent of (n)

Critical Exponents for $T=T_c$ (η, d)

- critical exponents are independent of n

- at T_c ~~$n \rightarrow \infty$~~ as $n \rightarrow \infty$ we have $\vec{H}^{(n)} \rightarrow \vec{H}^*$

$\Downarrow$

renormalization makes no difference

(same system)

exponent η

$$G_c(x) \sim x^{-(d-2+\eta)} \quad x \rightarrow \infty \quad T=T_c$$

$$\frac{G_c(x_1)}{G_c(x_2)} = \left(\frac{x_1}{x_2} \right)^{-(d-2+\eta)}$$

$$G_c^{(n+1)}(x) = b^{2d(1-\omega)} G_c^{(n)}(bx) \quad (\text{derived above})$$

at T_c ~~$G_c^{(n+1)}$~~ as $n \rightarrow \infty$ $G_c^{(n+1)}$ and $G_c^{(n)}$ apply to same system

$$\cancel{G_c^{(n+1)}(x)} = b^{2d(1-\omega)} \cancel{G_c^{(n)}(bx)}$$

$$\frac{G_c(x_1)}{G_c(x_2)} = \frac{G_c^{(n+1)}(x_1)}{G_c^{(n)}(bx_2)} = b^{2d(1-\omega)} = \cancel{x^{-(d-2+\eta)}}$$

$$\cancel{2d(1-\omega)} = \cancel{-(d-2+\eta)}$$

$$= \left(\frac{x_1}{bx_2} \right)^{-(d-2+\eta)} = b^{d-2+\eta}$$

$$2d(1-\omega) = d-2+\eta$$

$$\eta = 2-d+2d-2d\omega$$

$$\boxed{\eta = 2+d-2d\omega}$$

expected δ can be obtained in a similar fashion

$$m \sim B^{1/\delta}$$

$$\frac{m_1}{m_2} = \left(\frac{b_1}{b_2} \right)^{1/\delta}$$

Since: $m^{(n+1)} = b^{d(1-w)} m^{(n)}$

$$B^{(n+1)} = b^{dw} B^{(n)}$$

$$\Downarrow$$
$$b^{d(1-w)} = b^{dw/\delta}$$

$$d(1-w) = \frac{dw}{\delta}$$

$$\delta = \frac{w}{1-w} = \frac{d+2-n}{d-2+n}$$