

Kadanoff Scaling

block correlation function

$$C(r_L, \varepsilon_L) = \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle$$

$$S_I' = \sum_{i \in I} S_i \quad \text{blocking}$$

$$\sum S_I = S_I'$$

$$\begin{aligned} C(r_L, \varepsilon_L) &= \mathbb{Z}^{-2} [\langle S_I' S_J' \rangle - \langle S_I' \rangle \langle S_J' \rangle] \\ &= \mathbb{Z}^{-2} \sum_{i \in I} \sum_{j \in J} [\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle] \end{aligned}$$

physical assumption: $L \gg \xi$

$$\begin{aligned} C(r_L, \varepsilon_L) &= \mathbb{Z}^{-2} L^{2d} [\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle] \\ &= \mathbb{Z}^{-2} L^{2d} C(r, \varepsilon) \end{aligned}$$

$$r_L = L^x r$$

$$\varepsilon_L = \varepsilon L^y$$

$$\begin{aligned} \text{since } \sum_i S_i &= \sum_{i \neq j}^{N/L^d} S_i = \sum_{i=1}^{N/L^d} S_i' = \sum_{i=1}^{N/L^d} S_i \\ &= \sum_{i=1}^{N/L^d} S_i' = \sum_{i=1}^{N/L^d} S_i \\ &\quad \downarrow \\ &\quad B_L = L^y B \quad \gamma = L^y \end{aligned}$$

$$C\left(\frac{r}{L}, \varepsilon L^y\right) = L^{2(d-y)} C(r, \varepsilon)$$

choose $L = r/a$

$$C(a, \varepsilon \left(\frac{r}{a}\right)^y) = \left(\frac{r}{a}\right)^{2(d-y)} C(r, \varepsilon)$$

$$\cancel{C(r, \varepsilon)} =$$

$$C(r, \varepsilon) = \left(\frac{r}{a}\right)^{2(\gamma-d)} C(a, \varepsilon \left(\frac{r}{a}\right)^x)$$

introduce $\xi \sim (T - T_c)^{-\gamma}$

and $\varepsilon \rightarrow 0 \Rightarrow C(r, 0) = \left(\frac{r}{a}\right)^{2(\gamma-d)} C(a, 0)$

expect. $\left(\frac{1}{r}\right)^{d-2+\eta}$

$$d-2+\eta = 2d-\gamma$$

$$\Rightarrow \boxed{\gamma = \eta - d - 2}$$

for $\nu \Rightarrow$

$$\frac{r}{\xi} = r \varepsilon^\nu = \frac{r}{a} \left(\frac{r}{a}\right)^{\nu x}$$

$$\frac{r}{\xi} = \frac{r}{a} \Rightarrow \frac{r}{a} = \left(\frac{r}{a}\right)^{\nu x} \Rightarrow \nu x = 1$$

$$\frac{1}{\nu} \varepsilon r^x \rightarrow \varepsilon^{\frac{1}{\nu} + x} \Rightarrow x = \frac{1}{\nu}$$

Examples of Models

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Ising model

- invented by Lenz
- solved in 1D by Ising
- solved in 2D by Onsager
- critical exponents contradict Landau theory
- in 3D — no exact solution
 - numerical solution: Monte Carlo

Lattice-gas model (for non-ideal gas)

- divide space into cubic grid
 - molecules/atoms occupy cubes, only one per cube
 - interaction:

$$Z \sum_{i,j} J_{ij} e_i e_j$$

$e_i \rightarrow 1$ (there is molecule at site i)

0 (there is no molecule at site i)

$$Z = \sum_{\{e_i\}} \exp[\mu \sum_i e_i - 2\beta \sum_{i,j} J_{ij} e_i e_j]$$

μ - chemical potential

β - inverse temperature

- equivalent to Ising model

- XY - Heisenberg models

$$\text{Hamiltonian: } H = \frac{1}{2} \sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j + B \sum_i \vec{S}_i$$

Correlation functions

* 1D Ising model

* interested in $\langle f(s_i) f(s_{i+1}) \rangle$

$f(s)$ - function of spin on site i

spins s_{i+1} and s_i are separated by

i lattice spacing

$$* \langle f(s_i) f(s_{i+1}) \rangle = \frac{\sum_{s_1} \dots \sum_{s_N} \prod_{i=1}^N e^{-\beta J s_i s_{i+1}} f(s_i) f(s_{i+1})}{Q}$$

$$* Q = \sum_{s_1} \dots \sum_{s_N} e^{-\beta J s_1 s_{i+1}} = \text{Tr } T^N$$

$$T = \begin{pmatrix} e^{-\beta J} & e^{\beta J} \\ e^{\beta J} & e^{-\beta J} \end{pmatrix}$$

diagonalize

$$\begin{vmatrix} e^{-\beta J} - \lambda & e^{\beta J} \\ e^{\beta J} & e^{-\beta J} - \lambda \end{vmatrix} = 0$$

$$(e^{-\beta J} - \lambda)^2 - e^{2\beta J} = 0$$

$$e^{-2\beta J} - 2\lambda e^{-\beta J} + \lambda^2 - e^{2\beta J} = 0$$

$$\lambda = \frac{e^{-\beta J} \pm \sqrt{4e^{-2\beta J} - 4(e^{-2\beta J} - e^{2\beta J})}}{2}$$

$$= 2 \cosh \beta J, -2 \sinh \beta J$$

$$Q = 2^N \cosh^N \beta J + (-1)^N 2^N \sinh^N \beta J$$

for $N \rightarrow \infty$ larger term dominates

$$Q = 2^N \cosh^N \beta J$$

correlation function:

$$* \langle f(s_i) f(s_{i+1}) \rangle = \frac{1}{Q} \sum_{s_1} \dots \sum_{s_N} \prod_{i=1}^N e^{-\beta J s_i s_{i+1}} f(s_i) f(s_{i+1})$$

write as: $\Rightarrow \frac{\sum_{s_1} \dots \sum_{s_N} f(s_1) \left[\prod_{i=1}^I e^{-\beta J s_i s_{i+1}} \right] f(s_{i+1}) \left[\prod_{i=i+1}^N e^{-\beta J s_i s_{i+1}} \right]}{Q}$

note: $f(s_i)$ can be represented as a diagonal matrix

$$F_{\mu\nu} = f(s_\mu) \delta_{\mu\nu}$$

$$\langle f(s_i) f(s_{i+1}) \rangle = \frac{\text{Tr}(F T^I F T^{N-I})}{\text{Tr} T^N}$$

* can switch representation

$T \rightarrow$ diagonal which will make F non-diagonal
but symmetric! (real eigenvalues)

$$\langle f(s_i) f(s_{i+1}) \rangle = \frac{\sum_i \sum_j F_{ij} \lambda_j^I F_{ji} \lambda_i^{N-I}}{\sum_i \lambda_i^N} \quad \lambda_0 \rightarrow \text{larger eigenvalue}$$

* either $I \gg N-I$ or $I < N-I$

we assume $I < N-I$

then the dominant term will be

$$F_{00}^2 + |F_{01}|^2 \left(\frac{\lambda_1}{\lambda_0} \right)^I \quad \lambda_1 \rightarrow \text{smaller eigenvalue}$$

$$F_{00} = \langle f(s_i) \rangle = \langle f(s_{i+1}) \rangle$$

$$G^{(2)}(I) = \langle f(s_i) f(s_{i+1}) \rangle - \langle f(s_i) \rangle \langle f(s_{i+1}) \rangle = |F_{01}|^2 \left(\frac{\lambda_1}{\lambda_0} \right)^I$$

* as a result $\Rightarrow$ no phase transition

$$G_I^{(n)} = |F_0|^n \left(\frac{\lambda_1}{\lambda_0} \right)^I = A e^{-I/\xi}$$

$$\frac{\lambda_1}{\lambda_0} = e^{-1/\xi}$$

$$\Rightarrow \xi = \frac{1}{\log \frac{\lambda_1}{\lambda_0}}$$

correlation length always finite!!!

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Mean-field theory

Ising model: (ferromagnetic)

~~$$H = -J \sum_{\langle i,j \rangle} S_i S_j$$~~

$$H = -J \sum_{\langle i,j \rangle} S_i S_j$$

write S_i as $S_i = \bar{S}_i + \delta S_i$

$\bar{S}_i$ - average of S_i

δS_i - deviation from average

$$H = -J \sum_{\langle i,j \rangle} (\delta S_i + \bar{S}_i)(\delta S_j + \bar{S}_j)$$

$$= -J \sum_{\langle i,j \rangle} (\delta S_i \delta S_j + \delta S_i \bar{S}_j + \delta S_j \bar{S}_i + \bar{S}_i \bar{S}_j)$$

approximation $\delta S_i \delta S_j$ - second order in deviation $\Rightarrow \rightarrow 0$

$$H_{MF} = -J \sum_{\langle i,j \rangle} (\delta S_i \bar{S}_j + \delta S_j \bar{S}_i + \bar{S}_i \bar{S}_j)$$

since $S_i = S_i - \bar{S}$

$$H_{MF} = -J \sum_{\langle i,j \rangle} [(S_i - \bar{S})\bar{S}_j + (S_j - \bar{S})\bar{S}_i + \bar{S}_i \bar{S}_j]$$

$$= -J \sum_{\langle i,j \rangle} [S_i \bar{S}_j + S_j \bar{S}_i - \bar{S}_i \bar{S}_j]$$

essential simplification: we no longer have a term $S_i S_j$, in which nearest neighbour spins are coupled

$$H_{MF} = -J \sum_i S_i \sum_j \bar{S} + J \sum_{\langle i,j \rangle} \bar{S}_i^2$$

since $\bar{S}_i = \bar{S}$ (same for all lattice sites)

z - number of nearest neighbors

$$H_{MF} = -J \sum_i z \bar{S} S_i = -J z \bar{S} \sum_i S_i + \frac{J N z \bar{S}^2}{2}$$

partition function

$$Q = \sum_{\{S_i\}} e^{-\beta J z \bar{S} \sum_i S_i - \frac{\beta J N z \bar{S}^2}{2}}$$

$$= \left[\sum_{S_i = \pm 1} e^{\beta J z \bar{S} S_i} \right]^N e^{-\frac{\beta J N z \bar{S}^2}{2}}$$

~~Free energy~~

Free energy: $F = -\frac{1}{\beta} \ln Q$

find $\bar{S}$ which minimizes free energy

$$\frac{\partial F}{\partial \bar{S}} = 0$$

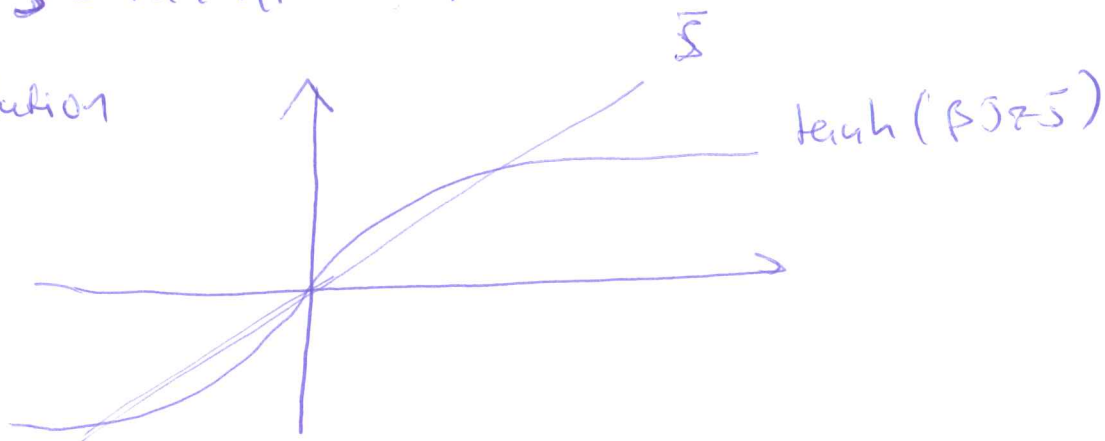
$$\Rightarrow \frac{1}{Q} \frac{\partial Q}{\partial \bar{S}} = 0 \Rightarrow$$

$$\frac{\sum_{S_i = \pm 1} e^{\beta J z \bar{S} S_i} S_i}{\sum_{S_i = \pm 1} e^{\beta J z \bar{S} S_i}} = \bar{S}$$

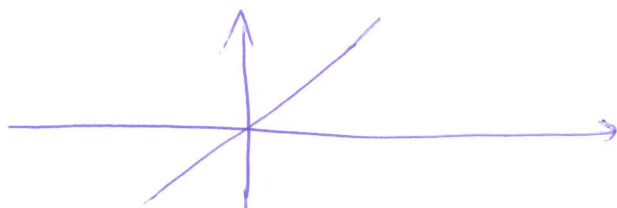
$$\bar{S} = \frac{\sum_{S_i = -1}^1 e^{\beta J \bar{S} S_i} S_i}{\sum_{S_i = -1}^1 e^{\beta J \bar{S} S_i}} = \frac{\sinh(\beta J \bar{S})}{\cosh(\beta J \bar{S})}$$

$$\bar{S} = \tanh(\beta J \bar{S})$$

solution



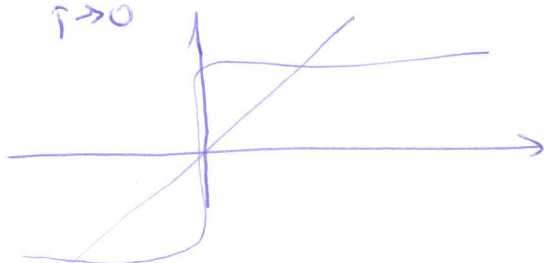
$$\beta \rightarrow 0 \quad (T \rightarrow \infty)$$



$$\bar{S} = 0$$

(disordered)

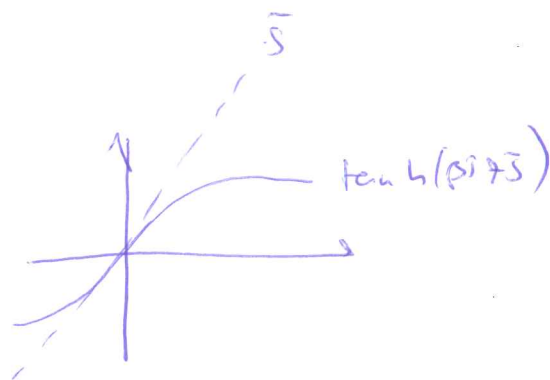
$$\beta \rightarrow \infty \quad T \rightarrow 0$$



$$\bar{S} \rightarrow 1$$

critical
phase transition occurs if

$$\beta J_c = 1$$



Critical Exponents in Mean-Field Theory

$$t = \frac{T - T_c}{T_c} \quad b = \beta B$$

$$\bar{S} = \tanh\left(\frac{\bar{S}}{1+t} + b\right) \quad \beta_c T_c = 1$$

spontaneous magnetization at $b = 0$

$$\tanh(x) = x - \frac{x^3}{3} + \dots$$

$$\bar{S} = \frac{\bar{S}}{1+t} - \frac{\bar{S}^3}{3(1+t)^3}$$

$\Downarrow$

nonzero solutions: $\frac{1+t-1}{1+t} = -\frac{\bar{S}^2}{3(1+t)^2}$

$$\bar{S}^2 = -3t(1+t)^2$$

$$\Rightarrow \bar{S} \sim |t|^{1/2}$$

$$\boxed{\beta = 1/2}$$

at finite b

$$\bar{S} = \tanh\left(\frac{\bar{S}}{1+t} + b\right)$$

expansion

$$\bar{S} = \frac{\bar{S}}{1+t} + b$$

(above T_c)

$\Rightarrow$ above T_c $\bar{S}$ depends linearly on b

$$\chi = \beta \frac{\partial \bar{S}}{\partial b}$$

$$\rightarrow \frac{\partial \bar{S}}{\partial b} = \frac{1}{1+t} \frac{\partial \bar{S}}{\partial b} + 1$$

$$\Rightarrow \frac{\partial \bar{S}}{\partial b} \sim \frac{1}{t}$$

below T_c

expand to third order

$$\bar{s} = \tanh\left(\frac{\bar{s}}{1+t} + b\right) \approx \frac{\bar{s}}{1+t} + b - \frac{1}{3}\left(\frac{\bar{s}}{1+t} + b\right)^3$$

$$\frac{\partial \bar{s}}{\partial b} = \frac{1}{1+t} \frac{\partial \bar{s}}{\partial b} + 1 - \left(\frac{\bar{s}}{1+t} + b\right)^2 \left[\frac{\partial \bar{s}}{\partial b} \frac{1}{1+t} + b \right]$$

set $b=0$

$$\frac{\partial \bar{s}}{\partial b} = \frac{1}{1+t} \frac{\partial \bar{s}}{\partial b} + 1 - \left(\frac{\bar{s}}{1+t}\right)^2 \left[\frac{\partial \bar{s}}{\partial b} \frac{1}{1+t} \right] + \frac{1}{1+t}$$

for $\bar{s}$ we use solution below T_c

$$\Rightarrow \bar{s}^2 = -3t(1+t)^2$$

$$\frac{t}{1+t} \frac{\partial \bar{s}}{\partial b} = 1 + \frac{3t(1+t)^2}{1+t} \left[\frac{\partial \bar{s}}{\partial b} \frac{1}{1+t} \right]$$

$$\frac{t - 3t(1+t)}{1+t} \frac{\partial \bar{s}}{\partial b} = 1$$

$$\Rightarrow \frac{\partial \bar{s}}{\partial b} \sim \frac{1}{t}$$

$$\gamma = \gamma' = 1$$

for C_B

$$Q = \left[\cosh(\beta T \gamma \bar{s}) e^{-\frac{\beta T \gamma \bar{s}^2}{2}} \right]^N$$

for $T > T_c$ $\bar{s} = 0$

$$Q = 1 \Rightarrow C_B = 0 \Rightarrow \alpha = 0$$

for $T < T_c \Rightarrow C_B$ can be shown to be a constant $\alpha' = 0$

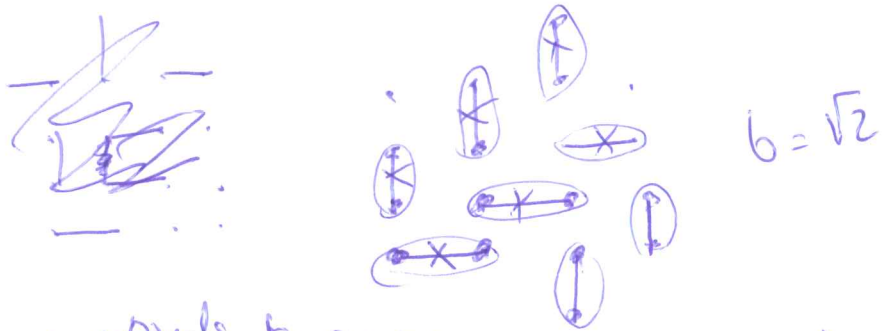
magnetization:

$$m = m + b - \frac{1}{3}m^3 - \frac{1}{3}b^3 \Rightarrow b \sim m^3$$

Renormalization

- blocking $\Rightarrow$ idea is to group lattice sites together in such a way that one obtains the same lattice

example:



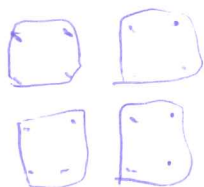
each block corresponds to a new lattice site in new system

(*)
original sites form a square lattice
new sites (x) form a square lattice

lattice parameter of original lattice: a
lattice parameter of new lattice: $a' = ba$

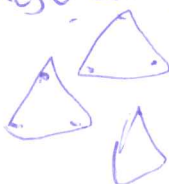
- all new dimensions are then reduced by b
- number of sites of lattice has shrunk by factor P
- $P = b^d$ d -dimensionality of system

other examples
square lattice



new lattice square
 $b = 2$

triangular lattice



new lattice triangular
 $b = \sqrt{3}$

Block variables

- suppose we have spins $\sigma_i^{(0)}$ for original lattice
- after blocking need spins which describe new

system:

$$\sigma_i^{(1)} = f(\sigma_j^{(0)}) \quad j \in S_i$$

$\sigma_i^{(1)}$ depends on spins of particular block

- if we keep blocking

$$\sigma_i^{(n+1)} = f(\sigma_j^{(n)}) \quad j \in S_i$$

- for f one can choose a number of functions

$$* \sigma_n^{(n+1)} = A^{(n)} \sum_{j \in S_n} \sigma_j^{(n)} \quad (\text{similar to Kadanoff})$$

or

$$* \sigma_n^{(n+1)} = \sum \sigma_i^{(n)}$$

$i \in S_n$ (randomly
chosen, or
regularly
chosen)

$\mathbb{Z}$ * majority rule

- constraint in choosing $f(\sigma)$

- set of new variables equal to original ones

original spins in Ising: $\sigma_i^{(0)} = \pm 1$

$\Rightarrow$ new ones have to be $\sigma_i^{(1)} = \pm 1$
also

Renormalization of Hamiltonian

at level 0 $\Rightarrow$ Hamiltonian is given

Ising: $H^{(0)} = \sum_{ij} J_{ij} \sigma_i^{(0)} \sigma_j^{(0)}$

in general $\Rightarrow P(\{s, \beta\}) = \exp[-\beta H(\{s, \beta\})]$

$\Downarrow$

$\Downarrow$

$$P^{(0)}(\{\sigma_i^{(0)}\}) = \exp[-\beta H^{(0)}(\{\sigma_i^{(0)}\})]$$

from $P^{(0)}$ and $H^{(0)} \Rightarrow$ we can work out effective Hamiltonian at $P^{(1)}, H^{(1)}$

$$P^{(1)}(\{\sigma_i^{(1)}\}) = \exp[-\beta H^{(1)}(\{\sigma_i^{(1)}\})]$$

$$= \sum_{\{\sigma_i^{(0)}\}} \exp[-\beta H^{(0)}(\{\sigma_i^{(0)}\})]$$

example: 1D

$$\exp[-\beta \sum_i J \sigma_i^{(0)} \sigma_{i+1}^{(0)}]$$



suppose we block in two member blocks

rule: if $\sigma_i^{(0)}$ and $\sigma_{i+1}^{(0)}$ are parallel $\Rightarrow \sigma_i^{(1)} = +1$
if not $\sigma_i^{(1)} = -1$

$$\exp[-\beta H^{(1)}(\{\sigma_i^{(1)}\})]$$

we know probability of each configuration

$$\exp[-\beta H^{(1)}(\uparrow, \uparrow, \uparrow, \uparrow, \dots)]$$

$$= \sum_{\sigma_i} \dots \sum_{\sigma_{N/2}} \exp[-\beta \sum_{i=1}^{N/2} J \sigma_i \sigma_{i+1}]$$

since we can do it for all config.

$\Rightarrow$ can define an effective Hamiltonian

- after renormalization $\rightarrow$ average quantities do not change

$$\langle X \rangle = \frac{1}{Z^{(n)}} \sum_{\sigma^n} X(\sigma^n) \exp[-H^{(n)}(\sigma^n)]$$

- can not change of $\ln$ of n

- model is the same at every step; only parameters of the Hamiltonian will change

- Ising model: coupling J

J_{nn}
 $\nearrow$
 nearest
neighbor

J_{nnn}
 $\nearrow$
 next nearest
neighbor

J_{nnnn}
 $\nearrow$
 next to next to
nearest neighbor

$$J_1 J_2 J_3 \rightarrow J_1' J_2' J_3' \rightarrow J_1'' J_2'' J_3'' \rightarrow \dots$$

one can write

$$H^{(n+1)} = Q(H^{(n)})$$

Hamiltonian parameters
are transformed

call $\vec{H}$ - vector of parameters of Hamiltonian

$$\vec{H}^{(n+1)} = \vec{Q}(\vec{H}^{(n)})$$

- parameters of Hamiltonian span a space

- renormalization transformations move us around in this space $\Rightarrow$ renormalization group transformations generate paths in the space of parameters of Hamiltonian

Fixed points

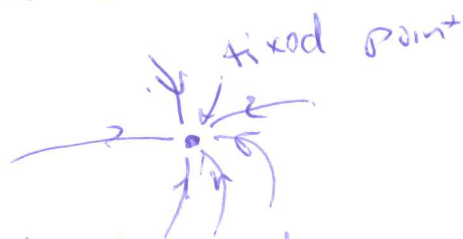
points which map onto themselves

$$\vec{H}^* = \vec{Q}(\vec{H}^*)$$

- we can keep renormalizing, but nothing changes

- attractive fixed point:

in neighborhood of point other points will tend toward attractive fixed point



- repulsive fixed point



- mixed fixed point $\Rightarrow$ attractive in some directions
repulsive in others

- to determine whether fixed point is attractive/repulsive

- consider point $\vec{H}'$ close to $\vec{H}^*$

$$\vec{H}^* = \vec{Q}(\vec{H}^*)$$

$$\vec{H}' \approx \vec{Q}(\vec{H}^* + \delta\vec{H}) = \vec{Q}(\vec{H}^*) + \bar{M}\delta\vec{H}$$

$$M_{ij} = \left. \frac{\partial Q_i}{\partial H_j} \right|_{\vec{H}=\vec{H}^*}$$

diagonalize $\bar{M}$

if eigenvalue $\lambda > 1 \Rightarrow$ repulsive fixed point

$\lambda < 1 \Rightarrow$ attractive fixed point