

Problem Set 9

1.) $c_{m_1}^\dagger \dots c_{m_l}^\dagger$

$$\langle 0 | c_{n_L} \dots c_{n_1} c_{m_1}^\dagger \dots c_{m_l}^\dagger | 0 \rangle$$

start with 1 $\rightarrow$ for now fermions

$$\begin{aligned} \langle 0 | c_{n_1} c_{m_1}^\dagger | 0 \rangle &= \langle 0 | (\delta_{m_1 n_1} - c_{m_1}^\dagger c_{n_1}) | 0 \rangle \\ &= \delta_{m_1 n_1} \end{aligned}$$

$$\begin{aligned} \langle 0 | c_{n_2} c_{n_1} c_{m_1}^\dagger c_{m_2}^\dagger | 0 \rangle &= \delta_{n_1 n_1} \langle 0 | c_{n_2} c_{m_2}^\dagger | 0 \rangle \\ &\quad - \langle 0 | c_{n_2} c_{m_1}^\dagger c_{m_1} c_{m_2}^\dagger | 0 \rangle \\ &= \delta_{m_1 n_1} \delta_{m_2 n_2} \\ &\quad - \langle 0 | c_{n_2} c_{m_1}^\dagger | 0 \rangle \delta_{m_1 n_2} \\ &= \delta_{m_1 n_1} \delta_{m_2 n_2} - \delta_{m_1 n_2} \delta_{m_2 n_1} \\ &= \begin{vmatrix} \delta_{m_1 n_2} & \delta_{m_2 n_1} \\ \delta_{m_1 n_1} & \delta_{m_2 n_2} \end{vmatrix} \end{aligned}$$

induction:

assume $\langle 0 | c_{n_L} \dots c_{n_1} c_{m_1}^\dagger \dots c_{m_l}^\dagger | 0 \rangle = \begin{vmatrix} \delta_{m_1 n_1} & \dots & \delta_{m_l n_l} \\ \vdots & & \vdots \\ \delta_{m_1 n_L} & \dots & \delta_{m_l n_L} \end{vmatrix}$

implies
then

$$\langle 0 | c_{n_{L+1}} \dots c_{n_1} c_{m_1}^\dagger \dots c_{m_{l+1}}^\dagger | 0 \rangle$$

$$\begin{aligned} &= \langle 0 | c_{n_{L+1}} \dots c_{n_2} c_{m_l}^\dagger \dots c_{m_{l+1}}^\dagger | 0 \rangle \delta_{m_1 n_1} \\ &\quad - \langle 0 | c_{n_{L+1}} \dots c_{n_2} c_{m_1}^\dagger c_{m_l} c_{m_2}^\dagger \dots c_{m_{l+1}}^\dagger | 0 \rangle \\ &= \end{aligned}$$

2nd

$$= \begin{vmatrix} \delta_{n_1 n_2} & \dots & \delta_{n_{L-1} n_2} \\ \vdots & & \\ \delta_{n_2 n_{L-1}} & \dots & \delta_{n_{L-1} n_{L-1}} \end{vmatrix} \delta_{n_1 n_1}$$

- L.O. $c_{n_{L-1}} \dots c_{n_2} c_{n_1}^\dagger c_{n_1} c_{n_2}^\dagger \dots c_{n_{L-1}}^\dagger |0\rangle$
 commute these two

$$= \begin{vmatrix} \delta_{n_1 n_2} & \dots & \delta_{n_{L-1} n_2} \\ i & & \\ \delta_{n_2 n_{L-1}} & \dots & \delta_{n_{L-1} n_{L-1}} \end{vmatrix} \delta_{n_1 n_1}$$

$$- \begin{vmatrix} \delta_{n_1 n_2} & \delta_{n_2 n_2} & \dots & \delta_{n_{L-1} n_2} \\ \vdots & & & \\ \delta_{n_2 n_{L-1}} & \dots & \delta_{n_{L-1} n_{L-1}} \end{vmatrix} \delta_{n_2 n_1}$$

+ L.O. $c_{n_{L-1}} \dots c_{n_2} c_{n_1}^\dagger c_{n_2}^\dagger c_{n_1} \dots c_{n_{L-1}}^\dagger |0\rangle$

$$= \begin{vmatrix} \delta_{n_1 n_1} & \dots & \delta_{n_{L-1} n_1} \\ \vdots & & \\ \delta_{n_1 n_{L-1}} & \dots & \delta_{n_{L-1} n_{L-1}} \end{vmatrix}$$

for bosons: permanent instead of determinant

2)

$$|\psi\rangle = C_{n_1}^\dagger C_{n_2}^\dagger |0\rangle$$

$$C_{n_1}^\dagger = \frac{1}{\sqrt{L}} \sum_{r_1} e^{i k_1 r_1} C_{r_1}^\dagger$$

$$|\psi\rangle = \frac{1}{L} \sum_{r_1, r_2} e^{i(k_1 r_1 + k_2 r_2)} C_{r_1}^\dagger C_{r_2}^\dagger |0\rangle$$

to get determinant form consider $r_1 = r_1$
two particular positions $r_2 = r_2$

$$R_1, R_2$$

at some point in summation:

$$\Rightarrow \frac{1}{L} e^{i k_1 R_1} e^{i k_2 R_2} C_{R_1}^\dagger C_{R_2}^\dagger |0\rangle$$

at some point in summation

$$\begin{aligned} & \frac{1}{L} e^{i k_1 R_2} e^{i k_2 R_1} C_{R_2}^\dagger C_{R_1}^\dagger |0\rangle \\ &= - \frac{1}{L} e^{i k_1 R_2} e^{i k_2 R_1} C_{R_1}^\dagger C_{R_2}^\dagger |0\rangle \end{aligned}$$

$$\frac{1}{L} (e^{i(k_1 R_1 + k_2 R_2)} - e^{i(k_1 R_2 + k_2 R_1)}) C_{R_1}^\dagger C_{R_2}^\dagger |0\rangle$$

$$|\psi\rangle = \frac{1}{L} \sum_{R_1, R_2} (e^{i(k_1 R_1 + k_2 R_2)} - e^{i(k_1 R_2 + k_2 R_1)}) C_{R_1}^\dagger C_{R_2}^\dagger |0\rangle$$

2: 2D Ising

- low-T expansion

$$Z_N = 2 \exp[2NK] \sum_p \exp[-K(r+2s)]$$

- simplest closed polygon $\rightarrow$ square: $r=2, s=0$

$$Z_N = 2 \exp[2NK] \cdot N \cdot \exp(-8K) + 2 \exp[2NK] = 2 \exp[NK] (1 + N \exp(-8K))$$

free energy: $F = -kT \ln Z_N$

$$= -kT \ln [2 \exp[NK] (1 + N \exp(-8K))] + 7N$$

for high -T

$$Z_N = 2^N \cosh^{2N} K \sum_p \tanh^{r+s} K$$

$$= 2^N \cosh^{2N} K (1 + N \tanh^4 K)$$

3.)

$$Z = \int_{y(0)=0}^{y(L)=0} D[y(x)] e^{-\frac{1}{2} \int_0^L dx (A y'^2 + B y^2)}$$

From Pathria / Beale

$$\langle q | e^{-\frac{1}{2} \int_0^L dx (A y'^2 + B y^2)} | q' \rangle = \left[\frac{m \omega}{2 \pi \hbar \sinh(\beta' \hbar \omega)} \right]^{1/2} \exp \left[-\frac{m \omega}{4 \hbar} \left[(q+q')^2 \tanh\left(\frac{\beta' \hbar \omega}{2}\right) + (q-q')^2 \coth\left(\frac{\beta' \hbar \omega}{2}\right) \right] \right]$$

for $H = \frac{m \dot{q}^2}{2} + \frac{m \omega^2}{2} q^2$

$$= \frac{m \dot{q}^2}{2} + \frac{m \omega^2}{2} q^2$$

set $\hbar = 1$

$$\Rightarrow \langle q | e^{-\frac{1}{2} \int_0^L dx (m \dot{q}'^2(x) + m \omega^2 q'^2(x))} | q' \rangle = \int_{q(0)=q}^{q(L)=q'} D[q(x)] \exp \left[-\int_0^L dx \left(\frac{m \dot{q}'^2(x)}{2} + \frac{m \omega^2 q'^2(x)}{2} \right) \right]$$

$$= \left[\frac{m \omega}{2 \pi \hbar \sinh(\beta' \hbar \omega)} \right]^{1/2} \exp \left[-\frac{m \omega}{4} \left[(q+q')^2 \tanh\left(\frac{\beta' \hbar \omega}{2}\right) + (q-q')^2 \coth\left(\frac{\beta' \hbar \omega}{2}\right) \right] \right]$$

set $\hbar = 1$ $\beta' = L$ $m \rightarrow \beta$ $x \rightarrow x$ $A = \frac{1}{2}$ $B = \frac{\omega^2}{2}$

$$\Rightarrow \langle y | e^{-\frac{1}{2} \int_0^L dx (\beta y'^2 + \beta y^2)} | y' \rangle = \left[\frac{\beta \beta}{2 \pi \sinh(\beta L \beta)} \right]^{1/2} \exp \left[-\frac{\beta \beta}{2} \left[(y+y')^2 \tanh(L \beta) + (y-y')^2 \coth(L \beta) \right] \right]$$

$$\langle y | e^{-LH} | y' \rangle$$

$$= \tilde{A} \exp[-\tilde{B}(y+y')^2 - \tilde{C}(y-y')^2]$$

$$= \tilde{A} \exp[-\tilde{B}(y^2 + y'^2) + \tilde{C}yy'] \Rightarrow \tilde{A}, \tilde{B}, \tilde{C} \text{ depend on length } (L)$$

$$\langle y(x) y(y') \rangle = \tilde{A}(x) \exp[-\tilde{B}(x)(y^2 + y'^2) - \tilde{C}(x)yy']$$

$$\int dy' \int dy \tilde{A}(x) \exp[-\tilde{B}(x)y^2]$$

$$\tilde{A}(x'-x) \exp[-\tilde{B}(x'-x)(y^2 + y'^2) - \tilde{C}(x'-x)yy']$$

$$\tilde{A}(L-x') \exp[-\tilde{B}(L-x')y'^2] yy'$$

$$\int dy' \int dy \tilde{A}(x) \exp[-\tilde{B}(x)y^2]$$

$$\tilde{A}(x'-x) \exp[-\tilde{B}(x'-x)(y^2 + y'^2) - \tilde{C}(x'-x)yy']$$

$$\tilde{A}(L-x') \exp[-\tilde{B}(L-x')y'^2]$$

normalization:

$$\int dy' \int dy \tilde{A}(x) \tilde{A}(x'-x) \tilde{A}(L-x') \exp[-\tilde{B}(x)y^2]$$

$$\langle y(x) y(y') \rangle = \frac{\tilde{C}(x'-x)}{y [\tilde{B}(x'-x) + \tilde{B}(L-x)]}$$

$$= \frac{1}{\left(\tilde{B}(x) + \tilde{B}(x-y') - \frac{\tilde{C}(x'-x)^2}{y [\tilde{B}(x'-x) + \tilde{B}(L-x)]} \right)}$$

$$\tilde{C}(x'-x) = \frac{\beta B}{2} [\tanh((y'-x)B) - \coth((y'-x)B)]$$

$$\tilde{B}(x) = \frac{\beta B}{2} [\tanh(xB) + \coth(xB)]$$