

Problem Set 8

①

$$\Phi(T, \psi, \eta) = \Phi_0(T, \psi) + \alpha_2 \eta^2 + \alpha_4 (T, \psi) \eta^4 + \alpha_6 (T, \psi) \eta^6$$

$$\frac{\partial \Phi}{\partial \eta} = 2\alpha_2 \eta + 4\alpha_4 \eta^3 + 6\alpha_6 \eta^5 = 0$$

solutions: $\eta = 0$

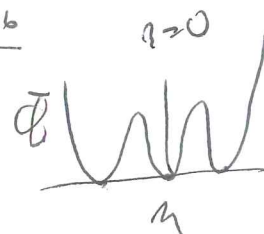
$$\eta^2 =$$

$$\alpha_2 + 2\alpha_4 \eta^2 + 3\alpha_6 \eta^4 = 0$$

$$\eta^2 = \frac{-2\alpha_4 \pm \sqrt{4\alpha_4^2 - 4 \cdot 3\alpha_6 \alpha_2}}{6\alpha_6}$$

$$= \frac{-\alpha_4 \pm \sqrt{\alpha_4^2 - 3\alpha_6 \alpha_2}}{3\alpha_6}$$

first-order transition occurs if



$$\Phi - \Phi_0 = 0 \quad \partial \Phi / \partial \eta = 0$$

$$\alpha_2 \eta^2 + \alpha_4 \eta^4 + \alpha_6 \eta^6 = 0$$

$$2\alpha_2 \eta + 4\alpha_4 \eta^3 + 6\alpha_6 \eta^5 = 0$$

$$\alpha_4 \eta^2 + 2\alpha_6 \eta^4 = 0$$

$$\eta^2 = -\alpha_4 / 2\alpha_6$$

$\alpha_4 < 0 \Rightarrow$ to get positive η^2

Since $\eta^2 = -\frac{\alpha_1}{2\alpha_6}$ and $\eta^2 = -\frac{\alpha_1}{2\alpha_6} \pm \frac{\sqrt{\alpha_1^2 - 3\alpha_2\alpha_6}}{3\alpha_6}$

$\Downarrow$

$$-\frac{\alpha_1}{2} = -\frac{\alpha_1}{3} \pm \frac{\sqrt{\alpha_1^2 - 3\alpha_2\alpha_6}}{3}$$

$$-\alpha_1 = -2\alpha_1 \pm 2\sqrt{\alpha_1^2 - 3\alpha_2\alpha_6}$$

$$-\alpha_1 = \pm 2\sqrt{\alpha_1^2 - 3\alpha_2\alpha_6}$$

$$\alpha_1^2 = 4\alpha_1^2 - 12\alpha_2\alpha_6$$

$$\alpha_2 = \frac{\alpha_1^2}{4\alpha_6}$$

continuous phase transitions

$$\alpha_2 \neq 0$$

tricritical point occurs when $\alpha_2 = \alpha_1 = 0$

2.) Berthelot equation of state

$$\left(1 + \frac{a}{Tv^2}\right)(v-b) = RT$$

use EOS in reduced variables

$$\bar{p} = \frac{8\bar{T}}{3\bar{v}-1} - \frac{3}{\bar{T}\bar{v}^2}$$

to determine critical exponents: introduce

$$\pi = \bar{p} - 1 = \frac{p-p_c}{p_c}$$

$$\varepsilon = \bar{T} - 1 = \frac{T-T_c}{T_c}$$

$$\omega = \bar{v} - 1 = \frac{v-v_c}{v_c}$$

$$\pi + 1 = \frac{8(\varepsilon + 1)}{3\omega + 2} - \frac{3}{(\varepsilon + 1)(\omega + 1)^2}$$

$$\pi = -1 + \frac{8(\varepsilon + 1)}{3\omega + 2} - \frac{3}{(\varepsilon + 1)(\omega + 1)^2}$$

degree of critical isotherm:

$$\varepsilon = 0$$

$$\pi = -1 + \frac{8}{3\omega + 2} - \frac{3}{(\omega + 1)^2}$$

$$= \frac{-(3\omega + 2)(\omega + 1)^2 + 8(\omega + 1)^2 - 3(3\omega + 2)}{(3\omega + 2)(\omega + 1)^2}$$

$$= \frac{(6 - 3\omega)(\omega + 1)^2 - 9\omega - 6}{(3\omega + 2)(\omega + 1)^2}$$

$$= \frac{(6 - 3\omega)(\omega^2 + 2\omega + 1) - 9\omega - 6}{(3\omega + 2)(\omega + 1)^2}$$

$$= \frac{6\omega^2 + 12\omega + 6 - 3\omega^3 - 6\omega^2 - 9\omega - 6}{(3\omega + 2)(\omega + 1)^2} \Rightarrow \boxed{\delta = 3}$$

$$\left(\frac{\partial \bar{U}}{\partial w}\right)_{w=0} = - \frac{8(\varepsilon+1) \cdot 3}{(3w+2)^2} + \frac{6}{(\varepsilon+1)(w+1)^3}$$

$$\frac{\partial \bar{U}}{\partial w} = - \frac{24(\varepsilon+1)}{4} + \frac{6}{(\varepsilon+1)}$$

$$= -6\varepsilon$$

$$\gamma = 1$$

③ Widom / Kadanoff scaling

$$f(T, B) = t^{1/\gamma} \psi(B/t^{x/\gamma})$$

$$t = \frac{T - T_c}{T_c}$$

magnetization

$$m|_{B=0} = - \frac{\partial f}{\partial B} = \frac{t^{1/\gamma} \psi'(B/t^{x/\gamma})}{t^{x/\gamma}} \Big|_{B=0}$$

$$= t^{\frac{1-x}{\gamma}} \psi'(0)$$

$$\Rightarrow m \sim B^{\beta} \quad \beta = \frac{1-x}{\gamma}$$

susceptibility

$$\chi|_{B=0} = - \left(\frac{\partial^2 f}{\partial B^2} \right)_T = t^{\frac{1-2x}{\gamma}} \psi''(0)$$

$$\chi \sim t^{-\gamma} \quad \gamma = \frac{2x-1}{\gamma}$$

$$C_B = -T \left(\frac{\partial^2 f}{\partial T^2} \right)_{B=0} = - \frac{\partial}{\partial T} \left[\frac{1}{T_c} t^{\frac{1}{\gamma}-1} \psi(B/t^{x/\gamma}) + t^{1/\gamma} \psi(B/t^{x/\gamma}) \left(-\frac{x}{\gamma} \right) \frac{B}{t^{x/\gamma}} \right]$$

$$= - \frac{1}{T_c} \left(\frac{1}{\gamma} - 1 \right) t^{\frac{1}{\gamma}-2} \psi(0)$$

$$\alpha = 2 - \frac{1}{\gamma}$$

$$m|_{T=T_c} = - \left(\frac{\partial f}{\partial B} \right)_T = \dots$$

$$f(T, B) = t^{1/\gamma} \psi(B/t^{x/\gamma}) \rightarrow B^{1/\gamma} \psi(B/t^{x/\gamma})$$

$$- \frac{\partial f}{\partial B} = \frac{1}{\gamma} B^{1/\gamma-1} \psi(\infty) + \dots$$

$$\delta = \frac{1}{\gamma} \quad \delta = \frac{x}{1-x}$$

$$\alpha + 2\beta + \gamma = 2 - \frac{1}{\gamma} + 2 \frac{(1-x)}{\gamma} + \frac{\gamma x - 1}{\gamma} = 2$$

$$\alpha + \beta(\delta+1) = 2 - \frac{1}{\gamma} + \frac{(1-x)}{\gamma} \left(\frac{1}{1-x} \right) = 2$$

Kardar & t

$$G_c^{(n)}(r, t) \approx \frac{\chi(r t^{(2-\alpha)/d})}{r^{d-2+\eta}}$$

in order to reproduce known behaviour

as $t \rightarrow 0$ $\chi(r t^{(2-\alpha)/d}) \rightarrow \text{constant}$

$$\text{as finite } t \Rightarrow \frac{\chi(r t^{(2-\alpha)/d})}{r^{d-2+\eta}} \rightarrow \frac{r^{d-2+\eta} t^{(2-\alpha)(d-2+\eta)}}{r^{d-2+\eta}} \times e^{-r t^{(2-\alpha)/d}}$$

$$\text{since } t \rightarrow 0 \rightarrow G_c^{(n)} \approx \frac{1}{r^{d-2+\eta}}$$

$$\text{finite } t \rightarrow G_c^{(n)} \approx e^{-r t} \text{ for large } r$$

if

$$s = t^{(2-\alpha)/d}$$

$$s = t^{-\nu}$$

$\Rightarrow$

$$2 - \alpha = \nu d$$

to derive other one: $\chi \sim \int G_c^{(n)}(r) dr$

$$t=0 \Rightarrow \chi \sim \int_0^\infty \frac{r^{d-2+\eta}}{r^{d-2+\eta}} dr = \int_0^\infty r^{1-\eta} dr$$

at $\neq 3$

$$\chi \sim s^{2-\eta}$$

$$s \sim |T - T_c|^{-\nu}$$

$$\chi \sim |T - T_c|^{-\gamma}$$

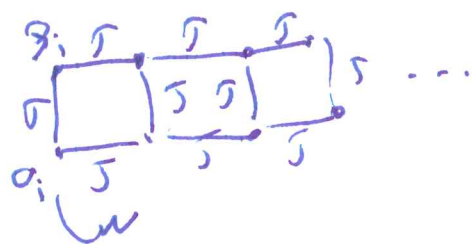
$$s^{-1/\nu} \sim |T - T_c|$$

$$\chi \sim |T - T_c|^{1/\nu}$$

$$\chi \sim s^{1/\nu}$$

$$2 - \eta = \gamma/\nu$$

4.) transfer matrix 4 dimensional



spin on upper leg σ
spin on lower leg S

$$e^{-\sum_{i=1}^N \left(\frac{J}{2} \sigma_i \sigma_{i+1} + \frac{J}{2} S_i S_{i+1} \right)}$$

4x4 matrix

divide by 2 to avoid double counting

$$\langle \sigma_i S_i | e^{-\frac{J}{2} \sigma_i \sigma_{i+1} - \frac{J}{2} S_i S_{i+1}} | \sigma_{i+1} S_{i+1} \rangle$$

element $\sigma_i \sigma_j$	1 1	1 -1	-1 1	-1 -1
1 1	e^{-3J}	1	1	e^J
1 -1	1	e^{-J}	e^J	1
-1 1	1	e^J	e^{-J}	1
-1 -1	e^J	1	1	e^{-3J}

$$T = \begin{pmatrix} e^{-3J} & 1 & 1 & e^J \\ 1 & e^{-J} & e^J & 1 \\ 1 & e^J & e^{-J} & 1 \\ e^J & 1 & 1 & e^{-3J} \end{pmatrix}$$

$$Q = \text{Tr } T^N$$

$$J = \beta K$$

$$E = - \frac{\partial \ln Q}{\partial \beta} \dots$$

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application of RG transformation

$$\sigma_1, \sigma_2, \sigma_3, \sigma_4 \rightarrow \sigma'_1, \sigma'_1$$

$$e^{-J[\frac{\sigma_1 \sigma_2}{2} + \sigma_2 \sigma_3 + \frac{\sigma_3 \sigma_4}{2}]} \rightarrow e^{-J' \sigma'_1 \sigma'_1}$$

12 \ 34	↑↑	↑↓	↓↑	↓↓
↑↑	e^{-2J}	e^{-J}	e^J	1
↑↓	e^J	e^{2J}	1	e^{-J}
↓↑	e^{-J}	1	e^{2J}	e^J
↓↓	1	e^J	e^{-J}	e^{-2J}

trace results

$$\begin{array}{ll} \sigma'_1 & \sigma'_1 \\ \uparrow & \uparrow \rightarrow 2(1 + e^{-2J}) \\ \uparrow & \downarrow \rightarrow 2(e^J + e^{-J}) \\ \downarrow & \uparrow \rightarrow 2(e^J + e^{-J}) \\ \downarrow & \downarrow \rightarrow 2(1 + e^{2J}) \end{array}$$

construct RG

$$\begin{aligned} 2(1 + e^{2J}) &= f(J) e^{-J'} \\ 2(1 + e^{-2J}) &= f(J) e^{-J' - K'} \\ 2(e^J + e^{-J}) &= f(J) e^{J'} \end{aligned}$$

K' - auxiliary "coupling" - acts if two neighboring spins point up

$$2(1 + e^{2J})(e^J + e^{-J}) = f^2(J)$$

$$\Rightarrow f(J) = 4e^{J/2} \cosh J$$

$$\Rightarrow \frac{2(1 + e^{2J})}{4e^{J/2} \cosh J} = e^{-J'}$$

$$\Rightarrow J' = J/2$$

$k' = 25$ but we forget about it

$\gamma' = \gamma'_2$ tends toward high- T fixed point