

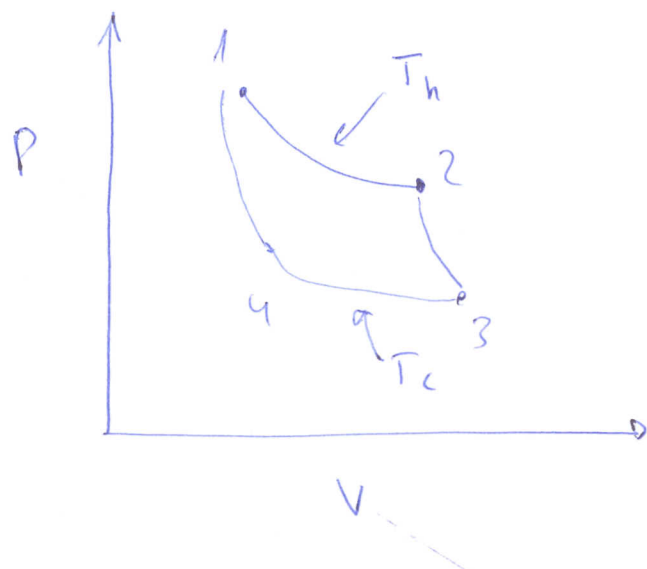
Problem Set 1:

①

1.) Carnot cycle / ideal gas

$$PV = N k_B T$$

$$E = \frac{3}{2} N k_B T$$



1 → 2 isothermal expansion

2 → 3 adiabatic expansion

3 → 4 isothermal compression

4 → 1 adiabatic compression

ΔU for the total process is zero

$$\Delta U_{\text{total}} = 0$$

1 → 2 $\Delta U_{12} = 0$ (isothermal process)

$$\begin{aligned} \Delta U_{12} &= \Delta Q_{12} - \Delta W_{12} = 0 \\ \Delta Q_{12} &= \Delta W_{12} = \int_{V_1}^{V_2} P dV = N k_B T_h \int_{V_1}^{V_2} \frac{dV}{V} \\ &= N k_B T_h \ln(V_2/V_1) \end{aligned}$$

2 → 3 $\Delta U = -\Delta W$
 $dU = -dW$

$$\frac{3}{2} N k_B dT = - \frac{N k_B T dV}{V}$$

$$\frac{3}{2} \ln(T_c/T_h) = - \ln(V_2/V_3)$$

$$\left(\frac{T_h}{T_c}\right)^{3/2} = \frac{V_2}{V_3} \Rightarrow T_h^{3/2} V_2 = T_c^{3/2} V_3$$

3 → 4 $\Delta Q_{34} = N k_B T_c \ln(V_4/V_3)$

4 → 1 $T_h^{3/2} V_1 = T_c^{3/2} V_4$

$$\begin{aligned} \frac{V_1}{V_4} &= \frac{V_2}{V_3} \\ \downarrow \\ \frac{V_3}{V_4} &= \frac{V_2}{V_1} \end{aligned}$$

efficiency: $\eta = 1 - \frac{\Delta Q_{43}}{\Delta Q_{12}} = 1 - \frac{T_c \ln(V_4/V_3)}{T_h \ln(V_2/V_1)}$
$$= 1 - \frac{T_c}{T_h}$$

(2)

2.) Magnetic engine

$$\left(\frac{\partial E}{\partial M} \right)_{N,T} = \left(\frac{\partial E}{\partial M} \right)_S + \left(\frac{\partial E}{\partial S} \right)_M \left(\frac{\partial S}{\partial M} \right)_T$$

(use of identity: $\left(\frac{\partial x}{\partial y} \right)_z = \left(\frac{\partial x}{\partial y} \right)_w + \left(\frac{\partial x}{\partial w} \right)_y \left(\frac{\partial w}{\partial y} \right)_z$)

$$dE = TdS - HdM$$

$$\left(\frac{\partial E}{\partial S} \right)_M = T \quad \left(\frac{\partial E}{\partial M} \right)_S = -H$$

$$\left(\frac{\partial E}{\partial M} \right)_{N,T} = -H + T \left(\frac{\partial S}{\partial M} \right)_T$$

$\left(\frac{\partial S}{\partial M} \right)_T$ can be obtained from a Maxwell relation for dA

$$dA = -SdT - HdM$$

$$\left(\frac{\partial A}{\partial T} \right)_M = -S$$

$$\left(\frac{\partial A}{\partial M} \right)_T = -H$$

$$\left[\frac{\partial}{\partial M} \left(\frac{\partial A}{\partial T} \right)_M \right]_T = - \left(\frac{\partial S}{\partial M} \right)_T \Rightarrow \left(\frac{\partial S}{\partial M} \right)_T = \left(\frac{\partial H}{\partial T} \right)_M$$

$$\left[\frac{\partial}{\partial T} \left(\frac{\partial A}{\partial M} \right)_T \right]_M = - \left(\frac{\partial H}{\partial T} \right)_M$$

from the equation of state: $\mu = \frac{NDH}{T} \Rightarrow H = \frac{\mu NT}{ND}$

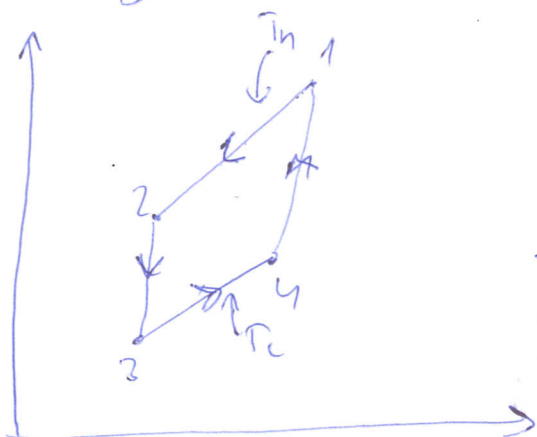
$$\left(\frac{\partial H}{\partial T} \right)_M = \frac{H}{T}$$

$$\frac{\partial H}{\partial T} = \frac{H}{T}$$

$$\Rightarrow \left(\frac{\partial E}{\partial M} \right)_{N,T} = -H + T \left(\frac{H}{T} \right) = 0$$

• Carnot cycle sketch

(3)



1 → 2 isothermal demagnetization
(straight line $M \propto 1/T$)

2 → 3 adiabatic demagnetization

3 → 4 isothermal magnetization (straight)

4 → 1 adiabatic magnetization

• heat/work/efficiency

$$\Delta U_T = 0$$

(1 → 2) energy is a linear function of temperature

$$dU = 0 \quad dQ = +dW \quad dW = -M dM$$

$$\Delta Q_{12} = - \int H dM$$

$$= - \int \left(\frac{T_h}{nD} \right) n dM = - \left(\frac{T_h}{nD} \right) \int_1^2 n dM$$

$$\Delta Q_{12} = - \left(\frac{T_h}{2nD} \right) (M_2^2 - M_1^2) \Rightarrow \text{heat absorbed}$$

$$(2 \rightarrow 3) \quad dU = C dT$$

$$\Rightarrow C dT = -dW$$

$$\int_{T_c}^{T_h} C dT = - \int_{M_2}^{M_1} \frac{T}{nD} n dM$$

$$2CnD \ln(T_h/T_c) = M_3^2 - M_2^2$$

$$(3 \rightarrow 4) \quad \Delta Q_{34} = - \frac{T_c}{2nD} (M_4^2 - M_3^2)$$

$$(4 \rightarrow 1) \quad 2CnD \ln(T_c/T_h) = M_1^2 - M_4^2$$

$$\frac{M_3^2 - M_2^2}{M_4^2 - M_1^2} = 1 \Rightarrow \frac{M_4^2 - M_3^2}{M_2^2 - M_1^2} = 1$$

$$\text{work done: } \Delta Q_{32} - \Delta Q_{43}$$

$$\text{efficiency: } 1 - \frac{\Delta Q_{32}}{\Delta Q_{12}} = 1 - \frac{T_c (M_4^2 - M_3^2)}{T_h (M_2^2 - M_1^2)}$$

$$= 1 - \frac{T_c}{T_h}$$

$$\mathcal{E}(n_1, n_2, n_3) = \frac{hc}{2L} (n_1^2 + n_2^2 + n_3^2)^{1/2}$$

$$E = \frac{hc}{2L} \sum_{i=1}^N (n_x^{(i)^2} + n_y^{(i)^2} + n_z^{(i)^2})^{1/2}$$

$$L = V^{1/3}$$

$$E = \frac{hc}{2V^{1/3}} \sum_{i=1}^N (n_x^{(i)^2} + n_y^{(i)^2} + n_z^{(i)^2})^{1/2}$$

$$V^{1/3} E = \text{constant} \quad \text{if } N \text{ is constant}$$

$$\Downarrow$$

$$P = - \frac{\partial E}{\partial V} = - \frac{1}{3} \frac{E}{V}$$

specific heat $C_V = \left(\frac{\partial E}{\partial T} \right)_V$

$$C_P = \left(\frac{\partial E + PV}{\partial T} \right)_P$$

$$= \left[\frac{\partial E}{\partial T} + \frac{\partial (PV)}{\partial T} \right]_P = \frac{\partial E}{\partial T} + \frac{1}{3} \frac{\partial E}{\partial T} = \frac{4}{3} \frac{\partial E}{\partial T}$$

$$\Rightarrow C_P/C_V = \frac{4}{3}$$

Pathria & Beale: Problem 1.9

(5)

show using extensivity that

$$S = \left(\frac{\partial S}{\partial N} \right)_{V,E} N + \left(\frac{\partial S}{\partial V} \right)_{N,E} V + \left(\frac{\partial S}{\partial E} \right)_{N,V} E$$

extensivity: S is a first-order homogeneous function of all other extensive variables

$$S(N, E, V) \rightarrow S(\lambda N, \lambda E, \lambda V) = \lambda S(N, E, V)$$

$$\Downarrow$$
$$\frac{\partial [\lambda S(N, E, V)]}{\partial \lambda} = \frac{\partial S(\lambda N, \lambda E, \lambda V)}{\partial \lambda}$$

$$\begin{aligned} &= \frac{\partial S(\lambda N, \lambda E, \lambda V)}{\partial(\lambda N)} \frac{\partial(\lambda N)}{\partial \lambda} \\ &+ \frac{\partial S(\lambda N, \lambda E, \lambda V)}{\partial(\lambda E)} \frac{\partial(\lambda E)}{\partial \lambda} \\ &+ \frac{\partial S(\lambda N, \lambda E, \lambda V)}{\partial(\lambda V)} \frac{\partial(\lambda V)}{\partial \lambda} \\ \downarrow \\ S(N, E, V) &= \frac{\partial S(\lambda N, \lambda E, \lambda V)}{\partial(\lambda N)} N \\ &+ \frac{\partial S(\lambda N, \lambda E, \lambda V)}{\partial(\lambda E)} E \\ &+ \frac{\partial S(\lambda N, \lambda E, \lambda V)}{\partial(\lambda V)} V \end{aligned}$$

set $\lambda = 1$

$$S(N, E, V) = \left(\frac{\partial S}{\partial N} \right) N + \frac{\partial S}{\partial E} E + \frac{\partial S}{\partial V} V \quad \text{Q.E.D.}$$

$$S_i = N_i k \ln V_i + \frac{3}{2} N_i k \left\{ 1 + \ln \left(\frac{2\pi m_i k T_i}{h^2} \right) \right\}$$

$$S_T = \sum_{i=1}^2 N_i k \ln V + \frac{3}{2} N_i k \left\{ 1 + \ln \left(\frac{2\pi m_i k T}{h^2} \right) \right\}$$

$$\Delta S = \sum_{i=1}^2 N_i k \ln \left(\frac{V_1 + V_2}{V_i} \right) + \sum_{i=1}^2 \frac{3}{2} N_i k \ln \left(\frac{T}{T_i} \right)$$

$$= \sum_{i=1}^2 N_i k \ln \left[\left(\frac{V_1 + V_2}{V_i} \right) \left(\frac{T}{T_i} \right)^{3/2} \right]$$

if the molecules are indistinguishable and the "mixing" occurs such that $\frac{V_1}{V_i} = \frac{V_2}{V_i} = \frac{N_1 + N_2}{V_1 + V_2}$ then the entropy must be shifted by $(-k \ln N!)$

in this case the entropy of "mixing" is simply that of thermal equilibration

$$\Delta S = \sum_{i=1}^2 \frac{3}{2} N_i k \ln \left[\frac{T}{T_i} \right]$$