Midterm, PHYS-552
Course: Advanced Statistical Mechanics

Fundamentals:

Problem 1) Suppose in a certain department the laser printer has a 50% chance of functioning properly on any given day. Calculate the probability that in a given workweek (5 days) the printer will function three of the five days (not necessarily consecutive). (5 pts.) (Extra credit: which department?)

Problem 2) The Hamiltonian of a quantum-mechanical system is given by $H(p_1, ..., p_N : q_1, ..., q_N)$, where N denotes the number of particles. Write the path-integral expression for the grand-canonical partition function. (5 pts.)

Problem 3) Given a classical system with Hamiltonian $H(p_1, ..., p_N; q_1, ..., q_N)$, and given some observable $A(p_1, ..., p_N; q_1, ..., q_N)$ write an expression for the time correlation function of this observable in the canonical ensemble. (5 pts.)

Problem 4) Given a classical system with Hamiltonian,

$$H(p_1, p_2; q_1, q_2) = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k}{2}(q_1 - q_2)^2$$

comment on the applicability of the virial and equipartition theorems. (5 pts.)

Applications:

Problem 1) Consider a single particle with mass *m* in some volume *V.* Write the probability density in phase space for this particle as a function of the three velocity components. Calculate the probability that the energy of the particle is some particular value *E*, in other words, calculate the probability *P(E)*. (20 pts.)

Problem 2) Consider a one-dimensional one-particle system with Hamiltonian $H(p,q)$, and some observable quantity $A(p,q)$. Show that

$$\int \mathrm{d}p\mathrm{d}q A(p.q)[\exp(-i\hat{L}t)\rho(p,q)] = \int \mathrm{d}p\mathrm{d}q[\exp(i\hat{L}t)A(p.q)]\rho(p,q)$$

where $\hat{L}$ denotes the Liouville operator, and $\rho(p.q)$ denotes the density in phase space. (20 pts.)

Problem 3) Consider a quantum system consisting of N independent spins. The energy for a single spin is given by ϵM with $M = -J, ..., J$. Calculate the canonical partition function, the internal energy, and the entropy if this system. Investigate the high and low temperature limits, as well as the limit $J \to \infty$. Comment on the possibility of negative temperature. (20 pts.)

Problem 4) Consider an ensemble of systems all with fixed particle number N, but with volume V and energy E which are allowed to fluctuate. Suppose that the ensemble consists of $\tilde{N}$ members in contact with each other, but otherwise isolated. Let n_{rs} denote the number of systems with energy E_r and volume V_s. Assume that the possible volumes (V_s) are a discrete set. (20 pts.)

A. Calculate the probability that the system occupies a state with energy E_r and volume V_s. in terms of the variables E_r and V_s, and the corresponding partition function Ω. You should have two unknown constants in these expressions.
B. Define a potential $\tilde{q} = \ln \Omega$. By expanding the differential $\mathrm{d}\tilde{q}$, ıdentify the unknown constants from part A., and show that $\tilde{q} = \frac{G}{k_B T}$, where G denotes the Gibbs free energy, k_B indicates the Boltzmann constant, and T stands for the temperature.
C. Suppose you have an ensemble of systems in which the particle number N and energy E are fixed, and only the volume is allowed to fluctuate. Derive an expression for the entropy in this case.