Solutions: Problem Set 1

$$U_i(K,L) = \begin{cases} P_j(K) = \exp(K s_j s_{j+1}) & \text{if } i = 2j \\ (2\sinh 2L)^{-1} Q_j(L) = \exp(L^* c_j) & \text{if } i = 2j-1 \end{cases}$$

$$Q_i(L) = e^{L} I_i + e^{-L} C_i = \begin{pmatrix}1&0\\0&1\end{pmatrix} \ldots \underset{i}{\begin{pmatrix} e^{L} & e^{-L} \\ e^{-L} & e^{L}\end{pmatrix}} \otimes \ldots \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

star-triangle relation:

$$U_{i+1}(K_1,L_1) U_i(L_2,K_2) U_{i+1}(K_3,L_3)$$

$$= U_i(K_3,L_3) U_{i+1}(L_2,K_2) U_i(K_1,L_1)$$

<u>i odd</u>

$i = 2j-1 \to$ odd

$i+1 = 2j \to$ even

$$\Rightarrow \quad P_j(K_1)\, Q_j(K_2)\, P_j(K_3)\, (2\sinh 2K_2)^{-1}$$

$$= (2\sinh 2L_3)^{-1} (2\sinh 2L_1)^{-1} Q_j(L_3)\, P_j(L_2)\, Q_j(L_1)$$

need tensor representation of P, Q

$$P_j(K) = \exp(K s_j s_{j+1}) = \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \ldots \underset{j\,/\,j+1}{\begin{pmatrix} e^{K} & 0 & 0 & 0 \\ 0 & e^{-K} & 0 & 0 \\ 0 & 0 & e^{-K} & 0 \\ 0 & 0 & 0 & e^{K}\end{pmatrix}} \otimes \ldots \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$Q_j(K) = e^{K} I_i + e^{-K} C_i = \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \ldots \otimes \underset{i}{\begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K}\end{pmatrix}} \otimes \ldots \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} e^{K} & e^{-K} & 0 & 0 \\ e^{-K} & e^{K} & 0 & 0 \\ 0 & 0 & e^{K} & e^{-K} \\ 0 & 0 & e^{-K} & e^{K}\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} \ldots \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

3

$$P_j(K_1)\, Q_j(K_2)\, P_j(K_3) =$$

$$\begin{pmatrix} e^{K_1} & & & \\ & e^{-K_1} & & 0 \\ & & e^{-K_1} & \\ 0 & & & e^{K_1} \end{pmatrix} \begin{pmatrix} e^{K_2} & e^{-K_2} & 0 & 0 \\ e^{-K_2} & e^{K_2} & 0 & 0 \\ 0 & 0 & e^{K_2} & e^{-K_2} \\ 0 & 0 & e^{-K_2} & e^{K_2} \end{pmatrix} \begin{pmatrix} e^{K_3} & 0 & 0 & 0 \\ 0 & e^{-K_3} & 0 & 0 \\ 0 & 0 & e^{-K_3} & 0 \\ 0 & 0 & 0 & e^{K_3} \end{pmatrix}$$

$$= \begin{pmatrix} e^{K_1} & 0 & 0 & 0 \\ 0 & e^{-K_1} & 0 & 0 \\ 0 & 0 & e^{-K_1} & 0 \\ 0 & 0 & 0 & e^{K_1} \end{pmatrix} \begin{pmatrix} e^{K_2+K_3} & e^{-K_2-K_3} & 0 & 0 \\ e^{-K_2+K_3} & e^{K_2-K_3} & 0 & 0 \\ 0 & 0 & e^{K_2-K_3} & e^{-K_2+K_3} \\ 0 & 0 & e^{-K_2-K_3} & e^{K_2+K_3} \end{pmatrix}$$

$$= \begin{pmatrix} e^{K_1+K_2+K_3} & e^{K_1-K_2-K_3} & 0 & 0 \\ e^{-K_1-K_2+K_3} & e^{-K_1+K_2-K_3} & 0 & 0 \\ 0 & 0 & e^{-K_1+K_2-K_3} & e^{-K_1-K_2+K_3} \\ 0 & 0 & e^{K_1-K_2-K_3} & e^{K_1+K_2+K_3} \end{pmatrix}$$

$$Q_j(L_3)\, P_j(L_2)\, Q_j(L_1) = \begin{pmatrix} e^{L_3} & e^{-L_3} & 0 & 0 \\ e^{-L_3} & e^{L_3} & 0 & 0 \\ 0 & 0 & e^{L_3} & e^{-L_3} \\ 0 & 0 & e^{-L_3} & e^{L_3} \end{pmatrix} \begin{pmatrix} e^{L_2} & 0 & 0 & 0 \\ 0 & e^{-L_2} & 0 & 0 \\ 0 & 0 & e^{-L_2} & 0 \\ 0 & 0 & 0 & e^{L_2} \end{pmatrix} \begin{pmatrix} e^{L_1} & e^{-L_1} & 0 & 0 \\ e^{-L_1} & e^{L_1} & 0 & 0 \\ 0 & 0 & e^{L_1} & e^{-L_1} \\ 0 & 0 & e^{-L_1} & e^{L_1} \end{pmatrix}$$

$$= \begin{pmatrix} e^{L_3} & e^{-L_3} & 0 & 0 \\ e^{-L_3} & e^{L_3} & 0 & 0 \\ 0 & 0 & e^{L_3} & e^{-L_3} \\ 0 & 0 & e^{-L_3} & e^{L_3} \end{pmatrix} \begin{pmatrix} e^{L_1+L_2} & e^{L_2-L_1} & 0 & 0 \\ e^{-L_1-L_2} & e^{L_1-L_2} & 0 & 0 \\ 0 & 0 & e^{L_1-L_2} & e^{-L_1-L_2} \\ 0 & 0 & e^{L_2-L_1} & e^{L_1+L_2} \end{pmatrix}$$

$$= \begin{pmatrix} 2\cosh(L_1+L_2+L_3) & 2\cosh(L_2+L_3-L_1) & 0 & 0 \\ 2\cosh(L_1+L_2-L_3) & 2\cosh(L_1-L_2+L_3) & 0 & 0 \\ 0 & 0 & 2\cosh(L_1+L_3-L_2) & 2\cosh(L_1+L_2+L_3) \\ 0 & 0 & 2\cosh(L_2+L_3-L_1) & 2\cosh(L_1+L_2+L_3) \end{pmatrix}$$

$\Rightarrow$

equating L.H.S. vs. R.H.S we obtain the $\star$-Δ relations

$$(2\sinh 2K_2)^{-1} e^{K_1+K_2+K_3} = (2\sinh 2L_1)^{-1}(2\sinh 2L_3)^{-1}\, 2\cosh(L_1+L_2+L_3)$$
$$(2\sinh 2K_2)^{-1} e^{K_1+K_2-K_3} = (2\sinh 2L_1)^{-1}(2\sinh 2L_3)^{-1}\, 2\cosh(L_2+L_3-L_1)$$
$$(2\sinh 2K_2)^{-1} e^{K_2-K_1-K_3} = (2\sinh 2L_1)^{-1}(2\sinh 2L_3)^{-1}\, 2\cosh(L_1+L_3-L_2)$$
$$(2\sinh 2K_2)^{-1} e^{K_3-K_1-K_2} = (2\sinh 2L_1)^{-1}(2\sinh 2L_3)^{-1}\, 2\cosh(L_1+L_2-L_3)$$

<u>i even</u>

$i = 2j \to$ even $\qquad i \to j$

$i-1 = 2j-1 \to$ odd $\qquad i+1 \to j+1$

$i+1 = 2j+1 \to$ odd

$\quad = 2(j+1)-1$

$$Q_{j+1}(K_1)\, P_j(K_2)\, Q_{j+1}(K_3)\, (2\sinh 2K_2)^{-1}$$
$$= (2\sinh 2L_1)^{-1}(2\sinh 2L_3)^{-1}\, P_j(L_3)\, Q_{j+1}(L_2)\, P_j(L_1)$$

L.H.S.

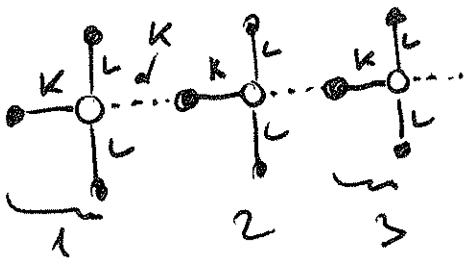
$$\begin{pmatrix} e^{K_1} & e^{-K_1} & 0 & 0 \\ e^{-K_1} & e^{K_1} & 0 & 0 \\ 0 & 0 & e^{K_1} & e^{-K_1} \\ 0 & 0 & e^{-K_1} & e^{K_1} \end{pmatrix} \begin{pmatrix} e^{K_2} & 0 & 0 & 0 \\ 0 & e^{-K_2} & 0 & 0 \\ 0 & 0 & e^{-K_1} & 0 \\ 0 & 0 & 0 & e^{K_2} \end{pmatrix} \begin{pmatrix} e^{K_3} & e^{-K_3} & 0 & 0 \\ e^{-K_3} & e^{K_3} & 0 & 0 \\ 0 & 0 & e^{K_3} & e^{-K_3} \\ 0 & 0 & e^{-K_3} & e^{K_3} \end{pmatrix}$$

$$\begin{pmatrix} e^{K_1} & e^{-K_1} & 0 & 0 \\ e^{-K} & e^{K_1} & 0 & 0 \\ 0 & 0 & e^{K_1} & e^{-K_1} \\ 0 & 0 & e^{-K_1} & e^{K} \end{pmatrix} \begin{pmatrix} e^{K_2+K_3} & e^{K_2-K_3} & 0 & 0 \\ e^{-K_2-K_3} & e^{K_3-K_2} & 0 & 0 \\ 0 & 0 & e^{K_3-K_2} & e^{-K_2-K_3} \\ 0 & 0 & e^{K_2-K_3} & e^{K_2+K_3} \end{pmatrix}$$

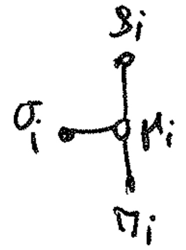
$$\Rightarrow \quad (2\sinh 2K_1)^{-1}(2\sinh 2K_3)^{-1}\, 2\cosh(K_1+K_2+K_3) = (2\sinh 2L_2)^{-1} \times e^{L_1+L_2+L_3}$$

etc.

2.)



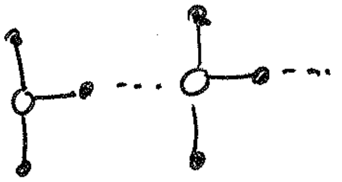
for now label as



transfer matrix $\bar{\bar{T}}_1 = \prod_{i=1}^{N} \sum e^{\left[-K\sigma_i\mu_i - K\mu_i\sigma_{i+1} - L\vartheta_i\mu_i - L\pi_i\mu_i\right]}$

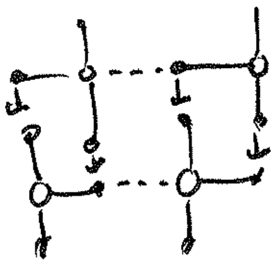
$$= \prod_{i=1}^{N} 2\cosh(K(\sigma_i + \sigma_{i+1}) + L(\vartheta_i + \pi_i))$$

to construct the lattice, construct another transfer matrix

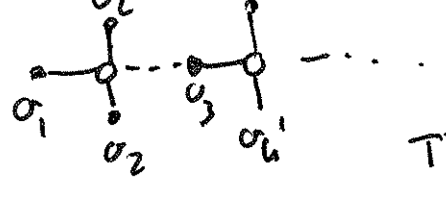


$$\bar{\bar{T}}_2 = \prod_{i=1}^{N} 2\cosh(K(\sigma_{i-1} + \sigma_i) + L(\pi_i + \vartheta_i))$$

the lattice can be formed using the two transfer matrices



easiest if we relabel transfer matrices



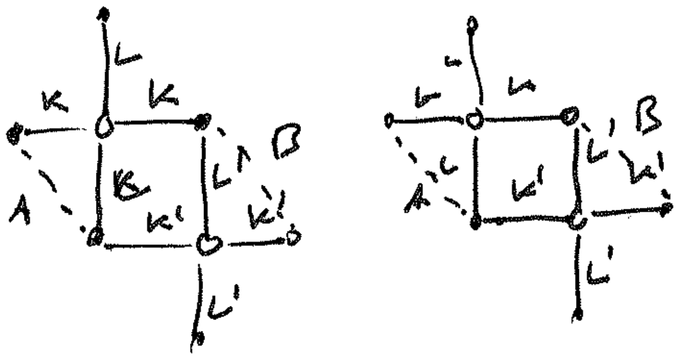
$T = \cancel{\prod_{i=1}^{N/2} 2\cosh(K(\sigma_{2i-1}}$

$$T_1 = \prod_{i=1}^{N/2} 2\cosh[K(\sigma_{2i-1}+\sigma_{2i+1}) + L(\sigma_i+\sigma_i')]\,\delta_{\sigma_{2i-1}\sigma'_{2i-1}}$$

$$T_2 = \prod_{i=1}^{N/2} 2\cosh[K(\sigma_{2i+1}+\sigma'_{2i+1}) + K(\sigma_{2i-2}+\sigma_{2i})]\,\delta_{\sigma_{2i}\sigma'_{2i}}$$

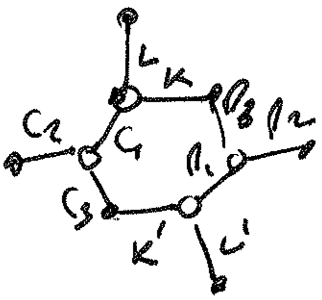
$\mathrm{Tr}\,[(T_1T_2)^{N/2}]$ gives an $N\times N$ lattice

3.)

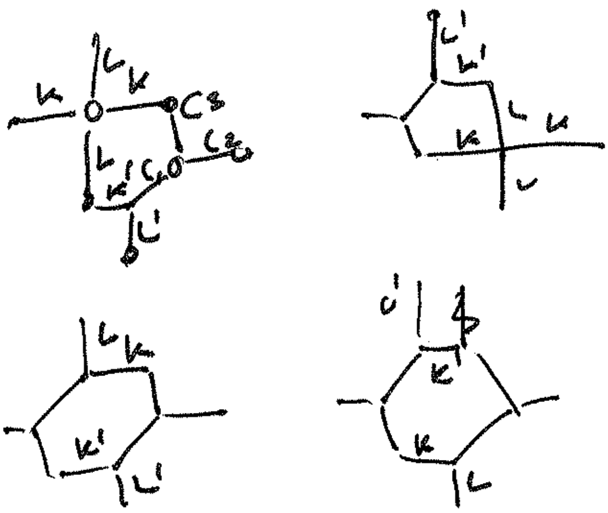


$A = -B$

★-Δ relation



→ apply one-by one



only works if $K = K'$
$L = L'$