Bethe ansatz (coordinate)

- method to solve certain 1D systems exactly
  * for finite systems: effort is reduced from exponential scaling with system size to linear scaling
  * also possible to take the thermodynamic limit → find exact solution
  * excited states / finite temperature extension
- types of models: contact potential → $\delta(x_i - x_j)$ or close-range spin-spin interactions
  examples: Bose gas / Heisenberg model (spin-1/2) / Hubbard model (fermionic)
- wavefunction has the form:

$$\psi = \sum_P A_P \, e^{i(k_{P_1} x_1 + \dots + k_{P_N} x_N)}$$

  - P - sum over permutations
  - A_P - phase factor
  - k_i → momenta ; x_i → coordinates

1D Bose Gas [from Korepin, Bogoliubov, and Izergin]

- described by field operators, commutation relations:

$$[\Psi(x,t), \Psi^+(x',t)] = \delta(x-x')$$

$$[\Psi(x,t), \Psi(x',t)] = [\Psi^+(x,t), \Psi^+(x',t)] = 0$$

- omit $\underline{t}$ since problem is not time dependent

- Hamiltonian: $H = \int dx\, \partial_x \Psi^+(x)\, \partial_x \Psi(x)$
$+ c \int dx\,dx'\, \Psi^+(x)\Psi^+(x')\Psi(x)\Psi(x)$

  c – coupling constant

- equation of motion: $i\partial_t \Psi = -\partial_x^2 \Psi + 2c\Psi^+\Psi\Psi$
(nonlinear Schrödinger eq'n)

- we consider case: $\boxed{c>0}$

- Fock vacuum: $|0\rangle : \Psi(x)|0\rangle = 0 \quad x \in \mathbb{R}^1$

$$\langle 0|\Psi^+(x) = 0$$

$$\langle 0|0\rangle = 1$$

- number operator: $Q = \int dx\, \Psi^+(x)\Psi(x)$

- momentum operator: $P = -\frac{i}{2}\int dx\,[\Psi^+(x)\,\partial_x\Psi(x) - [\partial_x\Psi^+(x)]\,\Psi(x)]$

Q, P are Hermitian operators and conserved quantities

- conserved quantities follows from:

$$[H, Q] = [H, P] = 0$$

- can be shown using commutation relations

$$H = \underbrace{\int dx\, \partial_x \Psi^\dagger(x)\, \partial_x \Psi(x)}_{T} + c \underbrace{\int dx\, \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x)}_{V}$$

$$Q = \int \Psi^\dagger(x) \Psi(x) dx$$

consider here only $[T, Q] =$

$$= \int dx\, \partial_x \Psi^\dagger(x)\, \partial_x \Psi(x) \int dy\, \Psi^\dagger(y) \Psi(y)$$

$$- \int dx dy\, \Psi^\dagger(y) \Psi(y)\, \partial_x \Psi^\dagger(x) \partial_x \Psi(x)$$

- consider 1st term

$$\int dy \int dx\, \partial_x \Psi^\dagger(x)\, \partial_x \Psi(x) \Psi^\dagger(y) \Psi(y)$$ integration by parts

$$= - \int dx dy\, \partial_x^2 \Psi^\dagger(x) \underbrace{\Psi(x) \Psi^\dagger(y)}_{\delta(x-y) + \Psi^\dagger(y)\Psi(x)} \Psi(y)$$

$$= - \int dx dy\, \partial_x^2 \Psi^\dagger(x)\, \delta(x-y) \Psi(y)$$

$$- \int dx dy\, \partial_x^2 \Psi^\dagger(x) \Psi^\dagger(y) \Psi(x) \Psi(y)$$

$$= - \int dx\, \partial_x^2 \Psi^\dagger(x) \Psi(x) \longrightarrow T$$

$$- \int dx dy\, \partial_x^2 \Psi^\dagger(x) \Psi^\dagger(y) \Psi(y) \Psi(x)$$

$$= T$$

$$- \int dx dy\, \Psi^\dagger(x) \Psi^\dagger(y) \Psi(y)\, \partial_x^2 \Psi(x)$$

$$= T - \int dx\,dy\, \Psi^\dagger(y) \underbrace{\Psi^\dagger(x)\Psi(y)}_{\Psi(y)\Psi^\dagger(x) - \delta(x-y)} \partial_x^2 \Psi(x)$$

$$= T + \overbrace{\int dx\, \Psi^\dagger(x)\, \partial_x^2 \Psi(x)}^{-T}$$

$$- \int dx\,dy\, \Psi^\dagger(y)\Psi(y)\Psi^\dagger(x)\, \partial_x^2 \Psi(x)$$

$$= \int dx\,dy\, \Psi^\dagger(y)\Psi(y)\, \partial_x \Psi^\dagger(x)\, \partial_x \Psi(x)$$

$$\Downarrow$$

$$[T, Q] = 0$$

Similarly one can prove $[H, Q] = [H, P] = 0$

$\Downarrow$ $\Downarrow$

can look for common eigenfunctions of

H, P, Q

$$|\Psi_N(\lambda_1, \dots, \lambda_N)\rangle = \frac{1}{\sqrt{N!}} \int dz_1 \dots dz_N\, \Psi_N(z_1, \dots, z_N | \lambda_1, \dots, \lambda_N)\, \Psi^\dagger(z_1) \dots \Psi^\dagger(z_N) |0\rangle$$

$$H|\Psi_N\rangle = E_N|\Psi_N\rangle$$

$$P|\Psi_N\rangle = P_N|\Psi_N\rangle$$

$$Q|\Psi_N\rangle = N|\Psi_N\rangle$$

in first quantization

$$H_N = \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial z_j^2}\right) + 2c \sum_{j<k} \delta(x_j - x_k)$$

$$P_N = \sum_{j=1}^{N} \left(-i \frac{\partial}{\partial z_j}\right)$$

$$H_N \Psi_E = E_N \Psi_N$$

relation between 1st and 2nd quantization

demonstrate for momentum operator

- for $N=1$ system

$$|\Psi_1\rangle = \int dz_1\, \psi_1(z_1;\lambda_1)\, \Psi^\dagger(z_1)|0\rangle$$

$$P|\Psi_1\rangle = i\int dx\, dz_1\, \psi_1(z_1;\lambda_1)\, [\partial_x \Psi^\dagger(x)]\, \underbrace{\Psi(x)\Psi^\dagger(z_1)}_{\delta(x-z_1)+\Psi^\dagger(z_1)\Psi(x)}|0\rangle$$

$$= i\int dz_1\, \psi_1(z_1;\lambda_1)\, [\partial_{z_1}\Psi^\dagger(z_1)|0\rangle + \int dx\, dz_1\, \psi_1(z_1;\lambda_1)\, \partial_x\Psi^\dagger(x)\Psi^\dagger(z_1)\overbrace{\Psi(x)|0\rangle}^{0}$$

$$= -i\int dz_1\, \partial_{z_1}\psi_1(z_1;\lambda_1)\, \Psi^\dagger(z_1)|0\rangle$$

$$= \int dz_1 \left[(-i\partial_{z_1})\psi_1(z_1;\lambda_1)\right]\Psi^\dagger(z_1)|0\rangle$$

momentum operator (1st quantization) — wavefct. (1st quantization)

- for N-particle system
  - every time we apply commutator we obtain a contribution of the form

$$\int dz_1 \dots dz_N \left[i\sum_{i=1}^{N}\partial_{z_i}\right]\psi_N(z_1,\dots,z_N;\lambda_1,\dots,\lambda_N)\times$$

$$\times\, \Psi^\dagger(z_1)\dots\Psi^\dagger(z_N)|0\rangle$$

- for Hamiltonian one can also show this

- consider H_N in 1st quantization Hamiltonian and χ_N symmetric in z_j

$\Downarrow$ sufficient to consider domain $\Downarrow$

$$z_1 < z_2 < z_3 < \dots < z_N$$

$\Downarrow$

$$\mathcal{H}_N^0 = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2}$$

$$H_N^0 \chi_N = E_N \chi_N$$

- boundary conditions:

$$\left(\frac{\partial}{\partial z_{j+1}} - \frac{\partial}{\partial z_j} - c\right)\chi_N = 0 \qquad z_{j+1} = z_j + 0$$

have to be satisfied by χ_N

- to show this: consider z_1, z_2, hold all others fixed
change variables: $z = z_2 - z_1$
$Z = z_2 + z_1$

$$\partial_{z_1} = \partial_z \left(\frac{\partial z}{\partial z_1}\right) + \partial_Z \frac{\partial Z}{\partial z_1} = -\partial_z + \partial_Z$$

$$\partial^2_{z_1} = \partial_z^2 - \partial^2_{zZ} + \partial_Z^2$$

$$\partial^2_{z_2} = \partial_z^2 + \partial^2_{zZ} + \partial_Z^2$$

$$\partial^2_{z_1} + \partial^2_{z_2} = 2\partial_z^2 + 2\partial_Z^2$$

$$H = -2\partial_z^2 - 2\partial_Z^2 - \sum_{j\neq 1,2} \partial^2_{z_j} + 2c\sum_{\substack{i<j\\ i,j\neq 1,2}} \delta(z_i - z_j) + 2c\,\delta(z)$$

$$H\chi_N(z, Z, z_3, \dots, z_N) = E\chi_N(z, Z, z_3, \dots, z_N)$$

- forgetting about fixed coordinates we have

$$\left[-2\partial_z^2 - 2\partial_Z^2 + 2c\,\delta(z)\right]\Psi(z,Z) = E\Psi(z,Z)$$

integrate in $\int_{-\epsilon}^{\epsilon} dz$ $\quad \epsilon \to$ infinitesimal

$$\to \partial_z \Psi(z_{+}, Z) - \partial_z \Psi(z_{-}, Z) = c\,\Psi(0, Z)$$

$$\partial_z = \frac{\partial z_1}{\partial z}\partial_{z_1} + \frac{\partial z_2}{\partial z}\partial_{z_2} \qquad z_2 = \frac{Z+z}{2}$$

$$z_1 = \frac{Z-z}{2}$$

$$\partial_z = \frac{1}{2}(\partial_{z_2} - \partial_{z_1})$$

$$\Downarrow$$

$$(\partial_{z_2} - \partial_{z_1})\Psi(z_1, z_2)\Big|_{z_2 = z_{1+}} - (\partial_{z_2} - \partial_{z_1})\Psi(z_1, z_2)\Big|_{z_2 = z_{1-}} =$$

$$= 2c\,\Psi(z_1, z_2)\Big|_{z_1 = z_2}$$

symmetry of wavefunction: (bosonic)

$$(\partial_{z_2} - \partial_{z_1} - c)\Psi(z_1, z_2) = 0 \qquad \cancel{z_1 = z_2}$$

$$z_2 = z_1 + 0$$

in general:

$$(\partial_{z_{j+1}} - \partial_{z_j} - c)\Psi_N = 0 \qquad z_{j+1} = z_j + 0$$

constructing the solution:

- consider 2-particle case

$$\chi_D = e^{i(\lambda_1 z_1 + \lambda_2 z_2)} - e^{i(\lambda_2 z_1 + \lambda_1 z_2)} \qquad \text{(determinant)}$$

$$\chi = (\partial_{z_2} - \partial_{z_1} + c)\chi_D = (i\lambda_2 - i\lambda_1 + c)\, e^{i(\lambda_1 z_1 + \lambda_2 z_2)}$$
$$- (i\lambda_1 - i\lambda_2 + c)\, e^{i(\lambda_2 z_1 + \lambda_1 z_2)}$$

χ not antisymmetric in $z_1 \longleftrightarrow z_2$

χ antisymmetric in $\lambda_1 \longleftrightarrow \lambda_2$

- boundary conditions: $(\partial_{z_2} - \partial_{z_1} - c)\chi = 0$ if $z_2 \to z_1$

$$(i\lambda_2 - i\lambda_1 - c)(i\lambda_2 - i\lambda_1 + c)\, e^{i(\lambda_1 z_1 + \lambda_2 z_2)}$$
$$- (i\lambda_1 - i\lambda_2 - c)(i\lambda_1 - i\lambda_2 + c)\, e^{i(\lambda_2 z_1 + \lambda_1 z_2)}$$
$$= [-(\lambda_2 - \lambda_1)^2 - c^2]\,[e^{i(\lambda_1 z_1 + \lambda_2 z_2)} - e^{i(\lambda_2 z_1 + \lambda_1 z_2)}]$$
$$= 0 \quad \text{for } z_2 \to z_1 \;!$$

in general:

$$\chi_N = \tilde{N}\left[\prod_{N \geq j > k \geq 1} (\partial_{z_j} - \partial_{z_k} + c)\right] \mathrm{Det}[\exp(i\lambda_j z_k)]$$

$$\chi_N = \tilde{N} \sum_P (-1)^P \prod_{N \geq j > k \geq 1} (i\lambda_{P_j} - i\lambda_{P_k} + c)\, e^{i\sum_{n=1}^{N} \lambda_{P_n} z_n}$$

normalization: $\tilde{N} = \left\{ N! \prod_{N \geq j > k \geq 1} [(\lambda_j - \lambda_k)^2 + c^2] \right\}^{-1/2}$

wave function can also be extended to $\mathbb{R}^N$

$$\chi_N = \{ N! \prod_{j>k} [(\lambda_j - \lambda_k)^2 + c^2] \}^{-1/2}$$

$$\sum_P (-1)^P \prod_{j>k} [\lambda_{P_j} - \lambda_{P_k} - ic\, \epsilon(x_j - x_k)]$$

$$\times\, e^{i \sum_{n=1}^{N} \lambda_{P_n} x_n}$$

$$\epsilon(x_j - x_k) = \text{sign of } (x_j - x_k)$$

$$\epsilon(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

for this wave function:

$$E_N = \sum_{j=1}^{N} \lambda_j^2 \qquad P_N = \sum_{j=1}^{N} \lambda_j \qquad Q_N = N$$

Periodic Boundary Conditions

- one can use either the extended wave function or the one defined for the domain $x_1 < x_2 < \ldots$

- two-particle example:

  use wavefunction valid for $x_1 < x_2$

  periodic boundary conditions means:

$$\chi_2(x_1, x_2) = \chi_2(x_2, x_1 + L)$$

substituting:

$$\chi_2(x_1, x_2) = (i(\lambda_2 - \lambda_1) + c)\, e^{i(\lambda_1 x_1 + \lambda_2 x_2)} - (i(\lambda_1 - \lambda_2) + c)\, e^{i(\lambda_2 x_1 + \lambda_1 x_2)}$$

$$\chi_2(x_2, x_1 + L) = (i(\lambda_2 - \lambda_1) + c)\, e^{i(\lambda_1 x_2 + \lambda_2 x_1)}\, e^{i\lambda_2 L} - (i(\lambda_1 - \lambda_2) + c)\, e^{i(\lambda_2 x_2 + \lambda_1 x_1)}\, e^{i\lambda_1 L}$$

$$\Downarrow$$

$$e^{i\lambda_1 L} = -\frac{(i(\lambda_2 - \lambda_1) + c)}{(i(\lambda_1 - \lambda_2) + c)} = -\frac{(\lambda_2 - \lambda_1) - ic}{(\lambda_1 - \lambda_2) \mp ic} = \frac{\lambda_1 - \lambda_2 + ic}{\lambda_1 - \lambda_2 - ic}$$

$$e^{i\lambda_2 L} = \frac{\lambda_2 - \lambda_1 + ic}{\lambda_2 - \lambda_1 - ic}$$

For the general case

$$\boxed{e^{i\lambda_j L} = -\prod_{k=1}^{N} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}} \Rightarrow \text{Bethe equations}$$

- N equations for N unknowns
  - equations are algebraic ⇒ can be expected to be easier to solve than the original nonlinear differential equation

Properties of the solutions

$\boxed{k_j \text{ - are real}}$

Proof: $|\exp(i\lambda L)| \leq 1$ if $\mathrm{Im}\,\lambda \geq 0$

$|\exp(i\lambda L)| \geq 1$ if $\mathrm{Im}\,\lambda \leq 0$

$\left|\frac{\lambda+ic}{\lambda-ic}\right| \geq 1$ if $\mathrm{Im}\,\lambda \geq 0$

$\left|\frac{\lambda+ic}{\lambda-ic}\right| \leq 1$ if $\mathrm{Im}\,\lambda \leq 0$

suppose: λ_{max} is λ_j with maximal imaginary part

$$\mathrm{Im}\,\lambda_{max} \geq \mathrm{Im}\,\lambda_j \qquad j=1,\dots,N$$

Bethe equations: for λ_{max}

$$|\exp i\lambda_{max} L| = \left|\prod_k \frac{\lambda_{max}-\lambda_k+ic}{\lambda_{max}-\lambda_k-ic}\right| \geq 1$$

$$\Downarrow$$

$$\mathrm{Im}\,\lambda_{max} \leq 0$$

if we choose λ_{min} to be λ_j with minimum imaginary part

$$\Downarrow$$

$\mathrm{Im}\,\lambda_{min} \geq 0$ from the same analysis

$$\Downarrow$$

λ's are real

logaritmic form of the Bethe ansatz equations (12)

$$e^{i\lambda_j L} = \prod_{k\neq j} \frac{\lambda_j - \lambda_k + ic}{\lambda_j - \lambda_k - ic}$$

$$2\pi \tilde{n}_j = \lambda_j L + \sum_{k\neq j} \varphi(\lambda_j - \lambda_k)$$

$$\varphi(\lambda) = i \ln\left(\frac{\lambda + ic}{\lambda - ic}\right) \qquad -2\pi < \varphi(\lambda) < 0$$

$$\operatorname{Im} \lambda = 0$$

can also use antisymmetric function

$$\Theta(\lambda) = \varphi(\lambda) + \pi \qquad \underline{\Theta(\lambda) = -\Theta(-\lambda)}$$

$$\Theta(\lambda) = i \ln\left(\frac{ic + \lambda}{ic - \lambda}\right)$$

$\Theta(\lambda)$ is monotonically increasing in λ

$$\Theta(\lambda_2) > \Theta(\lambda_1) \qquad \lambda_2 > \lambda_1$$

proof $\quad \Theta'(\lambda) = \dfrac{2c}{c^2 + \lambda^2} > 0$

Bethe equations $\quad \lambda_j L + \sum_{k=1}^{N} \Theta(\lambda_j - \lambda_k) = 2\pi n_j$

$$n_j = \tilde{n}_j - \frac{N-1}{2}$$

n_j – integers or half-integers

$$\boxed{\begin{array}{l}\text{Solutions of the Bethe ansatz exist and can be} \\ \text{uniquely parametrized by the integers/} \\ \text{half-integers } n_j.\end{array}}$$

Proof: Define Yang-Yang action

$$S = \frac{1}{2} L \sum_{j=1}^{N} \lambda_j^2 - 2\pi \sum_{j=1}^{N} n_j \lambda_j + \frac{1}{2} \sum_{j,k}^{N} \tilde{\Theta}_1(\lambda_j - \lambda_k)$$

$$\tilde{\Theta}_1(\lambda) = \int_0^{\lambda} \Theta(\mu)\, d\mu$$

$$\frac{\partial S}{\partial \lambda_j} = 0 \Rightarrow \text{ gives logarithmic form of Bethe ansatz equations}$$

- are the solutions of $\frac{\partial S}{\partial \lambda_j} = 0$ a minimum?

$$\frac{\partial S}{\partial \lambda_j} = L\lambda_j - 2\pi n_j + \frac{1}{2} \sum_{k=1} \Theta(\lambda_j - \lambda_k) - \frac{1}{2} \sum_{k=1}^{N} \Theta(\lambda_k - \lambda_j)$$

$$= L\lambda_j - 2\pi n_j + \sum_{k=1}^{N} \Theta(\lambda_j - \lambda_k)$$

$$\frac{\partial^2 S}{\partial \lambda_j \lambda_k} = L\delta_{jk} + \sum_{l=1}^{N} K(\lambda_j, \lambda_l)\, \delta_{jk} - K(\lambda_j, \lambda_k)$$

$$K(\lambda_j, \lambda_k) = \frac{\partial \Theta(\lambda_j - \lambda_k)}{\partial(\lambda_j - \lambda_k)} = \frac{2c}{c^2 + (\lambda_j - \lambda_k)^2}$$

matrix of second derivatives $\frac{\partial^2 S}{\partial \lambda_j \lambda_k}$ is positive definite

$$\sum_{j,l} \frac{\partial^2 S}{\partial \lambda_j \partial \lambda_l} v_j v_l = \sum_{j=1}^{N} L v_j^2 + \sum_{j>l} K(\lambda_j, \lambda_l)(v_j - v_l)^2 \geq 0$$

$\Rightarrow$ unique minimum

$$\boxed{\text{If } \quad n_j > n_k \quad \Rightarrow \quad \lambda_j > \lambda_k}$$

to show this one considers two different Bethe ansatz solutions and subtracts one from other

$$L(\lambda_j - \lambda_k) + \sum_{l=1}^{N} [\Theta(\lambda_j - \lambda_l) - \Theta(\lambda_k - \lambda_l)] = 2\pi(n_j - n_k)$$

– due to monotonic increase of $\Theta(\lambda)$ the statement If $n_j > n_k \Rightarrow \lambda_j > \lambda_k$ holds

y

If $n_j = n_k \Rightarrow \lambda_j = \lambda_k$, but in this case wavefunction is zero

$\Rightarrow$ Solution: 1.) choose a set of n_j all different

2.) solve Bethe ansatz equations (one choice is to solve them iteratively)

thermodynamic limit

$$L(\lambda_j - \lambda_k) + \sum_{\ell=1}^{N} [\theta(\lambda_j - \lambda_\ell) - \theta(\lambda_k - \lambda_\ell)] = 2\pi(n_j - n_k)$$

$\Downarrow$ from $\theta(\lambda)$ being monotonic it follows that

$$\frac{2\pi}{L}(n_j - n_k) \geq |\lambda_j - \lambda_k|$$

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we can write: $\theta(\lambda) - \theta(\mu) = \int_{\mu}^{\lambda} K(\nu, 0)\, d\nu$

$$K(\nu, 0) = \cancel{\frac{2c}{c^2 + (\lambda - \mu)}} \quad \frac{2c}{c^2 + \nu^2} = \theta'(\nu)$$

since $\frac{2c}{c^2+\nu^2} \leq \frac{2}{c}$

$$\theta(\lambda) - \theta(\mu) \leq \frac{2}{c}(\lambda - \mu)$$

$$\Downarrow$$

$$L(\lambda_j - \lambda_k) + \frac{2N}{c}(\lambda_j - \lambda_k) \geq 2\pi(n_j - n_k)$$

$$(\lambda_j - \lambda_k)\left[1 + \frac{2}{c}D\right] \geq \frac{2\pi(n_j - n_k)}{L} \qquad D = \frac{N}{L}$$

$$\Downarrow$$

$$\frac{2\pi}{L}(n_j - n_k) \geq |\lambda_j - \lambda_k| \geq \frac{2\pi(n_j - n_k)}{L(1 + \frac{2D}{c})} \geq \frac{2\pi}{L(1 + \frac{2D}{c})}$$

interval $|\lambda_j - \lambda_k|$ is bounded above

interval $|\lambda_j - \lambda_k|$ is bounded below

- energy functional $\sum_j \lambda_j^2$ will be minimized for n_j symmetrically spread around zero (obvious for $c = \infty$)
- define continuous function $\lambda(x)$

$$L\lambda(x) + \sum_{n=1}^{N} \theta(\lambda(x) - \lambda_k) = 2\pi L x$$

$$S = \frac{1}{2} L \lambda^2(x) + \sum_{n=1}^{N} \theta_1(\lambda(x) - \lambda_k) - 2\pi L x \lambda(x)$$

$\lambda\left(\frac{n_j}{L}\right) = \lambda_j \Rightarrow$ establishes contact with discrete Bethe ansatz equations

$\cancel{n_j \to m}$ λ_j - momenta

$\lambda_m = \lambda\left(\frac{m}{L}\right)$

- solution: take n's, solve Bethe equations, obtain λ_m
  for $m \in n_j \Rightarrow \lambda\left(\frac{m}{L}\right)$ is the momentum of a particle
  for $m \notin n_j \Rightarrow \lambda\left(\frac{m}{L}\right)$ is the momentum of a hole

particles + holes $\Rightarrow$ referred to as vacancies

$\rho_t(\lambda(x)) = \frac{dx(\lambda)}{d\lambda} \rightarrow$ density of vacancies

- derivative of the Bethe ansatz eq'ns for $\lambda(x)$ with respect to $\lambda(x)$

$$1 + \frac{1}{L}\sum_{k=1}^{N} K(\lambda(x), \lambda_k) = 2\pi \rho_t(\lambda(x))$$

- thermodynamic limit

$N \to \infty \quad L \to \infty \quad D = \frac{N}{L}$ remains finite

- start with Bethe equations

$$L\lambda_j + \sum_{k=1}^{N} \theta(\lambda_j - \lambda_k) = 2\pi\left[j - \left(\frac{N+1}{2}\right)\right]$$

as $L \to \infty \quad |\lambda_{j+1} - \lambda_j| \to O(\frac{1}{L})$

since $\frac{2\pi(n_{j+1} - n_j)}{L} \geq |\lambda_{j+1} - \lambda_j| \geq \frac{4\pi(n_{j+1} - n_j)}{L(1 + \frac{2\rho}{c})}$

λ's will fill a symmetric interval $[-q, q]$ $\quad q = \lim \lambda_N$

momentum density $\rho(\lambda_k) = \frac{1}{L(\lambda_{k+1} - \lambda_k)} > 0$

definition of density $\quad \rho(\lambda) = \rho_t(\lambda) = \frac{dx}{d\lambda} > 0$

$\rho(\lambda) > 0 \quad L\int \rho(\lambda)d\lambda = L\int dx$

$= N$ - number of particles

can also convert sum $\frac{1}{L}\sum_e K(\lambda(x), \lambda_e)$ to integral

$$\frac{1}{L}\sum_e K(\lambda(x), \lambda_e) = \int_{-N/2L}^{N/2L} K(\lambda(x), \lambda(y))dy$$

$$= \int_{-a}^{a} K(\lambda(x), \mu)\, \rho(\mu)\, d\mu$$

$$\Downarrow$$

Bethe equation in the thermodynamic limit

$$\boxed{\rho(\lambda) - \frac{1}{2\pi}\int_{-a}^{a} K(\lambda, \mu)\, \rho(\mu)\, d\mu = \frac{1}{2\pi}}$$

$$\boxed{\text{condition:} \quad \int_{-a}^{a} \rho(\lambda)\, d\lambda = D}$$

energy: $\quad \dfrac{E_L}{L} = \displaystyle\int_{-a}^{a} \lambda^2 \rho(\lambda)\, d\lambda$