Bethe-ansatz (II)

quasi-momenta k_i => complex quantities
phases $\theta(k_i, k_j)$

convention:

$$0 \leq \mathrm{R}(k_i) \leq 2\pi$$

$$-\pi \leq \mathrm{R}(\theta(k_i, k_j)) \leq \pi$$

equation for phases:

$$\cot \frac{\theta(k_i, k_j)}{2} = \frac{\Delta\left(\cot\frac{k_i}{2} - \cot\frac{k_j}{2}\right)}{1+\Delta - (1-\Delta)\cot\frac{k_i}{2}\cot\frac{k_j}{2}}$$

- in solving the Bethe ansatz equations it is more convenient to transfer variables from quasi-momenta to rapidities

here: give example for $\Delta = 1$, explain for $\Delta = -1$ and give overview for general case

$\Delta = 1$:

$$\rightarrow \quad \cot \frac{\theta(k_i, k_j)}{2} = \frac{1}{2}\left(\cot\frac{k_i}{2} - \cot\frac{k_j}{2}\right)$$

$$v_i = \frac{1}{2}\cot\frac{k_i}{2} \qquad \text{(rapidity)}$$

advantage: 1. variables (other ones) appear in simpler form

v_i, $v_i - v_j$ etc.

2.) $-\infty < v < \infty$

$$k_i = 2\cot^{-1} 2v_i = \frac{1}{i}\ln\frac{v_i + i/2}{v_i - i/2}$$

$$\frac{1}{2}\cot\frac{k_i}{2} = v_i \Rightarrow 2v_i = i\,\frac{e^{ik_i/2} + e^{-ik_i/2}}{e^{ik_i/2} - e^{-ik_i/2}}$$

$$\Rightarrow -2iv_i = \left(\frac{e^{ik_i} + 1}{e^{ik_i} - 1}\right)$$

$$\Rightarrow -2iv_i(e^{ik_i} - 1) = e^{ik_i} + 1$$

$$-2iv_i e^{ik_i} + 2iv_i = e^{ik_i} + 1$$

$$2iv_i - 1 = (1 + 2iv_i)\,e^{ik_j}$$

$$e^{ik_i} = \frac{2iv_i - 1}{2iv_i + 1} = \frac{v_i + i/2}{v_i - i/2}$$

$$\Rightarrow k_i = \frac{1}{i}\ln\frac{v_i + i/2}{v_i - i/2}$$

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can also write

$$\theta(k_i, k_j) = 2\cot^{-1}(v_i - v_j) = \frac{1}{i}\ln\frac{v_i - v_j + i}{v_i - v_j - i}$$

~~$v_i = \frac{1}{2}\cot\frac{k_i}{2} = \frac{i}{2}\frac{e^{ik_i/2} + e^{-ik_i/2}}{e^{ik_i/2} - e^{-ik_i/2}} = \frac{i}{2}\frac{e^{ik_i} + 1}{e^{ik_i} - 1}$~~

$$a = \cot^{-1} b \Rightarrow a = \frac{1}{2i}\ln\frac{b + i}{b - i}$$

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$$\theta(k_i, k_j) = 2\cot^{-1}(v_i - v_j) = \frac{1}{i}\ln\frac{v_i - v_j + i}{v_i - v_j - i}$$

Bethe ansatz equations can be written in the form

$$e^{ik_i N} = \prod_{j=1, j\neq i}^{n} e^{iG(k_i, k_j)}$$

$$\boxed{\left(\frac{v_i + i/2}{v_i - i/2}\right)^N = \prod_{j=1\, j\neq i}^{n} \frac{v_i - v_j + i}{v_i - v_j - i}}$$

energy eigenvalue:

$$E_M = \sum_i \cos k_i - 1 = \sum_i \left[-\frac{1}{2}\frac{1}{v_i^2 + 1/4}\right] = \frac{1}{2}\sum_{i=1}^{n}\frac{dk_i}{dv_i}$$

for $\Delta = -1$ rapidity is defined as

$$v_i = \frac{1}{2}\tan\frac{k_i}{2} \longrightarrow k_i = \frac{1}{i}\ln\frac{v_i - i/2}{v_i + i/2}$$

since

$$\cot\frac{G(k_i, k_j)}{2} = \frac{\Delta\left[\cot\left(\frac{k_i}{2}\right) - \cot\left(\frac{k_j}{2}\right)\right]}{1+\Delta - (1-\Delta)\cot\frac{k_i}{2}\cot\frac{k_j}{2}}$$

$$= \frac{-1\left[\cot\left(\frac{k_i}{2}\right) - \cot\left(\frac{k_j}{2}\right)\right]}{-2\cot\left(\frac{k_i}{2}\right)\cot\left(\frac{k_j}{2}\right)}$$

$$= \frac{1}{2}\left[\tan\left(\frac{k_j}{2}\right) - \tan\left(\frac{k_i}{2}\right)\right]$$

Bethe-ansatz equations:

$$\left(\frac{v_i - i/2}{v_i + i/2}\right)^N = \prod_{j=1\, j\neq i}^{M} \frac{v_i - v_j + i}{v_i - v_j - i}$$

$$E_M = \sum_i -\frac{1}{2}\frac{1}{v_i^2 + 1/4}$$

anisotropic spin chain $\Delta \neq \pm 1$

$\Delta = \cos 2\eta \qquad \eta$ - can be real/imaginary

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$$e^{ik_i} = \frac{\sin(v_i+\eta)}{\sin(v_i-\eta)}$$

$$e^{i\theta(k_i,k_j)} = -\frac{e^{i(k_i+k_j)} - 2\Delta e^{ik_i} + 1}{e^{i(k_i+k_j)} - 2\Delta e^{ik_j} + 1}$$

$$e^{i\theta(v_i,v_j)} = \frac{\sin(v_i-v_j+2\eta)}{\sin(v_i-v_j-2\eta)}$$

$$\left[\frac{\sin(v_i+\eta)}{\sin(v_i-\eta)}\right]^N = \prod_{j=1, j\neq i}^{M} \frac{\sin(v_i-v_j+2\eta)}{\sin(v_i-v_j-2\eta)}$$

$$E_M = \frac{1}{2}\sum_{i=1}^{M}\left[\frac{\sin(v_i+\eta)\sin(v_i-\eta)}{\sin(v_i-\eta)\sin(v_i+\eta)} - 2\Delta\right]$$

for $\Delta > 1$

$$\eta = i\frac{\Phi}{2} \qquad v_i = \frac{\varphi_i}{2}$$

$$E_M = -\sum_{i=1}^{M} \frac{\sinh^2\Phi}{\cosh\Phi - \cos\varphi_i}$$

for $-1 < \Delta < 1$

$$\eta = -\frac{\gamma}{2} \qquad v_i = i\frac{\psi_i}{2}$$

$$E_M = -\sum_{i=1}^{M} \frac{\sin^2\Psi}{\cosh\psi_i - \cos\Psi}$$

for $\Delta < -1 \qquad \eta = \frac{\pi}{2} - i\frac{\tilde{\Phi}}{2} \qquad v_i = -\frac{\varphi_i}{2} \qquad \Delta = -\cosh\tilde{\Phi}$

$$E_M = \sum_{i=1}^{M} \frac{\sinh^2\tilde{\Phi}}{\cosh\tilde{\Phi} - \cos\varphi_i}$$

Ground state of XXZ spin chain

$\Delta \leq -1$ ferromagnetic spin chain

- ground state $M = 0$ $E_0 = 0$

$\Delta \geq -1$

can prove (Marshall; Lieb, Schultz, Mattis; Lieb, Mattis)

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ground state $S = 0$, $S^z = 0$

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$M = N/2$

Bethe guessed: $M = N/2$ $(S^z = 0)$

chose: $\lambda_i = (2i - 1)$ $i = 1, 2, \ldots, N/2$

(proven by Yang and Yang, 1966)

1.) replace $i \rightarrow x_i = \frac{\lambda_i}{N}$

$N \rightarrow \infty$ $\lambda = Nx$

it follows that $k_i \rightarrow k(x)$ as $N \rightarrow \infty$

$$\theta(k_i, k_j) = \theta(k(x), k(y)) \equiv \Theta(x, y)$$

Bethe ansatz equations

$$N k_i = 2\pi \lambda_i + \sum_{j \neq i}^{n} \theta(k_i, k_j) \longrightarrow k(x) = 2\pi x + \frac{1}{2} \int_0 \Theta(x, y) \, dy$$

- next step is to parametrize x according to $\xi(x)$ which are the variables which depend on anisotropy

$$\Rightarrow \quad k(\xi(x)) = 2\pi x + \frac{1}{2}\int_0^1 dy\, \Theta(\xi(x), \xi(y))$$

$$\frac{dx}{d\xi} = -\rho(\xi)$$

$$\frac{dk(\xi)}{d\xi} = -2\pi\rho(\xi) - \frac{1}{2}\int_{\xi_0}^{\xi_1} d\eta\, \rho(\eta) \frac{\partial\Theta(\xi,\eta)}{\partial\xi}$$

$\Rightarrow \quad \xi_1 = -\xi_0 \quad \Rightarrow$ due to even-ness of $\rho(\xi)$

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$$-\frac{dk(\xi)}{d\xi} = 2\pi\rho(\xi) - \frac{1}{2}\int_{-\xi_0}^{\xi_0} d\eta\, \rho(\eta)\, \frac{\partial\Theta(\xi,\eta)}{\partial\xi}$$

ξ_0 - determined from condition:

$$\int_0^1 dx = \int_{-\xi_0}^{\xi_0} d\xi\, \rho(\xi) = 1$$

- make calculation specific to isotropic case

$\Delta = 1 \quad \Rightarrow \quad \xi(x) = v(x) \qquad -\infty < v(x) < \infty$

$$k(v) = 2\cot^{-1} 2v$$

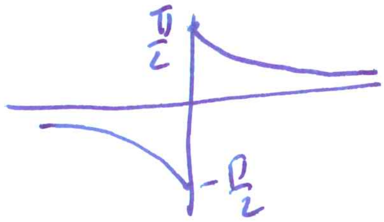
$$\Theta(v, v') = 2\cot^{-1}(v - v')$$

$$\frac{dk}{dv} = -\frac{1}{v^2 + 1/4}$$

$$\frac{1}{v^2+1/4} = 2\pi g(v) - \frac{1}{2}\int_{-\infty}^{\infty} dv' g(v') \frac{\partial \Theta(v,v')}{\partial v}$$

$$\Theta(v,v') = 2\cot^{-1}(v-v')$$

$$\frac{\partial \Theta(v-v')}{\partial v}$$ – has a jump at $v=0$



δ-fn. contribution needs to be accounted for

$$\frac{\partial \Theta(v-v')}{\partial v} = 2\pi\delta(v-v') - \frac{2}{1+(v-v')^2}$$

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$$\frac{1}{v^2+1/4} = \pi g(v) + \int_{-\infty}^{\infty} dv' \frac{g(v')}{1+(v-v')^2}$$

integral equation

integral equation solved by Fourier transforming

integral formula:

$$\int_{-\infty}^{\infty} dv \frac{e^{iuv}}{1+v^2} = \pi e^{-|u|}$$

left-hand side:

$$\frac{1}{v^2+1/4} = \frac{4}{4v^2+1} = 4\frac{1}{(2v)^2+1}$$

$$\text{FT.: } 4\int_{-\infty}^{\infty} dv \frac{e^{iuv}}{(2v)^2+1} = 2\int_{-\infty}^{\infty} dw \frac{e^{i\frac{uw}{2}}}{w^2+1} = 2\pi e^{-|u|/2}$$

$$w = 2v \Rightarrow v = \frac{w}{2}$$

right-hand side term 1:

$$\pi g(v) \rightarrow \pi \int_{-\infty}^{\infty} dv\, e^{iuv} g(v) = \pi \tilde{g}(u)$$

right-hand side term 2:

$$\int_{-\infty}^{\infty} dv' \frac{g(v')}{1+(v-v')^2} \Rightarrow \int_{-\infty}^{\infty} dv\, e^{iuv} \int_{-\infty}^{\infty} \frac{dv'\, g(v')}{1+(v-v')^2}$$

$$= \int_{-\infty}^{\infty} dv'\, g(v') \int_{-\infty}^{\infty} dv \frac{e^{iuv}}{1+(v-v')^2}$$

$$w = v - v' \qquad v = w + v'$$

$$= \int_{-\infty}^{\infty} dv'\, g(v')\, e^{iuv'} \int_{-\infty}^{\infty} dw \frac{e^{iuw}}{1+w^2}$$

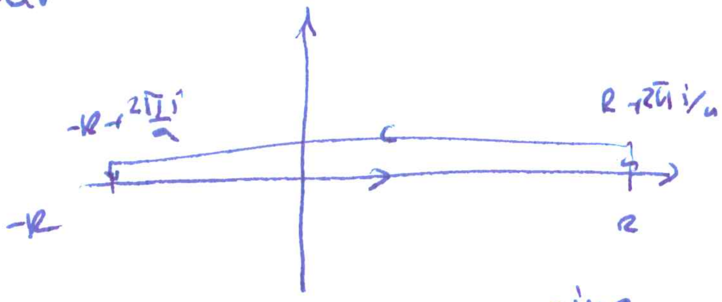
$$= \pi \tilde{g}(u) e^{-|u|}$$

$$2\pi e^{-|u|/2} = \pi \tilde{g}(u)(1+e^{-|u|}) \Rightarrow \tilde{g}(u) = \frac{1}{\cosh\frac{u}{2}}$$

$\tilde{g}(u) \to g(v) = \frac{1}{2\pi}\int e^{-iuv}\,\tilde{g}(u)\,du$

$= \frac{1}{2\pi}\int \frac{e^{-iuv}}{\cosh\frac{u}{2}}\,du$

contour: rectangular



$$(1+e^{2\pi v})\int_{-\infty}^{\infty}\frac{e^{-iuv}}{\cosh\frac{u}{2}}\,du = \lim_{R\to\infty}\oint_{C_R}\frac{e^{-ivz}}{\cosh z/2}\,dz$$

residue theorem:

contour integral equal to $2\pi i$ times residue of integrand at $z = \pi i$ (simple pole)

$$\mathrm{Res}\,\frac{e^{-ivz}}{\cosh\frac{z}{2}}\Big|_{z=\pi i} = \frac{e^{-ivz}}{2\sinh\frac{z}{2}}\Big|_{z=\pi i} = \frac{e^{\pi v}}{2i}$$

$$\to \text{F.T.}: \frac{2\pi i}{2(1+e^{\pi v})}\,\frac{1}{2\pi}\,\frac{e^{\frac{\pi v}{2}}}{i} = \frac{1}{\cosh\frac{\pi v}{2}}$$

$$g(v) = \frac{1}{\cosh \pi v}$$

$$E_{\frac{N}{2}} = -\frac{1}{2} \int_{-\infty}^{\infty} dv \, \frac{\rho(v)}{v^2 + 1/4}$$

$$= -N \int_{-\infty}^{\infty} \frac{dx}{(x^2+1) \cosh \frac{\pi x}{2}}$$

$$\boxed{E_{\frac{N}{2}} = -N \ln 2}$$

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exact energy of Heisenberg model