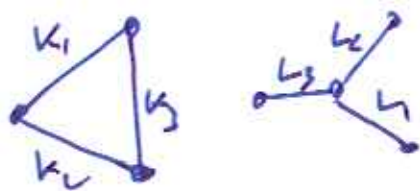


Lecture 4: Diagonalizing the Transfer ①

Matrix for $T=T_c$

Review:

1) \star - Δ relation:

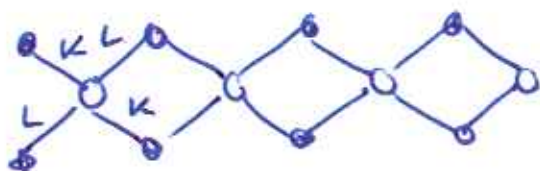


$$\Rightarrow \sinh 2K_1 \sinh 2L_1 = \sinh 2K_2 \sinh 2L_2 \\ = \sinh 2L_3 \sinh 2K_3$$

we will use it shortly

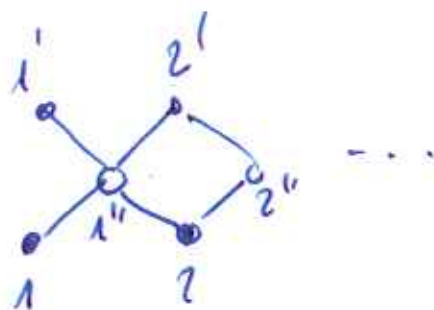
transfer matrix is constructed by rotating

lattice \Rightarrow



- periodic boundary conditions in both directions

- labelling:



- two different transfer matrices

one between lower $\bullet \bullet \dots$ sequence
and $0 0 \dots$ sequence $\Rightarrow V$

One between $0 0 \dots$ sequence and upper $\bullet \bullet \bullet \dots$
sequence $\Rightarrow W$

$V_{\psi, \psi''} \rightarrow \psi \rightarrow$ all lower \bullet -spins
if n units $\Rightarrow 2^n$ states

(2)

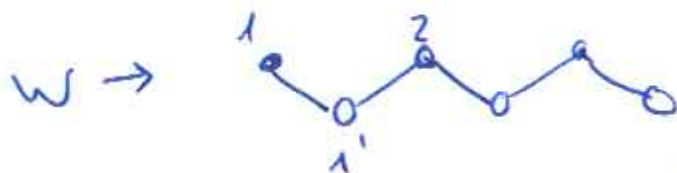
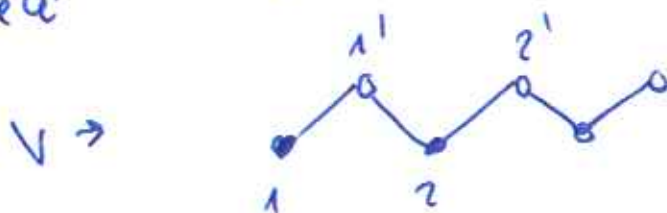
$\Rightarrow V_{\psi, \psi''} \rightarrow 2^n \times 2^n$ matrix

brute force diagonalizing for a system of 100
lattice sites ($n=100$) is already
prohibitively expensive

$$Z_N(T) = \sum_{\psi_1} \dots \sum_{\psi_m} V_{\psi_1, \psi_1} W_{\psi_2, \psi_1} \dots V_{\psi_m, \psi_m} W_{\psi_m, \psi_1}$$

$$V_{\psi, \psi'} = \exp \left[L \sum_j \sigma_j \sigma'_j + K \sum_j \sigma_j \sigma'_{j+1} \right]$$

$$W_{\psi, \psi'} = \exp \left[K \sum_j \sigma_j \sigma'_j + L \sum_j \sigma_j \sigma'_{j+1} \right]$$



$$Z_N(T) = \text{Tr}(VW)^{m/2} = \sum_{i=1}^{2^m} \lambda_i^{m/2}$$

$VW \rightarrow$ eigenvalues λ_i^2

- to diagonalize one can simplify it using the commutation and inversion relations, as well as symmetries

②

Commutation

consider: $V(K, L) W(K', L')$

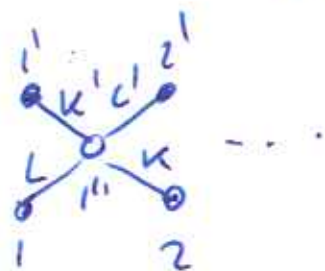
under what conditions is it true that

$$V(K, L) W(K', L') = V(K', L') W(K, L)?$$

- answer: $\sinh^2 K \sinh^2 L = \sinh^2 K' \sinh^2 L'$

- proof:

$$V(K, L) W(K', L') = \sum_{\sigma_j'' = \pm 1} \dots \sum_{\sigma_n'' = \pm 1} \prod_{j=1}^n \exp[\sigma_j'' (K' \sigma_j' + L' \sigma_{j+1}' + L \sigma_j + K \sigma_{j+1})]$$



$$V(K, L) W(K', L') = \prod_{j=1}^n X(\sigma_j, \sigma_j'; \sigma_{j+1}, \sigma_{j+1}')$$

$$w \quad \sigma_{n+1} \rightarrow \sigma_1$$

$$\sigma_{n+1}' \rightarrow \sigma_1'$$

$V(k, L)W(k', L')$ does not change if

(8)

$$X(\sigma_j, \sigma'_j; \sigma_{j+1}, \sigma'_{j+1}) \xrightarrow{\text{replaced by}} \rightarrow \rightarrow \rightarrow$$

$$e^{M\sigma_j \sigma'_j} X(\sigma_j, \sigma'_j; \sigma_{j+1}, \sigma'_{j+1}) e^{M\sigma_{j+1} \sigma'_{j+1}}$$

- can be shown easily by substituting into $V(k, L)W(k', L')$ and using periodic boundary conditions

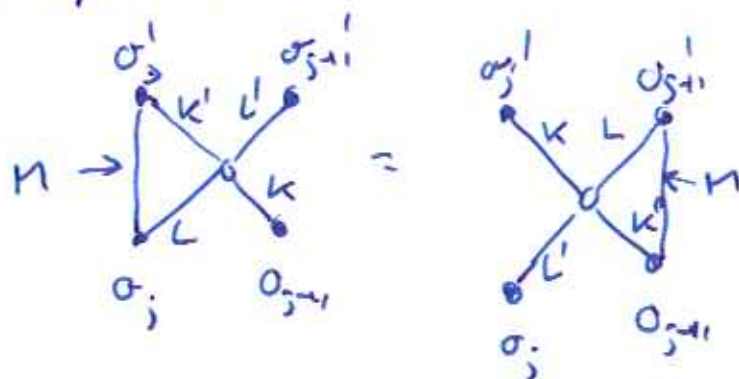
$$\rightarrow V(k, L)W(k', L') = V(k', L')W(k, L)$$

$$\text{if } e^{M\sigma_j \sigma'_j} X(\sigma_j, \sigma'_j; \sigma_{j+1}, \sigma'_{j+1})$$

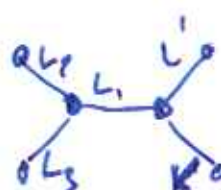
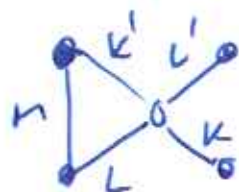
$$= X'(\sigma_j, \sigma'_j; \sigma_{j+1}, \sigma'_{j+1}) e^{M\sigma_{j+1} \sigma'_{j+1}}$$

for some number M

- draw: this equation can be drawn as

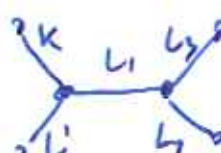
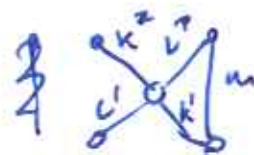


\rightarrow $\star - \Delta$ can be applied



equivalent \Rightarrow

$$L' = L_3$$



\Rightarrow

$$L_2 = k$$

$$\star - \Delta \Rightarrow \sinh 2L, \sinh 2M = \sinh 2L_2 \sinh 2K' \\ = \sinh 2L_3 \sinh 2L \\ \rightarrow \sinh 2K \sinh 2K' = \sinh 2L \sinh 2L'$$

(5)

Inversion

- Question: given K, L in $V(K, L)W(K', L')$

can we choose K', L' so that $V(K, L)W(K', L')$ is diagonal? \rightarrow No.

nearly diagonal? \rightarrow Yes.

\rightarrow 'inversion' will allow us to considerably simplify the eigenvalue equation!!!

\rightarrow answer to 1st question would hold if

$$\text{for } \sigma_j \neq \sigma'_j \quad X = 0$$

$$\text{or for } \sigma_{j+1} \neq \sigma'_{j+1} \quad X = 0.$$

but we can not choose K', L' this way

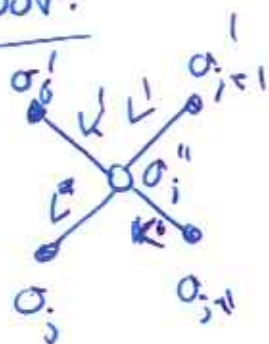
- however: if $\sigma_j \neq \sigma'_j$ and $\sigma_{j+1} \neq \sigma'_{j+1} \Rightarrow X = 0$

can be satisfied by choosing K', L' appropriately

$$\text{answer: } K' = L + \frac{i\pi}{2} \quad L' = -K$$

Proof:

(6)



$$\rightarrow X = 2 \cosh(L\sigma_i + K\sigma_{i+1} + K'\sigma_i' + L'\sigma_{i+1}') = 0$$

$$\begin{array}{cccc} \sigma_i & \sigma_{i+1} & \sigma_i' & \sigma_{i+1}' \\ 1 & 1 & -1 & 1 \end{array}$$

$$2 \cosh(L + K - K' + L') = 0$$

$$2 \cosh(L - K - K' - L') = 0$$

$$\cosh x = 0 \quad \text{if } x = \pm \frac{i\pi}{2}$$

$$\frac{e^x + e^{-x}}{2} = \frac{e^{\frac{i\pi}{2}} + e^{-\frac{i\pi}{2}}}{2} = 0$$

$$\Rightarrow L + K - K' + L' = \frac{i\pi}{2}$$

$$L - K - K' - L' = \frac{i\pi}{2}$$

$$\Rightarrow \cancel{L} \quad K' = L + \frac{i\pi}{2}$$

$$L' = -K$$

\Rightarrow this means that there are two possibilities for X to be finite

$$1.) \sigma_i = \sigma_i' \quad \text{and} \quad \sigma_{i+1} = \sigma_{i+1}'$$

$\rightarrow X_{\text{like}}$

$$2.) \sigma_i \neq \sigma_i' \quad \text{and} \quad \sigma_{i+1} \neq \sigma_{i+1}'$$

$\rightarrow X_{\text{unlike}}$

$$V(K, L) W(L + \frac{i\pi}{2}, -K)$$

(1)

$$= X_{\text{like}}^n I + X_{\text{unlike}}^n R$$

I - identity

$$R = \delta(\sigma_1, -\sigma'_1) \dots \delta(\sigma_n, -\sigma'_n)$$

$$X_{\text{like}} = 2 \cosh(L)$$

calculate X_{like}

$$\sigma_j = \sigma'_j, \quad \sigma_{j+1} = \sigma'_{j+1}$$

$$X_{\text{like}} = 2 \cosh(L\sigma_j + K\sigma_{j+1} + L\sigma_j + \frac{i\pi}{2}\sigma_j - K\sigma_{j+1})$$

$$= 2 \cosh(2L\sigma_j + \frac{i\pi}{2}\sigma_j)$$

$$= 2 \frac{e^{2L\sigma_j + \frac{i\pi}{2}\sigma_j} + e^{-2L\sigma_j - \frac{i\pi}{2}\sigma_j}}{2}$$

$$= 2 e^{\frac{i\pi}{2}\sigma_j} \sinh 2L = 2i \sinh 2L$$

calculate X_{unlike}

$$X_{\text{unlike}} = 2 \cosh(L\sigma_j + K\sigma_{j+1} - L\sigma_j - \frac{i\pi}{2}\sigma_j + K\sigma_{j+1})$$

$$= 2 \cosh(2K\sigma_{j+1} - \frac{i\pi}{2}\sigma_j)$$

$$= 2 \left(\frac{e^{2K\sigma_{j+1} - \frac{i\pi}{2}\sigma_j} + e^{-2K\sigma_{j+1} + \frac{i\pi}{2}\sigma_j}}{2} \right)$$

$$= 2 e^{-\frac{i\pi}{2}\sigma_j} \sigma_{j+1} \sinh 2K = -2i \sigma_j \sigma_{j+1} \sinh 2K$$

$$V(K, L) W(L + \frac{i\pi}{2}, -K) = (2i \sinh 2L)^n I$$

$$+ (-2i \sinh 2K)^n R \quad \text{if } n = \text{even!}$$

notice that $R^2 = I$

(8)

- question: does $V(k, L) W(L + \frac{i\pi}{2}, -k)$
obey $A - \Delta$ relation?

$$\sinh(8k) \sinh(2L + i\pi) = \sinh(2L) \sinh(2k) \quad ?$$

$$\rightarrow -\sinh(2k) \sinh(2L) = -\sinh(2k) \sinh(2L)$$

\Rightarrow yes

Symmetry relations

$$W(k, L) = V(L, k)^T$$

$$V(k, L) W(k, L) = [V(L, k) W(L, k)]^T$$

Proof: $V(k, L) = \exp \left[L \sum_j \sigma_j \sigma_j' + k \sum_j \sigma_j \sigma_{j+1}' \right]$

$$V(L, k) = \exp \left[k \sum_j \sigma_j \sigma_j' + L \sum_j \sigma_j \sigma_{j+1}' \right]$$

$$[V(L, k)]^T = \exp \left[k \sum_j \sigma_j' \sigma_j + L \sum_j \sigma_{j+1}' \sigma_j \right]$$

also for second equation

$$V(k, L) R = R V(k, L) = V(-k, -L) \quad \text{also for } W$$

$$\rightarrow V(k \pm \frac{i\pi}{2}, L \pm \frac{i\pi}{2}) = V(k, L)$$

Proof: consider



if all spins up $\Rightarrow V_{u,u'} = \exp[n(k+L)]$

restrict n to be even $\Rightarrow n = 2p$

$$V_{u,u'} = \exp(2p(K+L)) \quad \text{if } u \rightarrow up$$

$$u' \rightarrow up$$

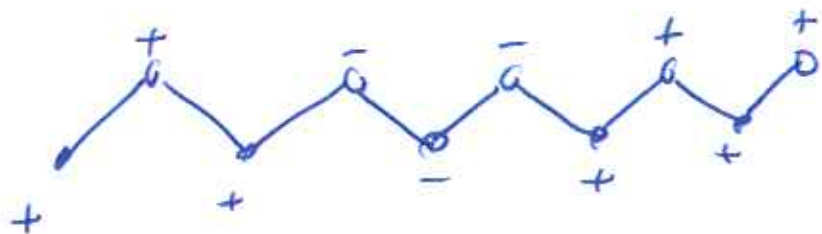
(9)

→ if there are bonds with anti-parallel spins

$$V_{u,u'} = \exp[2K(p-r) + 2L(p-s)]$$

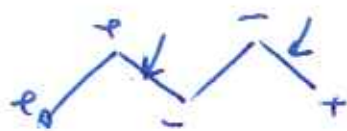
$r = K$ -bonds with anti-parallel spins

$s = L$ -bonds with anti-parallel spins



effect of one --- domain on +++ background

either



both contributions
K-bonds or
both L-bonds

Or



one K-bond and
one L-bond

it follows that r, s are either both even
or both odd

$$V_{u,u'} = \exp[2K(p-r) + 2L(p-s)]$$

$$\rightarrow \exp[2(K + \frac{i\pi}{2})(p-r) + 2(L + \frac{i\pi}{2})(p-s)]$$

$$= V_{u,u'} \underbrace{\exp i\pi(p-r) \exp i\pi(p-s)}$$

1 (since $= \exp i\pi m$ &
w/ m integer)

Commutation Relations for Transfer matrices

(10)

define operator $C = \delta(\sigma_1, \sigma'_1) \dots \delta(\sigma_n, \sigma'_n)$

shift operator

$$V(K, L) = C^{-1} V(K, L) C$$

$$W(K, L) = C^{-1} W(K, L) C$$

Proof: $V(K, L) = \exp [E \sum_j \sigma_j \sigma'_j + K \sum_j \sigma_{j+1} \sigma'_j]$
 $V(K, L) C = \exp [L \sum_j \sigma_j \sigma'_{j+1} + K \sum_j \sigma_{j+1} \sigma'_{j+1}]$
 $C^{-1} V(K, L) C = \exp [L \sum_j \sigma_{j+1} \sigma'_{j+1} + K \sum_j \sigma_{j+1} \sigma'_{j+1}]$

$$W(K, L) = V(K, L) C$$

proof: see above

\Downarrow

$$\text{if } \sinh 2K \sinh 2L = \sinh 2K' \sinh 2L'$$

$$\Rightarrow V(K, L) W(K', L') = V(K', L') V(K, L)$$

→ transfer matrices commute

$$\begin{aligned} \rightarrow V(K, L) V(L + \frac{i\pi}{2}, -K) C \\ = (2i \sinh 2L)^n I + (-2i \sinh 2K)^n R \end{aligned}$$

→ easy to show that

$$C^{-1} V C = V$$

Functional relations for Eigenvalues

(1)

$$\text{let } k = (\sinh 2K \sinh 2L)^{-1}$$

k - fixed \Rightarrow corresponds to an infinite set of k', L'

-but \Rightarrow transfer matrices all commute, so they have a common set of eigenvalues

\Downarrow

label this set of eigenvalues by $\underline{k} \Rightarrow X(k)$

$$V(K, L) X(k) = v(K, L) X(k)$$

$$C X(k) = c X(k)$$

$$R X(k) = r X(k)$$

$$c^n = r^2 = 1$$

can also write the functional relation

$$v(K, L) v(L + i\frac{\pi}{2}, -K) c$$

$$= (2i \sinh 2L)^n + (-2i \sinh 2K)^n r$$

$$\rightarrow \text{or for } \lambda's \quad \begin{aligned} VW &\rightarrow \lambda^2 \\ v^2 c &= \lambda^2 \end{aligned}$$

$$v(K, L) = \frac{\lambda(K, L)}{c^{1/2}}$$

$$\lambda(K, L) \lambda(L + i\frac{\pi}{2}, -K) =$$

$$= (2i \sinh 2L)^n + (-2i \sinh 2K)^n r$$

Obtaining eigenvalues for $T=T_c$

(12)

$$\rightarrow T=T_c \Rightarrow k=1$$

$$\sinh 2K \sinh 2L = 1$$

reparametrise variables

$$e^{2K} = \frac{\cancel{1 + \cosh u}}{\sinh u} = \frac{(1 + \sinh u)}{\cosh u}$$

$$e^{-2K} = \frac{\cosh u}{1 + \sinh u} = \frac{\cosh u (1 - \sinh u)}{1 - \sinh^2 u} = \frac{1 - \sinh u}{\cosh u}$$

$$e^{2L} = \frac{1 + \cosh u}{\sinh u}$$

$$e^{-2L} = \frac{1 - \cosh u}{\sinh u}$$

$$\sinh 2K = \frac{e^{2K} - e^{-2K}}{2} = \frac{1}{2} \left[\frac{2 \sinh u}{\cosh u} \right] = \tanh u$$

$$\sinh 2L = \coth u \Rightarrow \sinh 2K \sinh 2L = 1$$

reparametrized functions are

- single-valued
- meromorphic
- periodic in 2π

(13)

- rewrite form of $\Lambda(u)$

- consider $V_{u,u'} = \exp[2K(p-r) + 2L(p-s)]$

$$= \frac{(1 + \sin u)^{p-r}}{\cos u^{p-r}} \frac{(1 + \cos u)^{p-s}}{\sin u^{p-s}}$$

$$= \frac{(1 + \sin u)^{p-r} \cos^r u (1 + \cos u)^{p-s} \sin^s u}{(\cos u \sin u)^p}$$

- can say the following: $(1 + \sin u)^{p-r} \cos^r u (1 + \cos u)^{p-s} \sin^s u$
 \Rightarrow a power series in e^{iu}

"largest" term: e^{iu2p}

"smallest" term: e^{-iu2p}

$$\rightarrow V_{u,u'} = \frac{e^{-i2pu} (c_0 + c_1 e^{iu} + \dots + c_n e^{iu4p})}{(\sin u \cos u)^p}$$

key point: all matrix elements have this form
 (all $V_{u,u'}$, regardless of element, u, u')

$$\text{equation } V(K, L) x(k) = o(K, L) x(k)$$

\Rightarrow linear set of equations

$$V_{11} x_1 + V_{12} x_2 + \dots = v x_1$$

x 's do not depend on $K, L \Rightarrow$ they are "just" coefficients

$\Rightarrow v(k; L)$ also has to have the same form (17)
 [power series in e^{ik} divided by
 $(\sin u \cos u)^0$]

also $\lambda(k; L)$ has this form \Rightarrow can write $\rightarrow \rightarrow$
 $\rightarrow \rightarrow v(u), \lambda(u)$

- suppose u is replaced by $u + \pi$

$$u \rightarrow u + \pi$$

$$e^{ik} = \left(\frac{1 + i \sin u}{\cos u} \right) = \frac{1 - \sin u}{-\cos u} = -e^{-ik}$$

$$k \rightarrow -k + \frac{i\pi}{2}$$

$$L \rightarrow -L + \frac{i\pi}{2}$$

\rightarrow we already know that a shift by $\frac{i\pi}{2}$ in
both k, L has no effect

$$v(k, L) R x(k) = v(u + \pi) x(k)$$

$$\Downarrow$$

$$v(u) R x(k) = v(u + \pi)$$

$$\lambda(u + \pi) = \lambda(u) R$$

$$\Lambda(u) = \frac{e^{-2\pi i u} (c_0 + c_1 e^{iu} + \dots + c_n e^{i n u})}{(\sin u \cos u)^p}$$

$$\Lambda(u+\pi) = e^{-i\pi p} (c_0 - c_1 e^{iu} + \dots + c_n e^{i n u})$$

odd coefficients switch sign

two possibilities: $\Lambda(u+\pi) = \Lambda(u)r$

$$r = 1 \quad r = -1$$

$r = 1 \rightarrow$ odd coefficients are all zero

$r = -1 \rightarrow$ even coefficients are all zero

\Downarrow

$$\Lambda(u) = \frac{g}{(\sin u \cos u)^p} \prod_{j=1}^l \sin(u - u_j)$$

$$r = 1 \quad l = 2p$$

$$r = -1 \quad l = 2p - 1$$

~~write equation for $\Lambda(u+\pi)$~~

$$\Lambda(k, u) \Lambda(L + i\frac{\pi}{2}, -k) = (2i \sinh 2L)^{\frac{1}{2}} + (-2i \sinh 2L)^{\frac{1}{2}}$$

$$e^{2k} = \frac{1 + \sin u}{\cos u}$$

$$k \rightarrow L + i\frac{\pi}{2}$$

$$e^{2k} \rightarrow -e^{2L} = -\frac{(1 + \cos u)}{\sin u}$$

$$u \rightarrow u + \frac{\pi}{2} + \frac{(1 + \sin u \cos \frac{\pi}{2} + \cos u \sin \frac{\pi}{2})}{\cos u \cos \frac{\pi}{2} \sin u \sin \frac{\pi}{2}}$$

can show for all four parametrizations

(16)

$$\text{that } k \rightarrow L + \frac{i\pi}{2} \quad L \rightarrow -k$$

is equivalent to shifting u by $\frac{\pi}{2}$

\Downarrow

\Downarrow

$$\boxed{\lambda(u) \lambda(u + \frac{i\pi}{2}) = (2i \sinh 2L)^n + (-2i \sinh 2L)^n \checkmark}$$

$$\frac{g^2 \prod_{j=1}^l \sin(u - u_j) \cos(u - u_j)}{(\sin u \cos u)^{2p}} = (2i \cot u)^{2p} + (-2i \tan u)^{2p}$$

$$g^2 \prod_{j=1}^l \sin(u - u_j) \cos(u - u_j) = (2i)^{2p} \cos^{4p} u + (2i)^{2p} \sin^{4p} u$$

$$= 2^{2p} [\cos^{4p} u + \sin^{4p} u]$$

rewrite using $z = e^{2iu} \quad z_j = e^{2iu_j}$

$$g^2 \prod_{j=1}^l \left(\frac{(z_j^{1/2} - z_j^{-1/2})}{(z_j^{1/2} + z_j^{-1/2})} \right) = \frac{2^{2p}}{2^{4p}} \left[\left(\frac{z_j^{1/2} + z_j^{-1/2}}{z_j^{1/2} - z_j^{-1/2}} \right)^{4p} + \left(\frac{z_j^{1/2} - z_j^{-1/2}}{z_j^{1/2} + z_j^{-1/2}} \right)^{4p} \right]$$

$$\frac{g^2}{i^p u^l} \prod_{j=1}^l \left(\frac{z_j - z_j^{-1}}{z_j} \right) = \frac{1}{2^{2p}} \left[(z_j + 1)^{4p} + (z_j - 1)^{4p} \right] \frac{1}{z_j^{2p}}$$

$$\boxed{\frac{g^2}{i^p u^l} \prod_{j=1}^l \left(\frac{z_j^2 - z_j^{-2}}{z_j} \right) = \frac{1}{2^{2p}} z^{l-2p} \left[(z+1)^{4p} + (z-1)^{4p} \right]}$$

z_j 's needed are the zeros of the right-hand side (17)

$$(z+1)^{4p} + r(z-1)^{4p} = 0$$

$$z_j = \pm i \tan \frac{\theta_j}{2}$$

$$\theta_j = \pi(j - \frac{1}{2})/2p \quad \text{for } r=1$$

$$\theta_j = \frac{\pi j}{2p} \quad r=-1$$

check!

$$z_j = \frac{e^{\frac{i\pi(j-1/2)}{4p}} - e^{-\frac{i\pi(j-1/2)}{4p}}}{e^{\frac{i\pi(j-1/2)}{4p}} + e^{-\frac{i\pi(j-1/2)}{4p}}}$$

$$z_j + 1 = \frac{2 e^{\frac{i\pi(j-1/2)}{4p}}}{(e^{\frac{i\pi(j-1/2)}{4p}} + e^{-\frac{i\pi(j-1/2)}{4p}})}$$

$$z_j - 1 = \frac{-2 e^{-\frac{i\pi(j-1/2)}{4p}}}{e^{\frac{i\pi(j-1/2)}{4p}} + e^{-\frac{i\pi(j-1/2)}{4p}}}$$

$$\begin{aligned} \text{numerator} &\Rightarrow \frac{e^{\frac{i\pi(j-1/2)}{4p}} - e^{-\frac{i\pi(j-1/2)}{4p}}}{e^{\frac{i\pi(j-1/2)}{4p}} + e^{-\frac{i\pi(j-1/2)}{4p}}} \\ &= \frac{e^{-i\pi/2} - e^{i\pi/2}}{e^{\frac{i\pi(j-1/2)}{4p}} + e^{-\frac{i\pi(j-1/2)}{4p}}} = 0 \end{aligned}$$



can rewrite as

(18)

$$\phi_j = \frac{1}{2} \ln \tan(\theta_j/2)$$

$$\tan(\theta_j/2) = e^{2\phi_j}$$

$$z_j = \pm i \tan(\theta_j/2)$$

$$= i e^{2\phi_j}$$

$$= e^{2\phi_j \pm \frac{i\pi}{2}} = e^{2i\phi_j}$$

$$u_j = \phi_j \pm \frac{i\pi}{4}$$

$\Rightarrow j=1, \dots, l \Rightarrow$ it seems that there are 2^l possibilities

since $\Rightarrow \pm$ sign can be chosen 2^l ways

\Rightarrow turns out that not all are valid choices

to show this, suppose: $u \rightarrow i\infty$

$$u \rightarrow i\infty \quad e^{2K} = \frac{1 + \frac{\sinh u}{\cosh u}}{\frac{e^{-u} + e^{-\infty}}{2}} \rightarrow \frac{1 + \frac{e^{-\infty} - e^{\infty}}{2i}}{\frac{e^{-\infty} + e^{-\infty}}{2}} \Rightarrow +i$$

$$u \rightarrow -i\infty \quad e^{2K} = \frac{1 + \frac{e^{\infty} - e^{-\infty}}{2i}}{\frac{e^{\infty} + e^{-\infty}}{2}} \Rightarrow -i$$

\Rightarrow we know that negating sign of e^{2K}
or e^{2K} do not change 1

$$\left(\begin{array}{l} K \rightarrow K + \frac{i\pi}{2} \\ L \rightarrow L + \frac{i\pi}{2} \end{array} \Rightarrow \text{does not change 1} \right)$$

$$\Lambda(i\omega) = \Lambda(-i\omega)$$

$$i(\omega) - i\omega_j$$

(19)

$$\Lambda(i\omega) = \mathcal{B} \left(\frac{(e^{-i\omega} - e^{-\omega})(e^{-\omega} + e^{\omega})}{u_i} \right)^{-P}$$

$$\prod_{j=1}^l \frac{e^{-\omega - i\omega_j} - e^{\omega + i\omega_j}}{2i}$$

$$= \mathcal{B} \left(-\frac{e^{i\omega}}{u_i} \right)^{-P} \prod_{j=1}^l (-1) \frac{e^{\omega + i\omega_j}}{2i}$$

$$\Lambda(-i\omega) = \mathcal{B} \left(\frac{(e^{\omega} - e^{-\omega})(e^{\omega} + e^{-\omega})}{u_i} \right)^{-P} \prod_{j=1}^l \frac{e^{\omega - i\omega_j} - e^{-\omega + i\omega_j}}{2i}$$

$$= \mathcal{B} \left(\frac{e^{i\omega}}{u_i} \right)^{-P} \prod_{j=1}^l \frac{e^{\omega - i\omega_j}}{2i}$$

$$e^{i\pi P} e^{i \frac{\pi}{2} \omega_j + i\pi l P + 2\pi i} = e^{-i \frac{\pi}{2} \omega_j}$$

$$\frac{\frac{\pi}{2} \omega_j}{\pi} = \frac{1}{2} + \frac{P}{2}$$

$$\Lambda(u) = \mathcal{B}(\sin u \cosh u)^{-P} \prod_{j=1}^l \sin(u + i\phi_j + \frac{1}{2} \frac{\pi}{2})$$