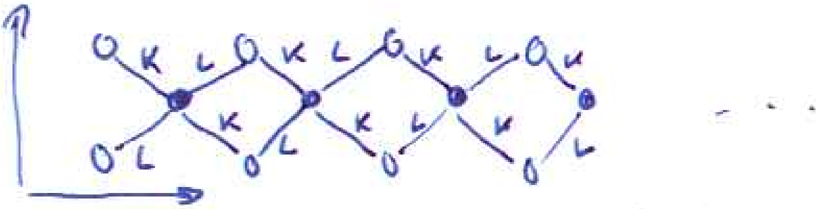
Square Lattice Ising Model (Baxter solution)

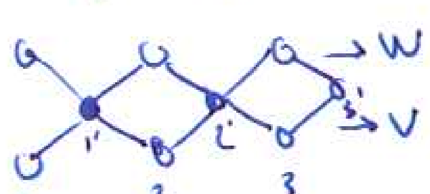
- transfer matrices V, W

- square lattice Ising, zero-field
  - draw lattice diagonally, label alternate rows differently



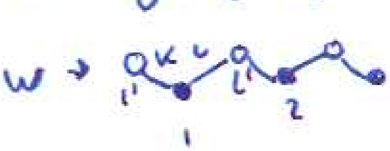
- periodic boundary conditions in both directions
  ↳ toroidal boundary conditions
- let m be the number of units in the vertical direction
- let n be the number of units in the horizontal direction
- φ_r $\quad r = 1, \dots, m$ ⇒ denotes all spins in a row
  2^n possible values ⇒ transfer matrix will be $2^n \times 2^n$

$$Z = \sum_{\varphi_1} \dots \sum_{\varphi_m} V_{\varphi_1 \varphi_2} W_{\varphi_2 \varphi_3} V_{\varphi_3 \varphi_4} W_{\varphi_4 \varphi_5} \dots$$



$$V_{\varphi \varphi'} = \prod_{j=1}^{n} \exp\left[L \sigma_j \sigma_j' + K \sigma_j' \sigma_{j+1} \right]$$

$$W_{\varphi \varphi'} = \prod_{j=1}^{n} \exp\left[K \sigma_j \sigma_j' + L \sigma_j \sigma_{j+1}' \right]$$



$$\Rightarrow Z_N = \mathrm{Tr}\left[(VW)^{m/2} \right]$$

$$= \Lambda_1^m + \Lambda_2^m + \dots \Lambda_{2^n}^m$$

$$\longrightarrow \Lambda_{max}^m$$

$\Lambda_1^2, \Lambda_2^2, \dots$ – eigenvalues of VW

Λ_{max} ~ largest eigenvalue

Significant Properties of V and W

V, W – functions of K, L: $V = V(K, L)$
$W = W(K, L)$

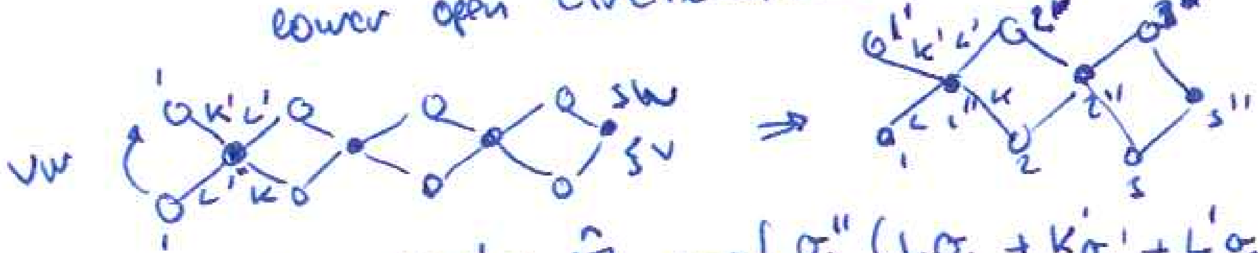
- matrix product $V(K,L)\,W(K,L) \Rightarrow$ defines partition function

- consider

$$V(K,L)\,W(K',L')$$

K, L, K', L' – can be any complex numbers

VW' – transfer matrix which takes you from lower open circle sites to upper one



can write $VW' = \prod_{j=1}^{n} \exp[\sigma_j''(L\sigma_j + K'\sigma_j' + L'\sigma_{j+1}' + K\sigma_{j+1})]$

$$VW' = \prod_{j=1}^{n} X(\sigma_j, \sigma_{j+1}; \sigma_j', \sigma_{j+1}')$$

$$X(a,b,c,d) = \sum_{s=\pm 1} \exp[s(La + Kb + K'c + L'd)]$$

- ask following question:

under what conditions is
$$V(K,L)\,W(K',L') = V(K',L')\,W(K,L)\ ?$$

- generalized commutation relation (as will be shown later) ③
- property will be used to obtain free energy

- to show it:
  1.) if we replace $X(a,b;c,d)$
  with $\to e^{\mu ac} X(a,b,c,d) e^{-\mu bd}$

  VW' does not change

$$\to X(\sigma_j \sigma_{j+1}; \sigma_j' \sigma_{j+1}') X(\sigma_{j+1} \sigma_{j+2}; \sigma_{j+1}' \sigma_{j+2}') \ldots$$

$$e^{\mu \sigma_j \sigma_j'} X(\sigma_j \sigma_{j+1}; \sigma_j' \sigma_{j+1}') \underbrace{e^{-\mu \sigma_{j+1} \sigma_{j+1}'} e^{\mu \sigma_{j+1} \sigma_{j+1}'}}_{\text{cancel}} X(\sigma_{j+1}, \sigma_{j+2}; \sigma_{j+1}' \sigma_{j+2}') \times e^{-\mu \sigma_{j+2} \sigma_{j+2}'}$$

- edge terms cancel due to periodic boundary conditions

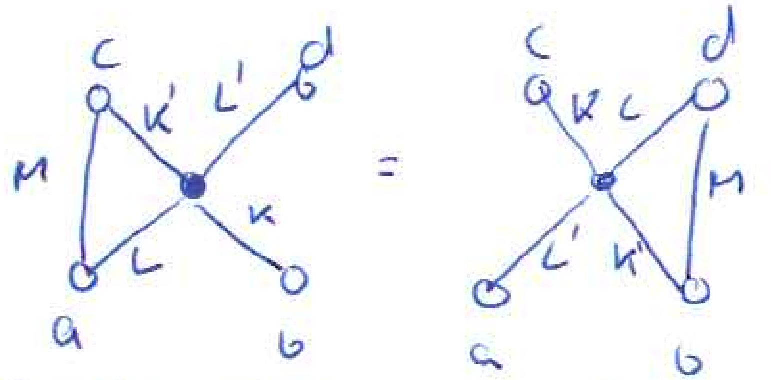
$VW' = V'W$ will be true if there exists a number μ

such that

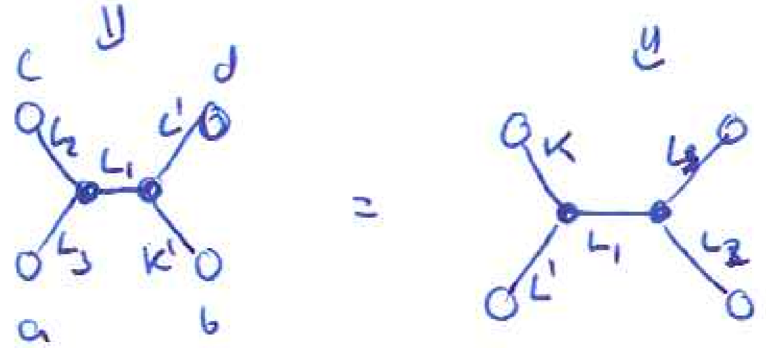
$$e^{\mu ac} X(a,b;c,d) = X'(a,b;c,d) e^{\mu bd}$$

$$X'(a,b,c,d) = \sum_{s=\pm 1} \exp[s(L'a + K'b + Kc + Ld)]$$

(X with $K \leftrightarrow K'$ and $L \leftrightarrow L'$ exchanged) graphically:



apply star-triangle relation → M-K'-L bonds make a triangle!



⇒ $\cancel{L_2} = L'$ $\quad L_2 = K$ ⇒ true if $\boxed{\sinh 2K \sinh 2L = \sinh 2K' \sinh 2L'}$

$L_3 = L'$

$VW' = V'W$ ~~for~~ if $\sinh 2K \sinh 2L = \sinh 2K' \sinh 2L'$

<u>Inversion</u>

- ~~can~~ consider VW'
  - can we choose $K' L'$ so that VW' is diagonal, or at least 'nearly diagonal'?

consider $X(a,b;c,d)$ ⑤

if there exists a set of K', L' such that

$X(a,b;c,d)=0$ if $a \neq b$ (or if $c \neq d$)

→ question answered

→ there exists no such $K' L'$ in general

→ One can satisfy a weaker condition

if $a \neq c$ and $b = d \Rightarrow X = 0$

for some K', L'

$X(a,b,c,d) = 2\cosh(La + Kb + K'c + L'd)$

$a \neq c$, $b = d$ $\Rightarrow$ $2\cosh(L - K' + K + L') = 0$

$2\cosh(L - K' - K - L') = 0$

$\cosh x = \frac{e^x + e^{-x}}{2} = 0$ if $x = -\frac{i\pi}{2}$

$\rightarrow \frac{e^{-\frac{i\pi}{2}} + e^{+\frac{i\pi}{2}}}{2} = \frac{i - i}{2} = 0$

$L - K' + K + L' = -\frac{i\pi}{2}$

$L - K' - K - L' = -\frac{i\pi}{2}$

$L - K' = -\frac{i\pi}{2}$ $\Rightarrow$ $\boxed{K' = L + \frac{i\pi}{2}}$

$\boxed{L' = -K}$

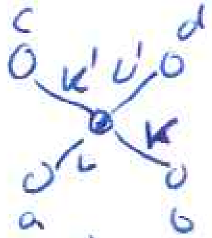
$X(a,b;c,d) = ?$

$a \neq c$ and $b = d$ can not

simultaneously

either $\underbrace{a \neq c \text{ and } b \neq d}$ or $\underbrace{a = c \text{ and } b = d}$

if $a = c$ and $b = d$



$$X(a,b;c,d) = X(a,b;a,b)$$
$$= 2\cosh(La + K'c + Kb + L'd)$$
$$= 2\cosh\left(La + La + \tfrac{i\pi}{2}a\right)$$
$$= 2\cosh\left(2La + \tfrac{i\pi}{2}a\right)$$
$$= e^{2La + \frac{i\pi}{2}a} + e^{-2La - \frac{i\pi}{2}a}$$
$$= e^{\frac{i\pi}{2}a}\left(e^{2La} + e^{-2La - i\pi a}\right) = 2e^{\frac{i\pi}{2}a}\sinh 2La$$
$$= 2(i\,a)\sinh 2La = \underline{2i\sinh 2L}$$

if $a \neq c$ and $b \neq d$

$\rightarrow c = -a \quad d = -b$

$$X(a,b;-a,-b)$$
$$= 2\cosh\left(La - La - \tfrac{i\pi}{2}a + Kb + Kb\right)$$
$$= 2\cosh\left(2Kb - \tfrac{i\pi}{2}a\right)$$
$$= e^{2Kb - \frac{i\pi}{2}a} + e^{-2Kb + \frac{i\pi}{2}a}$$
$$= 2e^{-\frac{i\pi}{2}a}\sinh 2Kb$$
$$= 2(-i)\,ab\sinh 2K = ab\,(-2i\sinh 2K)$$

we can write:

$$V(K,L)\,W\left(L + \tfrac{i\pi}{2}, -K\right)$$
$$= (2i\sinh 2L)^n\,\bar{\bar{I}} + (-2i\sinh 2K)^n\,\bar{\bar{R}}$$

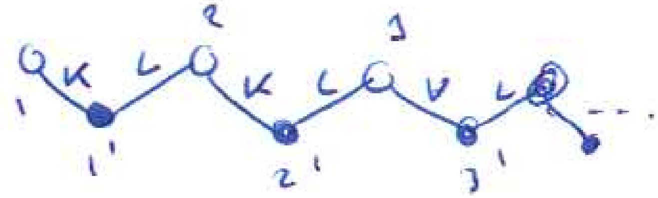
$\bar{\bar{R}} = \begin{pmatrix}0&1\\1&0\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\Rightarrow$ note that $\bar{\bar{R}}^2 = \bar{\bar{I}}$

Symmetry relations

- simple symmetry properties of transfer matrix

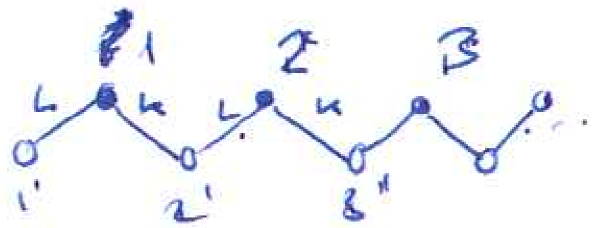
$$W(K,L) = V^T(L,K)$$

- check $W(K,L) \dots\to$



$$W(K,L) = \prod_{j=1}^{n} \exp[K\sigma_j\sigma_j' + L\sigma_{j+1}\sigma_j']$$

$V(K,L) \dots$



$$V(K,L) = \prod_{j=1}^{n} \exp(L\sigma_j'\sigma_j + K\sigma_{j+1}'\sigma_j)$$

$$V(K,L) \to V(L,K) = \prod_{j=1}^{n} \exp(K\sigma_j'\sigma_j + L\sigma_{j+1}'\sigma_j)$$

transpose by switching $\sigma_j \leftrightarrow \sigma_j'$

$$\Downarrow$$

$$W(K,L) = V^T(L,K)$$

- let r be the number of unlike pairs
- let s be the number of like pairs

$$V(K,L) = \exp(nL - 2Ls + nK - 2Kr)$$

assume n is even

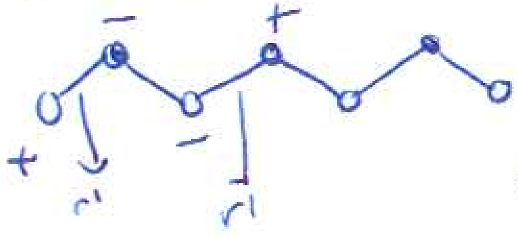
$$\to V(K,L) = \exp[2L(p-s) + 2K(p-r)]$$

$$= \exp[2Ls' - 2Kr']$$

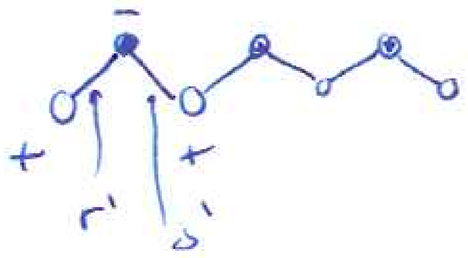
$s' = p - s$

$r' = p - r$

r', s' - either both even or both odd



$\Rightarrow$ $\exp[-2Kr' - 2Ls']$ does not change if $\exp 2K \rightarrow -\exp 2K$ and $\exp 2L \rightarrow -\exp 2L$



$$V(K \pm \tfrac{\pi i}{2}, L \pm \tfrac{\pi i}{2}) = V(K, L)$$

Commutation relations for transfer matrices

define shift operator (total spin shift)

$$C = \delta(\sigma_1, \sigma_2') \, \delta(\sigma_2, \sigma_2') \ldots \delta(\sigma_n, \sigma_1')$$

$A \rightarrow C^{-1} A C \Rightarrow$ replaces spin labels

$\sigma_1 \rightarrow \sigma_2$ plus $\sigma_1' \rightarrow \sigma_2'$
$\sigma_2 \rightarrow \sigma_3$ $\quad \sigma_2' \rightarrow \sigma_3'$
$\vdots$ $\quad \vdots$

C can not be written as

$$(\ )_1 \otimes \ldots \otimes (\ )_n$$

since it is not a local operator

to write it as a matrix let's use 3 spins as an example:

states:

①	②	③	④
1 1 1	1 1 -1	1 -1 1	-1 1 1
⑤	⑥	⑦	⑧
-1 -1 1	-1 1 -1	1 -1 -1	-1 -1 -1

C matrix elements

$$
\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
7 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

does not separate into direct products of single-site operators

due to periodic boundary conditions

$$V(K,L) = C^{-1}\, V(K,L)\, C$$
$$W(K,L) = C^{-1}\, W(K,L)\, C$$

$$\Rightarrow [V, C] = [W, C] = 0 \quad \Big\}\ \text{for any } K, L$$

$$W = VC$$

$$V = \exp\Big[\sum_{j=1}^{n} L\sigma_j\sigma_j' + K\sigma_j'\sigma_{j+1}'\Big]$$

$$VC = \exp\Big[\sum_{j=1}^{n} L\sigma_j\sigma_{j+1}' + K\sigma_{j+1}'\sigma_{j+1}'\Big] = W(K,L)$$

so $\Rightarrow$ if $\sinh 2K \sinh 2L = \sinh 2K' \sinh 2L'$

$$\Rightarrow\ VV' = V'V$$

if $\sinh 2K \sinh 2L = \sinh 2K' \sinh 2L'$

$$\Rightarrow \quad V(K,L)W(K'L') = V(K',L')W(K,L)$$

$$V(K,L)V(K',L')C = V(K',L')V(K,L)C$$

$$= \quad VV' = V'V$$

$\Rightarrow$ commutation relation for transfer matrices

$$V(K,L)W(L+\tfrac{i\pi}{2},-K) = V(K,L)V(L+\tfrac{i\pi}{2},-K)C$$
$$= (2i\sinh 2L)^n \bar{\bar{I}} + (-2i\sinh 2K)^n \bar{\bar{R}}$$

$$\bar{\bar{R}}^{-1} V \bar{\bar{R}} = V$$
$$\bar{\bar{R}}^{-1} W \bar{\bar{R}} = W$$

Functional relations for the eigenvalues

let $k = (\sinh 2K \sinh 2L)^{-1}$

k – some constant $\Rightarrow$ can generate an infinite set of commuting transfer matrices

$$- \cancel{\sinh(L+\tfrac{i\pi}{2})\sinh K}$$

$$\cancel{\frac{e^{L+\frac{i\pi}{2}} - e^{-L-\frac{i\pi}{2}}}{2}} = \cancel{e^{\frac{i\pi}{2}}\cosh L}$$

$$\sinh(2L+i\pi) * \sinh(-2K) = -\frac{e^{2L+i\pi} - e^{-2L-i\pi}}{2}\sinh 2K$$

$$= (-1)^2 \ +\frac{e^{2L}-e^{-2L}}{2}\sinh 2K = \sinh 2K \sinh 2L$$

$$L \to -K \qquad K \to L + \frac{i\pi}{2}$$

are also part of set

$V(K,L)$, C, R all commute

$$\Rightarrow \quad V(K,L)\,x(k) = v(K,L)\,x(k)$$
$$C\,x(k) = c\,x(k)$$
$$R\,x(k) = r\,x(k)$$

$C^n = R^2 = 1 \Rightarrow$ eigenvalues are unimodular

$$c^n = r^2 = 1$$

$$v(K,L)\,v(L+\tfrac{i\pi}{2}, -K) = (2i\sinh 2L)^n + (-2i\sinh 2K)^n r$$

Λ_j^2 – are eigenvalues of $V(K,L)W(K,L)$

$$\Lambda^2(K,L) = v^2(K,L)\,c$$
$$\Lambda(K,L) = v(K,L)\,c^{1/2}$$

c – constant

$$\Rightarrow \boxed{\begin{aligned}&\Lambda(K,L)\,\Lambda(L+\tfrac{i\pi}{2}, -K)\\ &\quad = (2i\sinh 2L)^n + (-2i\sinh 2K)^n r\end{aligned}}$$

equation for eigenvalues