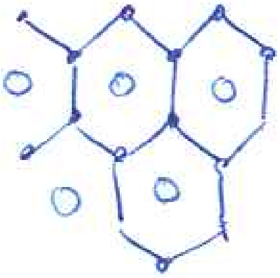
Honeycomb - Triangular Duality

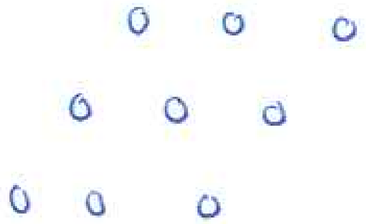
honeycomb lattice: • - lattice sites

o - dual lattice sites

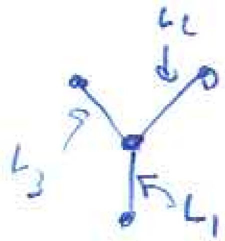


⇓

dual lattice is triangular

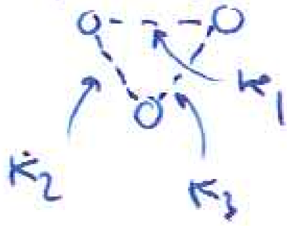


- honeycomb lattice w/ N sites:
  - edges grouped into three classes



$$Q_N^H[L] = \sum_{\sigma_1}\dots\sum_{\sigma_N} \left[\prod^{(1)}_{\langle i,j\rangle} e^{+L_1\sigma_i\sigma_j}\right]\left[\prod^{(2)}_{\langle i,j\rangle} e^{+L_2\sigma_i\sigma_j}\right]\left[\prod^{(3)}_{\langle i,j\rangle} e^{+L_3\sigma_i\sigma_j}\right]$$

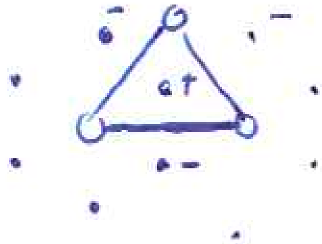
- triangular lattice w/ N sites:
  - edges grouped also into three classes



$$\Rightarrow Q_N^T[K] = \sum_{\sigma_1}\dots\sum_{\sigma_N} \cancel{e^{+K_1\sigma_i\sigma_j - K_2\sigma_i\sigma_j - K_3}}$$

$$\left[\prod^{(1)}_{\langle i,j\rangle} e^{+K\sigma_i\sigma_j}\right]\left[\prod^{(2)}_{\langle i,j\rangle} e^{+K_2\sigma_i\sigma_j}\right]\left[\prod^{(2)}_{\langle i,j\rangle} e^{K_3\sigma_i\sigma_j}\right]$$

apply duality:



- polygons are over the triangular lattice

$$Q_N^H[L] = \underbrace{2}_{d} e^{(L_1+L_2+L_3)N} \sum_P e^{-2L_1 r_1 - 2L_2 r_2 - 2L_3 r_3}$$

forget the two! → no effect on free energy in thermodynamic limit

high-T procedure applied to triangular lattice

$$\Rightarrow \quad Q_N^T[K] = (2\cosh K_1 \cosh K_2 \cosh K_3)^N \sum_P v_1^{r_1} v_2^{r_2} v_3^{r_3}$$

$$v_i = \tanh K_i$$

- can equate two partition functions:

$$\tanh K_i = e^{-2L_i}$$

$$e^{L_i} = \frac{1}{\sqrt{\tanh K_i}}$$

$$Q_N^H[L] = A^N \, Q_N^T[K]$$

$$A = \frac{e^{L_1+L_2+L_3}}{2\cosh K_1 \cosh K_2 \cosh K_3} = \frac{1}{2\sqrt{\prod_{i=1} \sinh K_i \cosh K_i}}$$

$$= \frac{\sqrt{2}}{\sqrt{\sinh 2K_1 \sinh 2K_2 \sinh 2K_3}}$$

Star-Triangle Relation

can map honeycomb lattice onto triangular lattice as follows

$$Q_N^H[L] = \sum_{\sigma_1}\cdots\sum_{\sigma_N}\Big[\prod_{\langle ij\rangle} e^{L_1\sigma_i\sigma_j}\Big]\Big[\prod_{\langle ij\rangle} e^{L_2\sigma_i\sigma_j}\Big]\Big[\prod_{\langle ij\rangle} e^{L_3\sigma_i\sigma_j}\Big]$$

↓ product over all vertical bonds

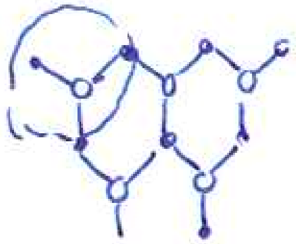
↓ product over all \ bonds

↓ product over all / bonds

- can also be rewritten in terms of "stars"

star →

honeycomb lattice is a product of such stars



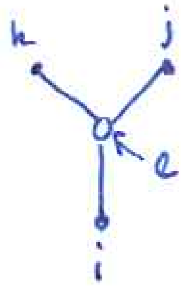
→ honeycomb lattice is also bipartite

o – A sublattice

• – B sublattice

- bi-partite: any given site on A sublattice only has ~~sites on B~~ neighbors on B sublattice

$$Q_N^H[L] = \sum_{\sigma_1}\cdots\sum_{\sigma_N}\Big[\prod_{\langle i,j,k\rangle}\Big] e^{L_1\sigma_e\sigma_i + L_2\sigma_e\sigma_j + L_3\sigma_e\sigma_k}$$



e → always on A sublattice

(i, j, k) → always on B sublattice

$\Rightarrow$ can sum over l-sites (A sublattice traced out) 4

$$Q_N^H[L] = \sum_{\sigma_1}^{B} \dots \sum_{\sigma_N}^{D} \prod_{\langle ijk \rangle} 2\cosh(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k)$$

remaining lattice is triangular, but it does not look Ising-like: can we map it to an Ising model?

$$Q_{\frac{N}{2}}^T[K] = \sum_{\sigma_1} \dots \sum_{\sigma_N} \prod_{\langle ijk \rangle} e^{K_1\sigma_j\sigma_k + K_2\sigma_i\sigma_k + K_3\sigma_i\sigma_j}$$

$$2\cosh(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k) = R\, e^{K_1\sigma_j\sigma_k + K_2\sigma_i\sigma_k + K_3\sigma_i\sigma_j}$$

has to hold for all configurations

$\Rightarrow$ both sides are invariant if $\sigma_i \to -\sigma_i$
$\sigma_j \to -\sigma_j$
$\sigma_k \to -\sigma_k$

need only consider configurations

σ_i	σ_j	σ_k
1	1	1
1	1	-1
1	-1	1
-1	1	1

resulting equations:

$$2\cosh(L_1+L_2+L_3) = Re^{K_1+K_2+K_3}$$
$$2\cosh(L_1+L_2-L_3) = Re^{-K_1-K_2+K_3}$$
$$2\cosh(L_1-L_2+L_3) = Re^{-K_1+K_2-K_3}$$
$$2\cosh(-L_1+L_2+L_3) = Re^{K_1-K_2-K_3}$$

multiply 1st one with 4th one / divide by 2nd one and 3rd one

$$\rightarrow \quad \frac{CC_1}{C_2C_3} = e^{4K_1} \qquad C = \cosh(L_1+L_2+L_3)$$
$$C_i = \cosh(-L_i+L_j+L_k)$$

$$\frac{CC_1 - C_2C_3}{C_2C_3} = e^{4K_1} - 1$$

$$\frac{e^{-2K_1}}{2}\left[\frac{CC_1 - C_2C_3}{C_2C_3}\right] = \sinh 2K_1$$

$$CC_1 = \cosh(L_1+L_2+L_3)\cosh(-L_1+L_2+L_3)$$
$$= \frac{1}{4}\left[e^{L_1+L_2+L_3} + e^{-L_1-L_2-L_3}\right]\left[e^{-L_1+L_2+L_3} + e^{L_1-L_2-L_3}\right]$$
$$= \frac{1}{4}\left[e^{2L_2+2L_3} + e^{2L_1} + e^{-2L_1} + e^{-2L_2-2L_3}\right]$$
$$= \frac{1}{2}\left[\cosh(2L_2+2L_3) + \cosh 2L_1\right]$$

$$C_2C_3 = \frac{1}{4}\left[e^{L_1-L_2+L_3} + e^{-L_1+L_2-L_3}\right]\left[e^{L_1+L_2-L_3} + e^{-L_1-L_2+L_3}\right]$$
$$= \frac{1}{4}\left[e^{2L_1} + e^{-2L_2+2L_3} + e^{2L_2-2L_3} + e^{-2L_1}\right] = \frac{1}{2}\left[\cosh 2L_1 + \cosh(2L_2-2L_3)\right]$$

⑥

$$\frac{cc_1 - c_2c_3}{c_2c_3} = \frac{\frac{1}{2}\left[\cosh(2L_2+2L_3) - \cosh(2L_2-2L_3)\right]}{c_2c_3}$$

$$= \frac{\left[\sinh 2L_2 \sinh 2L_3\right]}{c_2 c_3}$$

$$\sinh 2K_1 \sinh 2L_1 = e^{-2K_1} \frac{\sinh 2L_2 \sinh 2L_3}{2c_2c_3} \sinh 2L_1$$

$$\sqrt{\frac{cc_1}{c_2c_3}} = e^{2K_1} \to e$$

$$\sinh 2K_1 \sinh 2L_1 = \underbrace{\frac{\sinh 2L_1 \sinh 2L_2 \sinh 2L_3}{2\sqrt{cc_1c_2c_3}}}_{\text{symmetric in permutations of } L_1, L_2, L_3}$$

$$\Downarrow$$

$$\sinh 2K_i \sinh 2L_i = \frac{\sinh 2L_1 \sinh 2L_2 \sinh 2L_3}{2\sqrt{cc_1c_2c_3}} = \frac{1}{k}$$

for $i = 1, 2, 3$

to determine R: multiply all four equations

$$\Rightarrow \quad 16\, cc_1c_2c_3 = R^4$$

$$R^2 = 4\sqrt{cc_1c_2c_3} = 2k \sinh 2L_1 \sinh 2L_2 \sinh 2L_3$$

$$= \frac{2}{k^2 \sinh 2K_1 \sinh 2K_2 \sinh 2K_3}$$

relation between partition functions:

$$\boxed{Q^H_N[L] = Q^T_{\frac{N}{2}}[K]\, R^{N/2}}$$

* have expressed K's in terms of L's

Operator form

- 2D lattice problems: useful to consider a row of spins $\sigma_1 \cdots \sigma_N$
  - transfer matrix: $2^N \times 2^N$ matrix, connects rows among each other
- examples of $2^N \times 2^N$ operators (matrices)

$$(S_i)_{\vec{\sigma}\vec{\sigma}'} = \sigma_i\, \delta(\sigma_1, \sigma_1') \dots \delta(\sigma_N, \sigma_N')$$ [Baxter notation]

$$(C_i)_{\vec{\sigma}\vec{\sigma}'} = \delta(\sigma_1, \sigma_1') \dots \delta(\sigma_i, -\sigma_i') \dots \delta(\sigma_N, \sigma_N')$$ [Baxter notation]

- $S_i \rightarrow$ returns spin on site "i", leaving all else untouched
- $C_i \rightarrow$ "spin-flip" at site "i", leaving all else untouched

- these matrices are both examples of direct products (8)

$A \otimes B \otimes C \otimes \cdots \Rightarrow$ one big matrix

example: $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \otimes \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} A_{11}\bar{\bar{B}} & A_{12}\bar{\bar{B}} \\ A_{21}\bar{\bar{B}} & A_{22}\bar{\bar{B}} \end{bmatrix} =$

$$= \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

$\cancel{2\times 2 \Rightarrow 4}$ $(2\times 2)\otimes(2\times 2) \rightarrow 4\times 4$

---

suppose we have four $N\times N$ matrices A, B, C, D

$$[A\otimes B][C\otimes D] = (AC)\otimes(BD)$$

for 2×2

$$\begin{pmatrix} A_{11}\bar{\bar{B}} & A_{12}\bar{\bar{B}} \\ A_{21}\bar{\bar{B}} & A_{22}\bar{\bar{B}} \end{pmatrix}\begin{pmatrix} C_{11}\bar{\bar{D}} & C_{12}\bar{\bar{D}} \\ C_{21}\bar{\bar{D}} & C_{22}\bar{\bar{D}} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{(A_{11}C_{11} + A_{12}C_{21})}\,\bar{\bar{B}}\cdot\bar{\bar{D}} & \\ & \end{pmatrix}$$

$$\Rightarrow \quad \bar{\bar{A}}\bar{\bar{C}} \otimes \bar{\bar{B}}\bar{\bar{D}}$$

change notation!

$$(S_i)_{\vec{\sigma}\vec{\sigma}'} = \underset{1}{\begin{pmatrix}1&0\\0&1\end{pmatrix}} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \cdots \underset{i}{\begin{pmatrix}1&0\\0&-1\end{pmatrix}} \otimes \cdots \underset{N}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}$$

$$(C_i)_{\vec{\sigma}\vec{\sigma}'} = \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \cdots\cdots \underset{i}{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \cdots \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

identities:

$$S_i^2 = \Big[\underset{1}{\begin{pmatrix}1&0\\0&1\end{pmatrix}} \otimes \cdots \underset{i}{\begin{pmatrix}1&0\\0&-1\end{pmatrix}} \otimes \cdots \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes\Big]\Big[\begin{pmatrix}1&0\\0&1\end{pmatrix} \cdots \underset{i}{\begin{pmatrix}1&0\\0&-1\end{pmatrix}} \cdots \begin{pmatrix}1&0\\0&1\end{pmatrix}\Big]$$

- need to multiply only the matrices at the same positions

$$S_i^2 = \Big[\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \cdots \otimes \underset{i}{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}} \otimes \cdots \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\Big]$$

$$= I$$

$$C_i^2 = I \quad \text{since} \quad \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$S_i C_i = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$C_i S_i = \cancel{\begin{pmatrix}1&0\\0&-1\end{pmatrix}(} \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$S_i C_i + C_i S_i = 0$$

can also prove that: $[S_i, C_j] = [S_i, S_j] = [C_i, C_j] = 0$

For Ising model

$P_1(K) \dots P_{N-1}(K),\ Q_1(L), \dots, Q_N(L)$

$$[P_i(K)]_{\vec\sigma\vec\sigma'} = \exp(K\sigma_i\sigma_{i+1})\,\delta(\sigma_1,\sigma_1')\dots\delta(\sigma_N,\sigma_N')$$

(Baxter notation)

$$[Q_i(L)]_{\vec\sigma\vec\sigma'} = \delta(\sigma_1,\sigma_1')\dots\exp(L\sigma_i\sigma_i')\dots\delta(\sigma_N,\sigma_N')$$

(Baxter notation)

$P_i(K)$ – places a bond along the row of spins $\sigma_1 \dots \sigma_N$
(diagonal operator/matrix)

$Q_i(L)$ – places a bond perpendicular to row $\sigma_1 \dots \sigma_N$
next to σ_i
(not diagonal operator/matrix)

tensor product notation

$$\sigma_i\sigma_{i+1} \to \underline{(S_i)_{\vec\sigma\vec\sigma'}(S_{i+1})_{\vec\sigma'\vec\sigma''}} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\dots\otimes\underset{i}{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\otimes\underset{i+1}{\begin{pmatrix}-1&0\\0&1\end{pmatrix}}\otimes\dots\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\dots\otimes\underset{i\otimes i+1}{\begin{pmatrix}1&&&0\\&-1&&\\&&-1&\\0&&&1\end{pmatrix}}\otimes\dots\otimes\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Since diagonal we can easily exponentiate

$$\to [P_i(K)]_{\vec\sigma\vec\sigma'} = \underset{1}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}\otimes\dots\underset{i-1}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}\otimes\begin{pmatrix}e^K&&&0\\&e^{-K}&&\\&&e^{-K}&\\0&&&e^K\end{pmatrix}\otimes\underset{i+2}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}\dots\underset{N}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}$$

$$[Q_i(L)]_{\vec\sigma\vec\sigma'} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes\dots\otimes\underset{i}{\begin{pmatrix}e^L&e^{-L}\\e^{-L}&e^L\end{pmatrix}}\otimes\dots\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$P_i(K) = \exp(K s_i s_{i+1})$ – easy to show, exponentiation of a diagonal matrix

$$Q_i(L) = \exp(L)\, I + \exp(-L)\, c_i$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad c_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

since $c_i^2 = I$

$$\Rightarrow \exp(L c_i) = \cosh L\, I + \sinh L\, c_i$$

if $\tanh L^* = e^{-2L}$

$$\begin{aligned}
\exp(L^* c_i) &= \cosh L^*\, I + \sinh L^*\, c_i \\
&= \cosh L^* \left(I + \tanh L^*\, c_i \right) \\
&= \cosh L^* \left(I + e^{-2L}\, c_i \right) \\
&= e^{-L} \cosh L^* \left(e^{L} I + e^{-L} c_i \right) \\
&= \sqrt{\sinh L^* \cosh L^*} \left(e^{L} I + e^{-L} c_i \right) \\
&= \sqrt{\frac{\sinh 2L^*}{2}} \left(e^{L} I + e^{-L} c_i \right) \\
&= (2 \sinh 2L)^{-1/2}\, Q_i(L)
\end{aligned}$$

$$\boxed{Q_i(L) = (2 \sinh 2L)^{1/2} \exp(L^* c_i)}$$

define "interlaced" operators

$$U_i(K,L) = \begin{cases} P_j(K) & i = 2j \quad \cancel{(i\text{ even})} \; (i \text{ even}) \\ Q_j(L)(2\sinh 2L)^{-1/2} & i = 2j-1 \quad \cancel{(i\text{ odd})} \; (i \text{ odd}) \end{cases}$$

star-triangle relation $\Rightarrow$ for K_1, K_2, K_3
$L_1\ L_2\ L_3$

$$U_{i+1}(K_1,L_1)\,U_i(L_2,K_2)\,U_{i+1}(K_3,L_3)$$

$$= U_i(K_3,L_3)\,U_{i+1}(K_2,K_2)\,U_i(K_1,L_1)$$

suppose i - odd $\rightarrow$ j - does not change

$$\begin{array}{l} i \rightarrow 2j-1 \\ i+1 \rightarrow 2j \end{array} \Rightarrow$$

L.H.S.: $P_j(K_1)\ Q_j(K_2)\ P_j(K_3)\ [2\sinh 2K_2]^{-1/2}$

can multiply out explicitly

$$P_j(K_1) = \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} e^{K_1} & & & 0 \\ & e^{-K_1} & & \\ & & e^{-K_1} & \\ 0 & & & e^{K_1} \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{array}{cc} j & j+1 \end{array}$$

$$Q_j(K_2) = \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} e^{K_2} & e^{-K_2} \\ e^{-K_2} & e^{K_2} \end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{array}{cc} j & j+1 \end{array}$$

$$P_j(K_3) = \begin{pmatrix}1&0\\0&1\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} e^{K_3} & & & 0 \\ & e^{-K_3} & & \\ & & e^{-K_3} & \\ 0 & & & e^{K_3} \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\begin{pmatrix} e^{K_1} & 0 & & \\ 0 & e^{-K_1} & & \\ & & e^{-K_1} & 0 \\ & & 0 & e^{K_1} \end{pmatrix} \begin{pmatrix} e^{K_2} & e^{-K_2} & & \\ e^{-K_2} & e^{K_2} & & \\ & & e^{K_2} & e^{-K_2} \\ & & e^{-K_2} & e^{K_2} \end{pmatrix} \begin{pmatrix} e^{K_3} & 0 & & \\ 0 & e^{-K_3} & & \\ & & e^{-K_3} & 0 \\ & & 0 & e^{K_3} \end{pmatrix}$$

$$= \begin{pmatrix} e^{K_1} & 0 & & \\ 0 & e^{-K_1} & & \\ & & e^{-K_1} & 0 \\ & & 0 & e^{K_1} \end{pmatrix} \begin{pmatrix} e^{K_2+K_3} & e^{-K_2-K_3} & & \\ e^{-K_2+K_3} & e^{K_2-K_3} & & \\ & & e^{K_2-K_3} & e^{-K_2+K_3} \\ & & e^{-K_2-K_3} & e^{K_2+K_3} \end{pmatrix}$$

$$= \begin{pmatrix} e^{K_1+K_2+K_3} & e^{K_1-K_2-K_3} & & \\ e^{-K_1-K_2+K_3} & e^{-K_1+K_2-K_3} & & \\ & & e^{-K_1+K_2-K_3} & e^{-K_1-K_2+K_3} \\ & & e^{K_1-K_2-K_3} & e^{K_1+K_2+K_3} \end{pmatrix}$$

L.H.S $\nearrow$ times $[2\sinh 2K_2]^{-1/2}$

R.H.S.

$$U_i(K_3, L_3)\, U_{i+1}(L_2 K_2)\, U_i(K_1, L_1)$$

$$= Q_j(L_3)\, P_j(L_2)\, Q_j(L_1) (2\sinh 2L_1)^{-1/2} (2\sinh 2L_3)^{-1/2}$$

$$\begin{pmatrix} e^{L_3} & e^{-L_3} & & \\ e^{-L_3} & e^{L_3} & & \\ & & e^{L_3} & e^{-L_3} \\ & & e^{-L_3} & e^{L_3} \end{pmatrix} \begin{pmatrix} e^{L_2} & 0 & & \\ 0 & e^{-L_2} & & \\ & & e^{-L_2} & 0 \\ & & 0 & e^{L_2} \end{pmatrix} \begin{pmatrix} e^{L_1} & e^{-L_1} & & \\ e^{-L_1} & e^{L_1} & & \\ & & e^{L_1} & e^{-L_1} \\ & & e^{-L_1} & e^{L_1} \end{pmatrix}$$

$$\times (2\sinh 2L_1)^{-1/2} (2\sinh 2L_3)^{-1/2}$$

$$\begin{pmatrix} e^{L_3} & e^{-L_3} & & \\ e^{-L_3} & e^{L_3} & & 0 \\ & & e^{L_3} & e^{-L_3} \\ 0 & & e^{-L_3} & e^{L_3} \end{pmatrix} \begin{pmatrix} e^{L_1+L_2} & e^{-L_1+L_2} & & \\ e^{-L_1-L_2} & e^{-L_2+L_1} & & 0 \\ & & e^{L_1-L_2} & e^{-L_1-L_2} \\ 0 & & e^{L_2-L_1} & e^{L_1+L_2} \end{pmatrix}$$

$$\begin{pmatrix} 2\cosh(L_1+L_2+L_3) & 2\cosh(-L_1+L_2+L_3) & 0 \\ 2\cosh(L_1+L_2-L_3) & 2\cosh(L_2-L_1-L_3) & \\ & 0 & \ddots \end{pmatrix}$$

compare matrices of LHS and RHS

$$e^{K_1+K_2+K_3} = \underbrace{\frac{(2\sinh 2K_2)^{1/2}}{(2\sinh 2L_1\, 2\sinh 2L_3)^{1/2}}}_{1/R}\, 2\cosh(L_1+L_2+L_3)$$

tensor form of the $\star-\Delta$ relation

$$U_i(K,L)\,U_j(K',L') = U_j(K',L')\,U_i(K,L)$$
$$\text{if } |i-j| \geq 2$$

succinct notation: $U_{i+1}\, U_i'\, U_{i+1}'' = U_i''\, U_{i+1}'\, U_i$

$$U_i\, U_j' = U_j'\, U_i \qquad |i-j| \geq 2$$

Significance of ✭-Δ relation

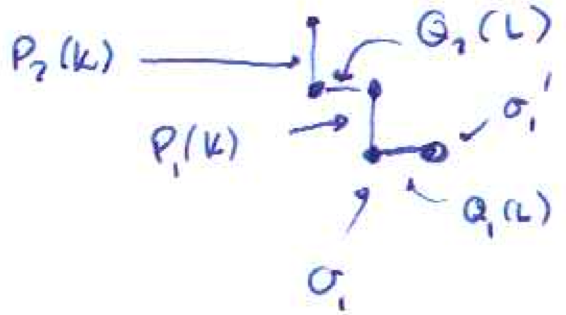
can use it to demonstrate commutation of transfer matrices

- diagonal transfer matrix can be constructed as

$$V(K,L) = U_1(K,L)\, U_2(K,L), \ldots, U_n(K,L)$$

- this is a diagonal transfer matrix → transfer matrix along the diagonal direction of the lattice (not a diagonal matrix!)

$$V(K,L) = Q_1(L)\, P_1(K)\, Q_2(L)\, P_2(K) \ldots$$



⇒ "generates" lattice points along diagonal direction

$$V = U_1\, U_2\, U_3 \ldots U_n$$

$$V' = U_1'\, U_2' \ldots U_n'$$

can show that

$$VV'(U_n'^{-1}\, U_n''\, U_n) = V(U_1\, U_1''\, U_1'^{-1})\, V'V$$

Show

$$U_1 \cdots U_n U_1' \cdots U_n' U_n^{-1} U_n'' U_n$$

$$= U_1 U_1'' U_1^{-1} U_1 \cdots U_n' U_1 \cdots U_n$$

$$U_2 \cdots U_n U_1' \cdots U_{n-1}' U_n'' = U_1'' U_2' \cdots U_n' U_1 \cdots U_{n-1}$$

- commute U_n as far as possible

$$\Rightarrow U_2 \cdots U_{n-2} U_1' \cdots U_{n-2}' \underbrace{U_n U_{n-1}' U_n''}_{\substack{\star\text{-}\Delta \\ U_{n-1}'' U_n' U_{n-1}}} = U_1'' U_2' \cdots U_n' U_1 \cdots U_{n-1}$$

$$\Rightarrow U_2 \cdots U_{n-1} U_1' \cdots U_{n-2}' U_{n-1}'' U_n' U_{n-1} = U_1'' U_2' \cdots U_n' U_1 \cdots U_{n-1}$$

- commute U_{n-1} as far as possible $\Rightarrow$ apply $\star$-Δ

→ if boundary conditions are properly implemented

$\Rightarrow \quad VV' = V'V \qquad$ (diagonal transfer matrices commute)

if $\star$-Δ are satisfied

- turns out to be a vital step in the solution of some models

- can put together triangular - triangular duality

$$Q^H_{2N}(L) = (2s_1 s_2 s_3)^{N/2} Q^T_N(K) = R^N Q^T_N(K^*)$$

$$s_i = \sinh 2L_i$$

$$Q^T_N(K) = \left[\frac{R}{\sqrt{2s_1 s_2 s_3}}\right]^N Q^T_N(K^*)$$

$$= k^{N/2} Q^T_N(K^*)$$

$$k = \frac{2\sqrt{c\, c_1 c_2 c_3}}{\sinh 2L_1 \sinh 2L_2 \sinh 2L_3}$$

$$\boxed{\sinh K_c = \frac{1}{3^{1/2}}}$$ for triangular lattice

$$\boxed{\sinh L_c = 3^{1/2}}$$ for hexagonal lattice