Luttinger Model (following Mahan, Giamarchi)

①

- variation on Tomonaga
- exactly solvable, in some respects identical to Tomonaga
- essential feature of Luttinger model:
  - introduces two types of particles

$$\varepsilon_k = k v_F \qquad \varepsilon_k = -k v_F$$

(right-going and left-going)

- Tomonaga:

- Luttinger:

- also in the Luttinger model occupied states stretch to $-\infty$ (unphysical, but aids in exact solution)

- two kinds of fermions in Luttinger model

$$\rightarrow a_{1k\sigma},\ a_{2k\sigma}$$

- assume that they obey fermionic anti-commutation relations

$$[a_{ik\sigma}, a^{\dagger}_{jk'\sigma'}] = \delta_{ij}\,\delta_{kk'}\,\delta_{\sigma\sigma'}$$

- define densities

$$\rho_i(p) = \sum_{k\sigma} a^\dagger_{ik+p\sigma} a_{ik\sigma}$$

$$\rho_i(-p) = \sum_{k\sigma} a^\dagger_{ik\sigma} a_{ik+p\sigma} = \rho_i^\dagger(p)$$

$$\sigma_i(p) = \sum_{k\sigma} \sigma a^\dagger_{ik+p\sigma} a_{ik\sigma}$$

$$\sigma_i(-p) = \sum_{k\sigma} \sigma a^\dagger_{ik\sigma} a_{ik+p\sigma} = \sigma_i^\dagger(p)$$

- commutation relations:

$$[\rho_1(-p), \rho_1(p')] = \cancel{\sum_{\substack{k\sigma \\ k'\sigma'}} [a^\dagger_{1k+p\sigma} a_{1k\sigma}, a^\dagger_1}$$

$$= \sum_{\substack{k\sigma \\ k'\sigma'}} [a^\dagger_{1k\sigma} a_{1k+p\sigma}, a^\dagger_{1k'+p'\sigma'} a_{1k'\sigma'}]$$

$$= \sum_{\substack{k\sigma \\ k'\sigma'}} (a^\dagger_{1k\sigma} a_{1k+p\sigma} a^\dagger_{1k'+p'\sigma'} a_{1k'\sigma'}$$
$$- a^\dagger_{1k'+p'\sigma'} a_{1k'\sigma'} a^\dagger_{1k\sigma} a_{1k+p\sigma})$$

$$= \sum_{\substack{kk' \\ \sigma\sigma'}} (a^\dagger_{1k\sigma} a_{1k'\sigma'} \delta_{\sigma\sigma'} \delta_{k+p,k'+p'}$$
$$- a^\dagger_{1k'+p'\sigma'} a_{1k+p\sigma} \delta_{\sigma\sigma'} \delta_{kk'})$$

$$= \sum_{k'\sigma} (a^\dagger_{1k'+p'-p\sigma} a_{1k'\sigma} - a^\dagger_{1k'+p'\sigma} a_{1k'+p\sigma})$$

if $p = p'$ $\quad [\rho_1(-p), \rho_1(p')] = \sum_{k\sigma} (n_{k\sigma} - n_{k+p\sigma})$

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normal ordering:

two operators are normal ordered if AB is such that the annihilation operators are to the right, the creation are to the left

$:AB: = AB - \langle 0|AB|0\rangle$

$\langle 0|AB|0\rangle$ - expectation value of AB over the vacuum which in this case stretches down to $-\infty$

- example: $a^+b \to :a^+b: = a^+b - \langle 0|a^+b|0\rangle = a^+b$

  $ba^+ \to :ba^+: = ba^+ - \langle 0|ba^+|0\rangle$

  $= ba^+ - \delta_{ab} = a^+b$

- for the commutator $[\rho_1(-p), \rho_1(p')]$

$$\sum_{ka} :a^+_{1k'+p'-pa} a_{1k'a}: - :a^+_{1k'+p'a} a_{1k'+pa}:$$

$$= \sum_{ka} :a^+_{1k'+p'-pa} a_{1ka}: - :a^+_{1k'+p'a} a_{1k'+pa}:$$

$$+ \sum_{ka} \langle 0|a^+_{k+p'-pa} a_{1ka}|0\rangle - \langle 0|a^+_{1k+pa} a_{1k+pa}|0\rangle$$

$= 0$ if $p \neq p'$

$= \sum_{ka} (n_{ka} - n_{k+pa}) = 2\frac{pL}{2\pi}$ if $p = p'$

$$\Rightarrow \quad [\rho_1(-p), \rho_1(p')] = \delta_{pp'} \frac{pL}{\pi} \qquad [\sigma_1(p), \sigma_2(p')] = 0$$

$$[\rho_2(p), \rho_2(-p')] = \delta_{pp'} \frac{pL}{\pi} \qquad [\sigma_i(p), \rho_j(p')] = 0$$

$$[\rho_1(p), \rho_2(-p')] = 0$$

$$[\sigma_1(-p), \sigma_1(p')] = \delta_{pp'} \frac{pL}{\pi}$$

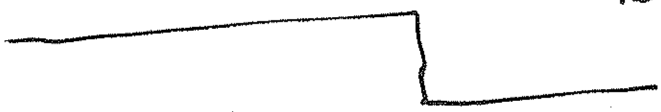
$$[\sigma_2(-p), \sigma_2(p')] = \delta_{pp'} \frac{pL}{\pi}$$

- these commutation relations hold indeed because of the assumption that k stretches down to $-\infty$

- consider $\sum_{k\alpha}(n_{k\alpha} - n_{k+p\,\alpha})$ for a normal Fermi liquid:

$\rightarrow \sum_{k\alpha}(n_{k\alpha} - n_{k+p\,\alpha}) = 0$

however, if we have



$$\sum_{k\alpha}(n_{k\alpha} - n_{k+p\,\alpha}) = \underbrace{2}_{\text{spin}}\,p\,\underbrace{\frac{L}{2\pi}}_{\substack{\text{volume}\\\text{associated}\\\text{with one}\\\text{state}}} = \frac{pL}{\pi}$$

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construct Hamiltonian using densities (which are bosonic operators)

$$H_0 = v_F \sum_{k\alpha} k\,(a^\dagger_{1k\alpha} a_{1k\alpha} - a^\dagger_{2k\alpha} a_{2k\alpha})$$

$$[H_0, \rho_1(p)] = v_F \sum_{\substack{k\alpha\\k'\alpha'}} k\,[a^\dagger_{1k\alpha} a_{1k\alpha}, a^\dagger_{1k'+p\,\alpha'} a_{1k'\alpha'}]$$

$$= v_F \sum_{\substack{k\alpha\\k'\alpha'}} k\,(a^\dagger_{1k\alpha} a_{1k\alpha} a^\dagger_{1k'+p\,\alpha'} a_{1k'\alpha'} - a^\dagger_{1k'+p\,\alpha'} a_{1k'\alpha'} a^\dagger_{1k\alpha} a_{1k\alpha})$$

$$= v_F \sum_{\substack{k\alpha\\k'\alpha'}} k\,(a^\dagger_{1k\alpha} a_{1k'\alpha'} \delta_{\alpha\alpha'} \delta_{k\,k'+p} - a^\dagger_{1k'+p\,\alpha'} a_{1k\alpha} \delta_{\alpha\alpha'} \delta_{kk'})$$

(5)

$$= v_F \sum_{ka} (k+p)\, a^{\dagger}_{1\,k+pa} a_{1ka} - \sum_{ka} k\, a^{\dagger}_{1k+pa} a_{1ka}$$

$= v_F\, p\, \rho_1(p)$ → in this case, unlike for the Tomonaga model, this relation is exact

similarly: One can show that

$$[H_0, \rho_1(p)] = v_F\, p\, \rho_1(p) \qquad [H_0, \sigma_1(p)] = v_F\, p\, \sigma_1(p)$$

$$[H_0, \rho_2(p)] = -v_F\, p\, \rho_2(p) \qquad [H_0, \sigma_2(p)] = -v_F\, p\, \sigma_2(p)$$

if we write:

$$H_0 = \frac{\pi}{L} v_F \sum_{p>0} \left[\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p) + \sigma_1(p)\sigma_1(-p) + \sigma_2(-p)\sigma_2(p)\right]$$

these commutation relations are obeyed

- transform to boson operators

$$\rho_1(-p) = b_{1p} \left(\frac{pL}{\pi}\right)^{1/2} \qquad \rho_1(p) = b^{\dagger}_{1p} \left(\frac{pL}{\pi}\right)^{1/2}$$

$$\rho_2(-p) = b^{\dagger}_{2-p} \left(\frac{pL}{\pi}\right)^{1/2} \qquad \rho_2(p) = b_{2-p} \left(\frac{pL}{\pi}\right)^{1/2}$$

$$\sigma_1(-p) = c_{1p} \left(\frac{pL}{\pi}\right)^{1/2} \qquad \sigma_1(p) = c^{\dagger}_{1p} \left(\frac{pL}{\pi}\right)^{1/2}$$

$$\sigma_2(-p) = c^{\dagger}_{2-p} \left(\frac{pL}{\pi}\right)^{1/2} \qquad \sigma_2(p) = c_{2-p} \left(\frac{pL}{\pi}\right)^{1/2}$$

$$\Rightarrow H_0 = v_F \sum_{p>0} p \left[b^{\dagger}_{1p} b_{1p} + b^{\dagger}_{2-p} b_{2-p} + c^{\dagger}_{1p} c_{1p} + c^{\dagger}_{2-p} c_{2-p} \right]$$

- $\rho_1(p)$ takes particles from state k to $k+p$ where $k<k_F$

$\underline{k+p > k_F}$ $\Rightarrow$ electron-hole pair

represented by bosonic operator b_{1p}^+

- $\rho_2(p)$ takes particles from ~~state~~ k to $k-p$ where

$-k_F < k$. $k-p < k_F \Rightarrow$ electron-hole pair

represented by bosonic operator b_{2p}^+

- interaction terms:

$$V_1 = \frac{1}{2L} \sum_{p>0} V_{1p} [\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)]$$

$$= \frac{1}{2\pi} \sum_{p>0} V_{1p}\, p\, [b_{1p}^+ b_{1p} + b_{2-p}^+ b_{2-p}]$$

$$V_2 = \frac{1}{2L} \sum_{p>0} V_{2p} [\rho_1(p)\rho_2(-p) + \rho_1(-p)\rho_2(p)]$$

$$= \frac{1}{2\pi} \sum_{p>0} V_{2p}\, p\, [b_{1p} b_{2-p} + b_{1p}^+ b_{2-p}^+]$$

Single-particle properties

- in these notes we will show how to calculate

$\underline{\text{static}}$ single-particle properties

- basic question: $\Psi_{1\sigma}(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} a_{k\sigma}$

field operator

- can we represent $\Psi_{1\sigma}(x)$ in terms of bosonic operators?

- consider the commutation relation:

$$[\Psi_{i\alpha}(x), \rho_j(p)] = \sum_{\substack{k \\ k'\alpha'}} \frac{e^{ikx}}{\sqrt{L}} [a_{ik\alpha}, a^{+}_{jk'+p\alpha'} a_{jk'\alpha'}]$$

$$= \sum_{kk'\alpha'} \frac{e^{ikx}}{\sqrt{L}} (a_{ik\alpha} a^{+}_{jk'+p\alpha'} a_{jk'\alpha'}$$

$$- a^{+}_{jk'+p\alpha'} a_{jk'\alpha'} a_{ik\alpha})$$

$$= \sum_{kk'\alpha'} \frac{e^{ikx}}{\sqrt{L}} a_{jk'\alpha'} \delta_{ij} \delta_{\alpha\alpha'} \delta_{kk'+p}$$

$$= \delta_{ij} \sum_{k'} e^{i(k'+p)x} a_{jk'\alpha}$$

$$= \delta_{ij} e^{ipx} \Psi_{i\alpha}(x)$$

similarly: $[\Psi_{i\alpha}(x), \sigma_j(p)] = \delta_{ij} \sigma e^{ipx} \Psi_{i\alpha}(x)$

- for bosonization one needs $\Psi_{i\alpha}(x)$ to be represented in terms of boson-operators (ρ's)

- "represent" here means construct some operator which has the same commutation properties with ρ_i as $\Psi_{j\sigma}$

in other words: $\Psi_{is}(x) = \Gamma_i[x, \rho_i(p)]$

~~$\Rightarrow [\Psi_{j\alpha}(x), \Gamma(x, \rho_i(p))] = \Psi_{j\alpha}(x)$~~

~~$\Psi_{i\alpha}(x) = F_i(x) \exp[J(x)]$~~

where Γ is such that $[\Gamma_i, \rho_j(p)] = e^{ipx} \Gamma_i \delta_{ij}$

→ it is possible to achieve this:

$$\Psi_{1\sigma}(x) = F_1(x)\exp(J(x))$$

$$J(x) = -\frac{\pi}{L}\sum_{p>0}\frac{1}{p}\left[e^{-ipx}\left[\rho_1(+p)+\sigma\sigma_1(+p)\right]\right.$$
$$\left.-e^{ipx}\left[\rho_1(-p)+\sigma\sigma_1(-p)\right]\right]$$

$F_1(x)$ - function of x, c-number ⇒ has to commute with $\rho_1(p)$

- consider commutator: $[\Psi_{1,\sigma}^{(x)}, \rho_1(p)]$

$$= F_1(x)\left[\exp(J(x)), \rho_1(p)\right]$$
$$= F_1(x)\left(\exp(J(x))\rho_1(p) - \rho_1(p)\exp(J(x))\right)$$
$$= F_1(x)\left(\exp(J(x))\rho_1(p)\exp(-J(x)) - \rho_1(p)\right)\exp(J(x))$$
$$= \underbrace{\left(\exp(J(x))\rho_1(p)\exp(-J(x)) - \rho_1(p)\right)}_{(e^J\rho_1e^{-J}-\rho_1)}\Psi_{1\sigma}(x)$$

expand: $\left(1+J+\frac{J^2}{2}\right)\rho_1\left(1-J+\frac{J^2}{2}\right)-\rho_1$

$$= \cancel{\rho_1} + [J,\rho_1] + \frac{J}{2}[J,\rho_1] - [J,\rho_1]\frac{J}{2} - \cancel{\rho_1}$$

if $[J,\rho_1]$ is also a c-number

$$\Rightarrow e^J\rho_1e^{-J} - \rho_1 = [J,\rho_1]$$

(proven here up to second order, but can be shown up to higher orders as well)

- now let's show that $[J, \varrho_i]$ is indeed a number (9)

$$[J(x), \varrho_i(p)] = \frac{\pi}{L} \sum_{p'>0} \frac{e^{ip'x}}{p'} [\varrho_i(-p'), \varrho_i(p)]$$
$$= e^{ipx}$$

$\Rightarrow$ if we represent $\psi_{i\sigma}(x)$ as $F_i(x)\exp(J(x))$ then the commutation relation $[\psi_{i\sigma}(x), \varrho_i(p)] = e^{ipx}\psi_{i\sigma}(x)$

holds

→ <u>note</u>: $F_i(x)\exp(J(x))$ is not a real field operator since it does not destroy a particle. This problem can be solved via particular ways of writing $F_i(x)$ (Klein factors), but here we will content ourselves with the fact that we have found a representation of $\psi_{i\sigma}(x)$ which obeys the commutation rule.

- $\varrho_i(p) + \sigma\sigma_i(p) = \sum_{n\sigma'} (1+\sigma\sigma') a^\dagger_{in+p\sigma'} a_{in\sigma'} = 2\sum_n a^\dagger_{in+p\sigma} a_{in\sigma}$
$= 2\varrho_{i\sigma}(p)$

define: $\rho_{ia}(p) = \sum_k a^\dagger_{i k+p a}\, a_{i k a}$

$$\rho_{ia}(-p) = \sum_k a^\dagger_{ika}\, a_{ik+pa}$$

$$\rho_i(p) = \sum_a \rho_{ia}(p)$$

$$\sigma_i(p) = \sum_\sigma \sigma \rho_{i\sigma}(p)$$

$$J_{is}(x) = -\frac{2\pi}{L} \sum_p \frac{e^{-ipx}}{p} \rho_{is}(p)$$

$$\Psi_{is}(x) = F_i(x) \exp(J_{is}(x))$$

commutation relations

$$[\rho_{1s}(-p), \rho_{1s}(p')] = \delta_{pp'} \left(\frac{pL}{2\pi}\right)$$

$$[\rho_{2s}(p), \rho_{2s}(-p')] = \delta_{pp'} \left(\frac{pL}{2\pi}\right)$$

represent by bosonic operators

$$\rho_{1s}(-p) = b_{1sp} \left(\frac{pL}{2\pi}\right)^{1/2} \qquad \rho_{1s}(p) = b^\dagger_{1sp} \left(\frac{pL}{2\pi}\right)^{1/2}$$

$$\rho_{2s}(-p) = b^\dagger_{2s-p} \left(\frac{pL}{2\pi}\right)^{1/2} \qquad \rho_{2s}(p) = b_{2s-p} \left(\frac{pL}{2\pi}\right)^{1/2}$$

$$H_0 = \frac{2\pi v_F}{L} \sum_{\substack{p>0 \\ a}} [\rho_{1s}(p)\rho_{1s}(-p) + \rho_{2s}(-p)\rho_{2s}(p)]$$

- with slight modifications: one can calculate quantities like $\langle \Psi_{1\sigma}^{\dagger}(x') \Psi_{1\sigma}(x) \rangle$ for an interacting system
- we show this here ~~via~~ the following steps: construct a procedure which allows the calculation of $\langle \Psi_{1\sigma}^{\dagger}(x') \Psi_{1\sigma}(x) \rangle$, for the ~~non~~ interacting case but such that it reproduces the known results for the non-interacting case
- $\langle \Psi_{1S}^{\dagger}(x') \Psi_{1S}(x) \rangle$ for the non-interacting case:

$$\langle \Psi_{1\sigma}^{\dagger}(x') \Psi_{1\sigma}(x) \rangle = \frac{1}{L} \sum_{kk'} e^{i(k'x'-kx)} \langle a_{1k\sigma}^{\dagger} a_{1k'\sigma} \rangle$$

$$= \frac{1}{L} \sum_{kk'} e^{i(k'x'-kx)} n_{1k\sigma} \delta_{kk'}$$

$$n_{1k\sigma} = \Theta(k_F - k)$$

$$\langle \Psi_{1\sigma}^{\dagger}(x') \Psi_{1\sigma}(x) \rangle = \frac{1}{L} \sum_{k} e^{ik(x'-x)} \Theta(k_F - k)$$

$$= \frac{1}{2\pi} \int dk \, e^{ik(x'-x)} \Theta(k_F - k)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{k_F} dk \, e^{ik(x'-x)}$$

integral does not converge, but can add a factor to aid convergence

$$\langle \Psi_{1\sigma}^{\dagger}(x') \Psi_{1\sigma}(x) \rangle = \frac{1}{2\pi} \lim_{\alpha \to 0} \int_{-\infty}^{k_F} dk \, e^{ik[(x-x')-i\alpha]}$$

$$= \lim_{\alpha \to 0} \frac{1}{2\pi} \frac{e^{ik[(x-x')-i\alpha]}}{ik[(x-x')-i\alpha]} \Big|_{-\infty}^{k_F} = \frac{1}{2\pi} \frac{e^{ik_F(x-x')}}{i[(x-x')-i\alpha]}$$

similarly: $\langle \psi_{2s}^{\dagger}(x) \psi_{2s}(x') \rangle = \dfrac{e^{ik_F(x-x')}}{2\pi i\,(x-x'-i\alpha)}$

→ can we reproduce this result using the bosonic representation of $\psi_{i\alpha}(x)$?

with some modifications, due to Luther and Peschel, yes. Need to add a factor to aid convergence

$$\Rightarrow \quad \psi_{1s}(x) = \frac{1}{(2\pi\alpha)^{1/2}} \exp[ik_F x + J_{1s}(\alpha, x)]$$

$$\psi_{2s}(x) = \frac{1}{(2\pi\alpha)^{1/2}} \exp[-ik_F x - J_{2s}(\alpha, x)]$$

$$J_{i\alpha}(\alpha, x) = \sum_{k>0} \frac{e^{-\alpha k/2}}{k} \left[e^{-ikx} \rho_{i\alpha}(k) - e^{ikx} \rho_{i\alpha}(-k) \right]$$

$$= -J_{i\alpha}^{\dagger}(\alpha, x)$$

in the limit $\alpha \to 0$ $\quad J_{i\alpha}(\alpha, x) \to J_{i\alpha}(x)$

- in terms of boson operators

$$J_{1s}(\alpha, x) = \sum_{p>0} e^{-\frac{\alpha p}{2}} \left(\frac{2\pi}{pL}\right)^{1/2} \left(b_{1s p}\, e^{ipx} - b_{1s,p}^{\dagger}\, e^{-ipx} \right)$$

$$J_{2s}(\alpha, x) = \sum_{p>0} e^{-\alpha p/2} \left(\frac{2\pi}{pL}\right) \left(b_{2s,-p}^{\dagger}\, e^{ipx} - b_{2s,p}\, e^{-ipx} \right)$$

write: $\langle \psi^\dagger_{rs}(x)\psi_{rs}(x')\rangle = \frac{1}{2\pi\alpha} e^{ik_F(x-x')} \underbrace{\langle e^{-\phi_{rs}(0,x)} e^{\phi_{rs}(0,x')}\rangle}$

work on this term $\boxed{13}$

$$\prod_{p>0} \langle \exp[e^{-\alpha p/2}\left(\frac{2\pi}{pL}\right)^{1/2}(e^{-ipx}b_p^\dagger - e^{ipx}b_p)]$$
$$\exp[e^{-\alpha p/2}\left(\frac{2\pi}{pL}\right)^{1/2}(e^{ipx'}b_p - e^{-ipx'}b_p^\dagger)]\rangle$$

use identity: $\exp(A+B) = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A,B]\right)$

- need commutator of the arguments in the exponential

$$\sum_{\substack{p>0\\p'>0}} e^{-\alpha p/2} e^{-\alpha p'/2}\left(\frac{2\pi}{pL}\right)^{1/2}\left(\frac{2\pi}{p'L}\right)^{1/2}$$
$$[e^{-ipx}b_p^\dagger - e^{ipx}b_p, e^{ip'x'}b_{p'} - e^{-ip'x'}b_{p'}^\dagger]$$

$$= \sum_{p,p'>0} e^{-\frac{\alpha}{2}(p+p')}\left(\frac{2\pi}{L}\right)\left(\frac{1}{pp'}\right)^{1/2}\left(e^{i(p'x'-px)}[b_p^\dagger, b_{p'}] + e^{i(px-p'x')}[b_{p'}^\dagger, b_p]\right)$$

$$= \sum_{pp'>0} e^{-\alpha p}\left(\frac{2\pi}{pL}\right)\left[e^{ip(x-x')} - e^{-ip(x-x')}\right]$$

commutator is just a number => can be pulled out of the average

$$\langle \Psi^{\dagger}_{\alpha}(x)\Psi_{\alpha}(x')\rangle = \frac{1}{2\pi\alpha} e^{ik_F(x-x')}$$

$$\times \exp\left[\frac{1}{2}\sum_{p>0} e^{-\alpha p}\left(\frac{2\pi}{pL}\right)\left(e^{-ip(x-x')} - e^{ip(x-x')}\right)\right]$$

$$\prod_{p>0} \left\langle \exp\left[e^{-\alpha p/2}\left(\frac{2\pi}{pL}\right)^{1/2}\left(e^{-ipx}b_p^{\dagger} - e^{ipx}b_p + e^{ipx'}b_p - e^{-ipx'}b_p^{\dagger}\right)\right]\right\rangle$$

- we now apply the identity $\exp(A+B) = \exp(A)\exp(B)\exp(-\frac{1}{2}[A,B])$

again to separate the exponential into ~~terms~~ two terms: one will have only creation the other only destruction operators

$$\left\langle \exp\left\{\sum_{p>0} e^{-\alpha p/2}\left(\frac{2\pi}{pL}\right)^{1/2}\left(e^{-ipx} - e^{-ipx'}\right)b_p^{\dagger}\right.\right.$$

$$\left.\left. + \sum_{p>0} e^{-\alpha p/2}\left(\frac{2\pi}{pL}\right)^{1/2}\left(e^{ipx'} - e^{ipx}\right) b_p\right\}\right\rangle$$

again need commutator

$$\sum_{pp'>0} e^{-\alpha p/2}\left(\frac{2\pi}{pL}\right)^{1/2} e^{-\alpha p'/2}\left(\frac{2\pi}{p'L}\right)^{1/2}\left(e^{-ipx} - e^{-ipx'}\right)\times$$

$$\times\left(e^{ip'x'} - e^{ip'x}\right)\left[b_p^{\dagger}, b_{p'}\right]$$

$$= \sum_{p} e^{-\alpha p}\left(\frac{2\pi}{pL}\right)\left[2 - e^{ip(x'-x)} - e^{-ip(x'-x)}\right]$$

$$\Rightarrow \langle \Psi^{+}_{1a}(x)\Psi_{1a}(x')\rangle = \frac{1}{2\pi\alpha} e^{ik_F(x-x')}$$

$$\exp\Big[-\frac{2\pi}{L}\sum_{p>0}\frac{e^{-\alpha p}}{p}\left(1-e^{ip(x-x')}\right)\Big]$$

$$\prod_{p>0}\Big\langle \exp\Big[\left(\frac{2\pi}{pL}\right)^{1/2} e^{-\alpha p/2}\left(e^{+ipx}-e^{-ipx'}\right) b_p^{+}\Big]$$

$$\times \exp\Big[\left(\frac{2\pi}{pL}\right)^{1/2} e^{-\alpha p/2}\left(e^{ipx'}-e^{ipx}\right) b_p\Big]\Big\rangle$$

- for a system at zero temperature b_p will operate on the vacuum $\Rightarrow b_p|0\rangle = 0$
- if we expand the exponentials we will always have annihilation operators to the right $\Rightarrow \langle\ldots\rangle$ is one.

$$\Rightarrow \langle \Psi^{+}_{1a}(x)\Psi_{1a}(x)\rangle = \frac{1}{2\pi\alpha} e^{ik_F(x-x')}$$

$$\exp\Big[-\frac{2\pi}{L}\sum_{p>0}\frac{e^{-\alpha p}}{p}\left(1-e^{ip(x-x')}\right)\Big]$$

the sum over $\frac{2\pi}{L}\sum_p \longrightarrow \int dp$ in the thermodynamic limit

$$\cancel{-\frac{2\pi}{L}\int_0^\infty dp}$$

$$\int_0^\infty \frac{dp}{p} e^{-\alpha p}\left(1 - e^{ip(x-x')}\right)$$

expand exponential: $e^{ip(x-x')}$

$$= \sum_{\ell=0}^{\infty} \frac{(ip(x-x'))^\ell}{\ell!}$$

$$\Rightarrow \sum_{\ell=1}^{\infty}\left[\int_0^\infty dp\, e^{-\alpha p} p^{\ell-1}\right]\frac{(i(x-x'))^\ell}{\ell!}$$

$$\int_0^\infty dp\, e^{-\alpha p} p^{\ell-1} = ?$$

$$\int_0^\infty dp\, e^{-\alpha p} = \frac{1}{\alpha}$$

$$-\frac{d}{d\alpha} \Rightarrow \quad \int_0^\infty dp\, e^{-\alpha p} p = \frac{1}{\alpha^2}$$

$$\int_0^\infty dp\, e^{-\alpha p} p^2 = \frac{2}{\alpha^3}$$

$$\Rightarrow \int_0^\infty dp\, e^{-\alpha p} p^{\ell-1} = \frac{(\ell-1)!}{\alpha^\ell}$$

$$\Rightarrow \sum_{\ell=1}^{\infty} \frac{1}{\ell}\left[\frac{i(x-x')}{\alpha}\right]^\ell = \ln\left[1 - \frac{i(x-x')}{\alpha}\right]$$

$$\Rightarrow \langle \Psi_{1a}^{+}(x)\, \Psi_{1a}(x')\rangle = \frac{1}{2\pi\alpha}\,\frac{\exp[-ik_F(x-x')]}{1 - i(x-x')/\alpha}$$

indeed the same as the well-known

result

Applying Luther/Peschel to an Interacting System

outline:

Luttinger Hamiltonian:

$$H = \sum_{p>0} \left[\bar{\omega}_p \left(b_{1p}^\dagger b_{1p} + b_{2-p}^\dagger b_{2-p} \right) + \bar{V}_p \left(b_{1p}^\dagger b_{2-p}^\dagger + b_{2-p} b_{1p} \right) \right]$$

$$\bar{\omega}_p = p v_F + p \frac{V_{1p}}{2\pi}$$

$$\bar{V}_p = p \frac{V_{2p}}{2\pi}$$

→ can be diagonalised using the transformation

$$b_{1p} = \beta_p \cosh(\lambda_p) - \alpha_p^\dagger \sinh(\lambda_p)$$

$$b_{1p}^\dagger = \beta_p^\dagger \cosh(\lambda_p) - \alpha_p \sinh(\lambda_p)$$

$$b_{2-p} = \alpha_p \cosh(\lambda_p) - \beta_p^\dagger \sinh(\lambda_p)$$

$$b_{2-p}^\dagger = \alpha_p^\dagger \cosh(\lambda_p) - \beta_p \sinh(\lambda_p)$$

$$\rightarrow H = \sum_{p>0} \left[\left(\beta_p^\dagger \beta_p + \alpha_p^\dagger \alpha_p \right) \left\{ \left[\cosh^2 \lambda_p + \sinh^2 \lambda_p \right] \bar{\omega}_p - 2 \bar{V}_p \cosh\lambda_p \sinh\lambda_p \right\} + \left(\beta_p^\dagger \alpha_p^\dagger + \alpha_p \beta_p \right) \left\{ \left[\cosh^2 \lambda_p + \sinh^2 \lambda_p \right] \bar{V}_p - 2 \bar{\omega}_p \cosh\lambda_p \sinh\lambda_p \right\} \right]$$

λ_p should be chosen so that the coefficient of $(\beta_p^\dagger \alpha_p^\dagger + \alpha_p \beta_p)$ is zero ⇒ we are then back to an easy problem of harmonic oscillators

$\Rightarrow \quad H_- = \sum_{p\alpha} E_p \left(\beta_p^\dagger \beta_p + \alpha_p^\dagger \alpha_p \right)$

$$E_p = \left(\bar{\omega}_0^2 - \bar{V}_p^2 \right)^{1/2}$$

$\Rightarrow \cosh(2\lambda_p) = \dfrac{\bar{\omega}_p}{E_p} \qquad \sinh(2\lambda_p) = \dfrac{\bar{V}_p}{E_p}$

$\rightarrow$ to solve for $\langle \Psi_{1\alpha}^\dagger(x)\, \Psi_{1\alpha}(x') \rangle$

need to substitute in terms of operators

$$b_{2p}^\dagger\, b_{2p}\, b_{1p}^\dagger,\ b_{1p} \longrightarrow \alpha_p,\ \beta_p\ \alpha_p^\dagger\ \beta_p^\dagger$$

$\vdots$

$$\langle \Psi_1^\dagger(x)\, \Psi_1(x') \rangle = \frac{e^{-ik_F(x-x')}}{2\pi[\alpha - i(x-x')]}\, e^{-\phi(x,x')}$$

term responsible for interaction

final note on Luttinger vs. Tomonaga:

<u>exchange</u>!!!

Tomonaga is better on exchange

consider δ-fn. potential

$$\delta(x) \Rightarrow \frac{1}{2\pi} \sum_p e^{ipx} V_p$$

in this case V_p = constant

→ in general, potential is

$$\frac{1}{2} \sum_q V_q \sum_{kk'} a^+_{k+q} a^+_{k'-q} a_{k'} a_k$$

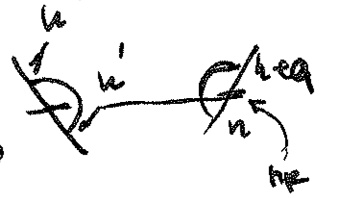
boson approximation:

$$\rightarrow \frac{1}{2} \sum_q V_q \left(\sum_k a^+_{k+q} a_k \right)\left(\sum_{k'} a^+_{k'-q} a_{k'} \right)$$

grouped in a particular way

→ can also group differently

Direct: $\frac{1}{2} \sum_q V_q \left(\sum_k a^+_{k+q} a_k \right)\left(\sum_{k'} a^+_{k'-q} a_{k'} \right) \Rightarrow$



Exchange: $-\frac{1}{2} \sum_{qkk'} V_q (a^+_{k+q} a_{k'})(a^+_{k'-q} a_k)$

exchange term can be rewritten as

$$-\sum_{Qpk} V_{p-k} (a^+_p a_{p+Q})(a^+_{k+Q} a_k)$$

if V_p is constant these two terms cancel, no effect

in Luttinger, because we are distinguishing the particles, $(1,2) \rightarrow$ exchange term is removed

- Only a direct term remains
  - → gives error
- in Tomonaga both terms are considered