Bosonization: Tomonaga model

(see Mahan "Many-body Physics")

- 1D electron gas
- approximate treatment
- excitations of electron gas are approximate bosons in this approximate treatment
- successful applications: - TTF-TCNQ conductivity
  - X-ray absorption problems

Tomonaga model

$$H = v_F \sum_{k\sigma} |k| \, a^\dagger_{k\sigma} a_{k\sigma} + \frac{1}{2L} \sum_k V_k \, \rho(k) \rho(-k)$$

$$\rho(k) = \sum_{p\sigma} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}$$

- summation over k-states turns into an integral

$$\sum_k f(k) = \frac{L}{2\pi} \int dk \, f(k)$$

- V_k - depends on the details of the potential

- basic step in Tomonaga model:

$$\rho_1(k) = \sum_{\sigma p>0} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}$$

$$\rho_2(k) = \sum_{\sigma p<0} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}$$

$$\rho(k) = \rho_1(k) + \rho_2(k)$$

- important thing: commutation relations

$$[\rho(k), \rho(k')] = 0 \text{ , but } [\rho_1(k), \rho_1(k')] \neq 0$$

- examine: $[\rho_1(k), \rho_1(k')] =$

$$\sum_{\substack{\sigma\sigma' \\ p,p'>0}} [a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}, a^\dagger_{p'-\frac{k'}{2}\sigma'} a_{p'+\frac{k'}{2}\sigma'}]$$

$$a^\dagger_{p-\frac{k}{2}\sigma} \underbrace{a_{p+\frac{k}{2}\sigma} a^\dagger_{p'-\frac{k'}{2}\sigma'}} a_{p'+\frac{k'}{2}\sigma'} - a^\dagger_{p'-\frac{k'}{2}\sigma'} a_{p'+\frac{k'}{2}\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}$$

$$\delta_{p+\frac{k}{2}\, p'-\frac{k'}{2}} \delta_{\sigma\sigma'} - a^\dagger_{p'-\frac{k'}{2}\sigma'} a_{p+\frac{k}{2}\sigma}$$

$$\rightarrow a^\dagger_{p-\frac{k}{2}\sigma} a_{p'+\frac{k'}{2}\sigma'} (\delta_{p+\frac{k}{2}\, p'-\frac{k'}{2}} \delta_{\sigma\sigma'})$$

$$- a^\dagger_{p-\frac{k}{2}\sigma} a^\dagger_{p'-\frac{k'}{2}\sigma'} a_{p+\frac{k}{2}\sigma} a_{p'+\frac{k'}{2}\sigma'}$$

$$- a^\dagger_{p'-\frac{k'}{2}\sigma'} a_{p'+\frac{k'}{2}\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}$$

$$= \delta_{p+\frac{k}{2},\, p'-\frac{k'}{2}} \delta_{\sigma\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p'+\frac{k'}{2}\sigma'}$$

$$- a^\dagger_{p'-\frac{k'}{2}\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p'+\frac{k'}{2}\sigma'} a_{p+\frac{k}{2}\sigma}$$

$$- a^\dagger_{p'-\frac{k'}{2}\sigma'} a_{p'+\frac{k'}{2}\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}$$

$$= \delta_{p+\frac{k}{2}\, p'-\frac{k'}{2}} \delta_{\sigma\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p'+\frac{k'}{2}\sigma'}$$

$$- \delta_{p-\frac{k}{2},\, p'+\frac{k'}{2}} \delta_{\sigma\sigma'} a^\dagger_{p'-\frac{k'}{2}\sigma'} a_{p+\frac{k}{2}\sigma}$$

commutator:

$$\sum_{\substack{\sigma\sigma' \\ p,p'>0}} [\delta_{p+\frac{k}{2}\, p'-\frac{k'}{2}} \delta_{\sigma\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p'+\frac{k'}{2}\sigma'} - \delta_{p-\frac{k}{2},\, p'+\frac{k'}{2}} \delta_{\sigma\sigma'} a^\dagger_{p'-\frac{k'}{2}\sigma'} a_{p+\frac{k}{2}\sigma}]$$

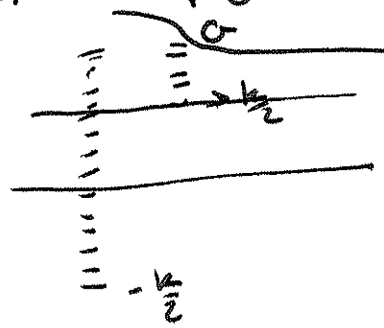
$$= \sum_{\substack{p>0\\ \sigma}} \Big[a^{\dagger}_{p-\frac{k}{2}\sigma}\, a_{p+k'+\frac{k}{2}\sigma} \quad \theta\!\left(p+\tfrac{k}{2}+\tfrac{k'}{2}\right)$$

$$- a^{\dagger}_{p-k'-\frac{k}{2}}\, a_{p+\frac{k}{2}\sigma} \quad \theta\!\left(p-\tfrac{k}{2}-\tfrac{k'}{2}\right)\Big]$$

consider special case

$k' = -k$

$$[\rho_1(k), \rho_1(-k)] = \sum_{\substack{p>0\\ \sigma}} \left[n_{p-\frac{k}{2}\sigma} - n_{p+\frac{k}{2}\sigma}\right] = \sum_{\substack{\frac{k}{2} > p \geq -\frac{k}{2}\\ \sigma}} \bar{n}_p$$



$$\Rightarrow \sum_{\substack{\frac{k}{2} > p > -\frac{k}{2}\\ \sigma}} \bar{n}_p$$

$$\sum_{p} n_{p\sigma} = 2\,\theta(k_F - |p|) = \begin{cases} 2\left(\frac{kL}{2\pi}\right) & k < 2k_F \\ \frac{2k_F L}{\pi} & k > 2k_F \end{cases}$$

- interested in excitations near k_F

$$\Rightarrow \quad [\rho_1(k), \rho_1(-k)] = \frac{kL}{\pi}$$

similarly one can show that

$$[\rho_2(k), \rho_2(-k)] = -\frac{kL}{\pi}$$

$$[\rho_1(k), \rho_2(-k)] = 0$$

→ in the Tomonaga model we assume

$$[\rho_1(k), \rho_1(-k')] = \delta_{kk'}\frac{kL}{\pi}$$

$$[\rho_2(k), \rho_2(-k')] = -\delta_{kk'}\frac{kL}{\pi}$$

$$[\rho_1(k), \rho_2(-k')] = 0$$

- we have not considered the off diagonal terms

$k' \neq -k$

in this case $[\rho_1(k), \rho_1(-k')]$ is not exactly

zero $\Rightarrow$ $[\rho_1(k), \rho_1(-k')]$

$$= \sum_{\substack{p>0 \\ \sigma}} [a^\dagger_{p-\frac{k}{2}\sigma} a_{p+k'+\frac{k}{2}\sigma} \Theta(p+\tfrac{k}{2}+\tfrac{k'}{2})$$

$$- a^\dagger_{p-k'-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma} \Theta(p-\tfrac{k}{2}-\tfrac{k'}{2})]$$

for $k \neq -k'$ $\langle a^\dagger_{p-\frac{k}{2}\sigma} a_{p+k'+\frac{k}{2}\sigma} \rangle = 0$, so

on average $\langle [\rho(k), \rho(-k')] \rangle = 0$. This is a

reasonable justification for our approximation.

→ ~~Since~~ our commutation relations can be rewritten

in terms of creation and destruction operators

define: $\rho_1(k) = b_k \left(\frac{kL}{\pi}\right)^{1/2}$

$$\rho_1(-k) = b_k^\dagger \left(\frac{kL}{\pi}\right)^{1/2}$$

$$\rho_2(k) = b_{-k}^\dagger \left(\frac{kL}{\pi}\right)^{1/2}$$

$$\rho_2(-k) = b_{-k} \left(\frac{kL}{\pi}\right)^{1/2}$$

$\Downarrow$

$$[b_k, b_{k'}^\dagger] = \delta_{kk'}$$

- potential term can now be written in terms of

creation and destruction operators

$$\frac{1}{2L} \sum_k V_k \, \rho(k) \rho(-k) = \sum_k \overline{V}_k (b_k + b_{-k}^\dagger)(b_k^\dagger + b_{-k})$$

where $\bar{V}_k = \dfrac{|k|\, V_k}{2\pi}$

- need to also express the kinetic energy in terms of the boson operators
- again consider commutator

$$[\rho_1(k), H_0] = ?$$

$$H_0 = \sum_{\substack{k' \\ \sigma}} |k'|\, a^\dagger_{k'\sigma} a_{k'\sigma}$$

$$= V_F \sum_{\substack{p>0\, k' \\ \sigma\sigma'}} [a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma} , a^\dagger_{k'\sigma'} a_{k'\sigma'}]\, |k'|$$

$$= V_F \sum_{\substack{k'\, p>0 \\ \sigma\sigma'}} \Big(a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma} a^\dagger_{k'\sigma'} a_{k'\sigma'}\, |k'|$$
$$- a^\dagger_{k'\sigma'} a_{k'\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}\, |k'| \Big)$$

1st term:

$$a^\dagger_{p-\frac{k}{2}\sigma} a_{k'\sigma'}\, \delta_{p+\frac{k}{2},k'} \delta_{\sigma\sigma'} - a^\dagger_{p-\frac{k}{2}\sigma} a^\dagger_{k'\sigma'} a_{p+\frac{k}{2}\sigma} a_{k'\sigma'}$$

$$= a^\dagger_{p-\frac{k}{2}\sigma} a_{k'\sigma'}\, \delta_{p+\frac{k}{2}k'}\, \delta_{\sigma\sigma'} - a_{k'\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{k'\sigma'} a_{p+\frac{k}{2}\sigma}$$

$$= \delta_{p+\frac{k}{2}k'}\, \delta_{\sigma\sigma'}\, a^\dagger_{p-\frac{k}{2}\sigma} a_{k'\sigma'} - a^\dagger_{k'\sigma'} a_{p+\frac{k}{2}\sigma}\, \delta_{p-\frac{k}{2}k'}\, \delta_{\sigma\sigma'}$$

$$+ a^\dagger_{k'\sigma'} a_{k'\sigma'} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma}$$

$\Rightarrow$ commutator:

$$[\rho_1(k), H_0] = V_F \sum_{\sigma p>0} a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma} \left[|p+\tfrac{k}{2}| - |p-\tfrac{k}{2}| \right]$$

if $k < 2p$ $\quad [\rho_1(k), H_0] = v_F \sum_{\delta p > 0} |k| \, a^\dagger_{p+\frac{k}{2}a} a_{p+\frac{k}{2}a}$

$$= v_F |k| \, \rho_1(k)$$

$$\Rightarrow [b_k, H_0] = v_F k \, b_k = \omega_k b_k$$

if we choose H_0 to be $H_0 = \sum_k \omega_k b_k^\dagger b_k$

$$[b_k, \sum_{k'} \omega_{k'} b_{k'}^\dagger b_{k'}] = \sum_{k'} \omega_{k'} [b_k, b_{k'}^\dagger b_{k'}]$$

$$= \sum_{k'} \omega_{k'} [b_k, b_{k'}^\dagger] b_{k'} = \omega_k b_k$$

for $\rho_2(k) \rightarrow [\rho_2(k), H_0] = -\omega_k \rho_2(k)$

total Hamiltonian:

$$\underbrace{H = \sum_k \omega_k b_k^\dagger b_k + \sum_k \bar{V}_k (b_k + b_{-k}^\dagger)(b_k^\dagger + b_{-k})}_{}$$

this Hamiltonian is quadratic in bosonic operators, hence easy to diagonalize

$$\rightarrow \quad Q_k = \frac{1}{(2\omega_k)^{1/2}} (b_k + b_{-k}^\dagger)$$

$$P_k = i \left(\frac{\omega_k}{2}\right)^{1/2} (b_k^\dagger - b_{-k})$$

$$[Q_k, P_{k'}] = i \delta_{kk'}$$

$$H = \frac{1}{2} \sum_k [P_k P_{-k} + Q_k Q_{-k} (\omega_k^2 + 4\omega_k \bar{V}_k)]$$

eigen frequencies: $E_k = (\omega_k^2 + 4\omega_k \bar{V}_k)^{1/2}$

$$= |k| \left(v_F^2 + \frac{2 V_k v_F}{\Omega}\right)$$

Spin waves

other collective excitations: spin waves (magnons)

$$\sigma(k) = \sum_{\sigma p>0} \sigma\, a^\dagger_{p-\frac{k}{2}\sigma}\, a_{p+\frac{k}{2}\sigma}$$

$$\sigma_1(k) = \sum_{\sigma p>0} \sigma\, a^\dagger_{p-\frac{k}{2}\sigma}\, a_{p+\frac{k}{2}\sigma}$$

$$\sigma_2(k) = \sum_{\sigma p<0} \sigma\, a^\dagger_{p-\frac{k}{2}\sigma}\, a_{p+\frac{k}{2}\sigma}$$

$$[\sigma_1(k), \sigma_1(-k')] = \sum_{\sigma, p>0} \sigma^2 \left(a^\dagger_{p-\frac{k}{2}\sigma}\, a_{p-k'+\frac{k}{2}\sigma} - a^\dagger_{p+k'-\frac{k}{2}\sigma}\, a_{p+\frac{k}{2}\sigma}\right)$$

$$[\sigma_1(k), \rho_1(-k')] = \sum_{\sigma, p>0} \sigma \left(a^\dagger_{p-\frac{k}{2}\sigma}\, a_{p-k'+\frac{k}{2}\sigma} - a^\dagger_{p+k'-\frac{k}{2}\sigma}\, a_{p+\frac{k}{2}\sigma}\right)$$

$$[\sigma_1(k), \sigma_1(-k')] = \delta_{kk'} \frac{kL}{\pi} \quad (\text{same as } [\rho_1(k), \rho_1(-k')])$$

$[\sigma_1(k), \rho_1(-k')] = 0$ if different spins have the same probability

$\Rightarrow$ one can write a Hamiltonian for spin waves

$$\sigma_1(k) = c_k \left(\frac{kL}{\pi}\right)^{1/2}$$

$$\sigma_1(-k) = c_k^\dagger \left(\frac{kL}{\pi}\right)^{1/2}$$

$$\sigma_2(k) = c_{-k}^\dagger \left(\frac{kL}{\pi}\right)^{1/2}$$

$$\sigma_2(-k) = c_{-k} \left(\frac{kL}{\pi}\right)^{1/2}$$

$$[c_k, c_{k'}^\dagger] = \delta_{kk'}$$

$$[c_k, b_k^\dagger] = 0$$

$$[\sigma_i(k), H_0] = v_F \sum_{\sigma, p>0} \sigma\, a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma} (|p+\tfrac{k}{2}| - |p-\tfrac{k}{2}|)$$

$$\simeq v_F\, k\, \sigma_i(k)$$

→ can write a spin-Hamiltonian

$$H_{sw} = \frac{v_F \pi}{L} \sum_{k>0} [\sigma_1(-k)\sigma_1(k) + \sigma_2(k)\sigma_2(-k)] = \sum_k \omega_k c_k^\dagger c_k$$

⇓

$$H_T = H + H_{sw} = \sum_k (\varepsilon_k \alpha_k^\dagger \alpha_k + \omega_k c_k^\dagger c_k)$$

H_T – excitation spectra of 1D electron gas

- some potentials also enter into spin wave Hamiltonian

example: $\mathcal{U} = \frac{1}{L} \sum_{k\sigma} U_k (\sum_p a^\dagger_{p-\frac{k}{2}\sigma} a_{p+\frac{k}{2}\sigma})(\sum_{p'} a^\dagger_{p'+\frac{k}{2}\sigma} a_{p'-\frac{k}{2}\sigma})$

$$= \frac{1}{L} \sum_{k\sigma} \rho_\sigma(k) \rho_\sigma(-k)$$

$$= \frac{1}{L} \sum_{k\sigma} [(\rho_\uparrow(k) + \rho_\downarrow(k))(\rho_\uparrow(-k) + \rho_\downarrow(-k))$$
$$+ (\rho_\uparrow(k) - \rho_\downarrow(k))(\rho_\uparrow(-k) - \rho_\downarrow(-k))]$$

$$= \frac{1}{L} \sum_k U_k [\rho(k)\rho(-k) + \sigma(k)\sigma(-k)]$$

→ in this case total Hamiltonian will also include a spin-wave term