Mermin-Wagner Theorem (source: Auerbach) 1

Spontaneously broken symmetry

- general phenomenon in statistical mechanics
- can occur at finite temperatures and when the order parameter does not commute with the Hamiltonian
- Spin density wave in the z-direction

$$S^z_{\vec{q}} = \sum_i e^{i\vec{q}\cdot\vec{x}_i} S^z_i$$

- consider a rotationally invariant Hamiltonian

$\mathcal{H}_0 \rightarrow$ add symmetry breaking term, here the spin-density wave coupled to a field

$$\mathcal{H}(h) = \mathcal{H}_0 - h S^z_{\vec{q}}$$

- magnetization per site:

$$m_{\vec{q}}(h) = \frac{1}{N\mathcal{Z}} \mathrm{Tr}\left[e^{-\beta\mathcal{H}(h)} S^z_{\vec{q}}\right]$$

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta\mathcal{H}(h)}\right]$$

- finite fields: $m_{\vec{a}}(h)$ becomes finite (induced by the ordering field)

<u>Definition:</u> <u>spontaneously broken symmetry</u>

occurs in the system if

$$\lim_{h\to 0^+} \lim_{N\to\infty} m_{\vec{a}}(h,N,T) \neq 0$$

in other words if, after taking the thermodynamic limit, the limit $h\to 0^+$ results in a finite magnetization

- in the absence of the ordering field: examine two-point correlation function

$$S^{\alpha\alpha}(\vec{q}) = \lim_{h\to 0^+} \frac{1}{ZN} [e^{-\beta H} S^{\alpha}_{\vec{q}} S^{\alpha}_{-\vec{q}}]$$

spontaneously broken symmetry implies true long range order: $\lim_{N\to\infty} \frac{S^{\alpha\alpha}(\vec{a})}{N} > 0$

or $\lim_{|\vec{r}_i-\vec{r}_j|\to\infty} \lim_{N\to\infty} \langle \vec{S}_i \cdot \vec{S}_j \rangle \neq 0$

"quasi-long-range order" → power law decay of correlation functions at large distances

- given these definitions we can state the Mermin-Wagner theorem:

for the quantum Heisenberg model

$$\mathcal{H} = \frac{1}{2} \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j - h S^{\dagger}_{\vec{q}}$$

with short range interactions which obey

$$\bar{J} = \frac{1}{2N} \sum_{ij} |J_{ij}| |\vec{x}_i - \vec{x}_j|^2 < \infty$$

there can be no spontaneously broken symmetry at finite temperatures in one and two dimensions

$$\lim_{h \to 0^+} \lim_{N \to \infty} m_{\vec{q}}(h, N) = 0.$$

To prove this: first define a <u>scalar product</u>

$$(A, B) = \frac{1}{Z} \sum_{m,n} \langle n | A^{\dagger} | m \rangle \langle m | B | n \rangle \left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m} \right)$$

$|n\rangle \to$ eigenstate of $\mathcal{H} \Rightarrow \mathcal{H}|n\rangle = E_n |n\rangle$

the summation excludes terms with $E_n = E_m$

$\left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m} \right)$ is non negative

$\to$ either $E_n > E_m$ or $E_m < E_n$

if $E_n > E_m \Rightarrow e^{-\beta E_m} - e^{-\beta E_n} > 0$ and $E_n - E_m > 0$ ①

$$\Rightarrow \left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m}\right) > 0$$

if $E_n < E_m \Rightarrow e^{-\beta E_m} - e^{-\beta E_n} < 0$ and $E_n - E_m < 0$

$$\Rightarrow \left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m}\right) > 0$$

$A = B \Rightarrow 0 \leq (A, A) = \chi_{AA}$ (susceptibility)
(indeed (A,A) is a scalar product)

- inequality: $\tanh x < x$ can be used to show that

$$0 < \left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m}\right) \leq \frac{\beta}{2}\left(e^{-\beta E_m} + e^{-\beta E_n}\right)$$

$$\Rightarrow \left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m}\right) = e^{-\frac{\beta}{2}(E_m + E_n)} \frac{\left(e^{-\frac{\beta}{2}(E_m - E_n)} - e^{-\frac{\beta}{2}(E_n - E_m)}\right)}{E_n - E_m}$$

$$= \frac{e^{-\frac{\beta}{2}(E_m + E_n)}}{(E_n - E_m)} \frac{\left(e^{\frac{\beta}{2}(E_n - E_m)} - e^{-\frac{\beta}{2}(E_n - E_m)}\right)}{\left(e^{\frac{\beta}{2}(E_n - E_m)} + e^{-\frac{\beta}{2}(E_n - E_m)}\right)} \times \left(e^{\frac{\beta}{2}(E_n - E_m)} + e^{-\frac{\beta}{2}(E_n - E_m)}\right)$$

$$= \frac{\beta}{2} \frac{2}{\beta(E_n - E_m)} \tanh\left(\frac{\beta}{2}(E_n - E_m)\right)\left(e^{-\beta E_m} + e^{-\beta E_n}\right)$$

$$\leq \frac{\beta}{2}\left(e^{-\beta E_m} + e^{-\beta E_n}\right)$$

it follows that:

$$(A,A) = \frac{1}{Z} \sum_{m,n} \langle n|A^+|m\rangle \langle m|A|n\rangle \left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{(E_n - E_m)} \right)$$

$$\leq \frac{1}{Z} \frac{\beta}{2} \sum_{m,n} \langle n|A^+|m\rangle \langle m|A|n\rangle \left(e^{-\beta E_m} + e^{-\beta E_n} \right)$$

$$= \frac{\beta}{2} \left(\langle A^+ A\rangle + \langle A A^+\rangle \right)$$

$$\Rightarrow \quad (A,A) \leq \frac{\beta}{2} \left(\langle A^+ A\rangle + \langle A A^+\rangle \right)$$

for scalar products the Cauchy-Schwartz inequality states that: $|(A,B)|^2 \leq (A,A)(B,B)$

$$\Rightarrow \quad |(A,B)|^2 \leq \frac{\beta}{2} \langle A^+ A + A A^+\rangle (B,B)$$

define: $B = [C^+, H]$

$$\Rightarrow (A,B) = \frac{1}{Z} \sum_{mn} \langle n|A^+|m\rangle \langle m|B|n\rangle \left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m} \right)$$

$$= \frac{1}{Z} \sum_{m,n} \langle n|A^+|m\rangle \langle m|[C^+,H]|n\rangle \left(\frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m} \right)$$

$$\langle m|C^+ H|n\rangle - \langle m|H C^+|n\rangle$$

$$= (E_n - E_m) \langle m|C^+|n\rangle$$

$$= \frac{1}{Z} \sum_{m,n} \langle n|A^+|m\rangle \langle m|C^+|n\rangle \left(e^{-\beta E_m} - e^{-\beta E_n} \right)$$

$$= \frac{1}{Z} \sum_{m,n} e^{-\beta E_m} \langle m|C^+|n\rangle \langle n|A^+|m\rangle$$

$$- \frac{1}{Z} \sum_{m,n} e^{-\beta E_n} \langle n|A^+|m\rangle \langle m|C^+|n\rangle$$

$$(A,B) = \langle [C^\dagger, A^\dagger] \rangle$$

it also follows that

$$(B,B) = \langle [C^\dagger, B^\dagger] \rangle = \cancel{\langle [C^\dagger, [H, C^\dagger]] \rangle}$$

$$= \langle [C^\dagger, [H, C]] \rangle$$

$$\Downarrow$$

Bogoliubov inequality

$$|\langle [C^\dagger, A^\dagger] \rangle|^2 \leq \frac{\beta}{2} \langle A A^\dagger + A^\dagger A \rangle \langle [C^\dagger, [H, C]] \rangle$$

choose: $C = S^x_{\vec{k}}$ $\quad A = S^y_{-\vec{k}-\vec{q}}$

$$|\langle [S^x_{-\vec{k}}, S^y_{\vec{k}+\vec{q}}] \rangle|^2 \leq \frac{\beta}{2} \langle S^y_{-\vec{k}-\vec{q}} S^y_{\vec{k}+\vec{q}} + S^y_{\vec{k}+\vec{q}} S^y_{-\vec{k}-\vec{q}} \rangle$$

$$\times \langle [S^x_{-\vec{k}}, [H, S^x_{\vec{k}}]] \rangle$$

$$\frac{\beta}{2} \langle S^y_{-\vec{k}-\vec{q}} S^y_{\vec{k}+\vec{q}} + S^y_{\vec{k}+\vec{q}} S^y_{-\vec{k}-\vec{q}} \rangle = \beta \langle S^y_{-\vec{k}-\vec{q}} S^y_{\vec{k}+\vec{q}} \rangle$$

$$= \beta \, S^{yy}_{\vec{k}+\vec{q}}$$

$$\langle [C^\dagger, A^\dagger] \rangle = \langle [S^x_{-\vec{k}}, S^y_{\vec{k}+\vec{q}}] \rangle$$

$$= \sum_i \sum_j e^{-i \vec{k} \cdot \vec{x}_i + i(\vec{k}+\vec{q}) \cdot \vec{x}_j} \langle [S^x_i, S^y_j] \rangle$$

$$= \sum_i e^{i \vec{q} \cdot \vec{x}_i} \langle [S^x_i, S^y_i] \rangle$$

$$= i \sum_i e^{i \vec{q} \cdot \vec{x}_i} \langle S^z_i \rangle = i N m^z_{\vec{q}}$$

$$\langle [C^\dagger, A^\dagger] \rangle = i N m^z_{\vec{q}}$$

define $F(\vec{k}) = N^{-1} \langle [S^x_{-\vec{k}}, [\mathcal{H}, S^x_{\vec{k}}]] \rangle$

Bogoliubov's inequality: $m^2_{\vec{a}} \leq \beta\, S^{xx}(\vec{k}+\vec{a})\, F(\vec{k})$

→ calculate $F(\vec{k})$

$$\mathcal{H} = \frac{1}{2}\sum_{\langle i,j \rangle} J_{ij}\, \vec{S}_i \cdot \vec{S}_j - h\, S^z_{\vec{a}}$$

first do the term $-h S^z_{\vec{a}}$

$$N^{-1} \langle [S^x_{-\vec{k}}, [S^z_{\vec{a}}, S^x_{\vec{k}}]] \rangle (-h)$$

$$= -\frac{h}{N} \sum_{jlm} \langle [S^x_j, \underbrace{[S^z_l, S^x_m]}_{[S^z_l, S^x_m] = i\delta_{lm} (S^y_l)}] \rangle\, e^{-i\vec{k}\cdot\vec{x}_j}\, e^{i\vec{a}\cdot\vec{x}_l}\, e^{i\vec{k}\cdot\vec{x}_m}$$

$$= -\frac{ih}{N} \sum_{jl} \langle [S^x_j, S^y_l] \rangle\, e^{-i\vec{k}\cdot\vec{x}_j}\, e^{i(\vec{a}+\vec{k})\cdot\vec{x}_m}$$

$$= \frac{h}{N} \sum_{l} e^{i\vec{a}\cdot\vec{x}_l} \langle S^z_l \rangle = h\, m^z_{\vec{a}}$$

$$F(\vec{k}) = h\, m^z_{\vec{a}} + \frac{1}{2N}\sum_{\langle j,l \rangle} J_{jl} \langle [S^x_{-\vec{k}}, [\vec{S}_j \cdot \vec{S}_l, S^x_{\vec{k}}]] \rangle$$

now let's do two-body term

$$\frac{1}{2N}\sum_{\langle j,l \rangle} J_{jl} \langle [S^x_{-\vec{k}}, [\vec{S}_j \cdot \vec{S}_l, S^x_{\vec{k}}]] \rangle$$

$$= \frac{1}{2N}\sum_{\langle j,l \rangle} J_{jl} \sum_{mn} e^{-i\vec{k}(\vec{x}_m - \vec{x}_n)} \langle [S^x_m, [\vec{S}_j \cdot \vec{S}_l, S^x_n]] \rangle$$

consider first the inside commutator

$$[\vec{S}_j \cdot \vec{S}_e , S_n^x]$$

$$= \underbrace{[S_j^x S_e^x , S_n^x]}_{0} + [S_j^y S_e^y , S_n^x] + [S_j^z S_e^z , S_n^x]$$

$$= S_j^y [S_e^y , S_n^x] + [S_j^y , S_n^x] S_e^y$$

$$+ S_j^z [S_e^z , S_n^x] + \cancel{S_j^z S_e^z} [S_j^z , S_n^x] S_e^z$$

$$= -i S_j^y S_e^z \delta_{en} - i S_j^z S_e^y \delta_{jn}$$

$$+ i S_j^z S_e^y \delta_{en} + i S_j^y S_e^z \delta_{jn}$$

$$= \frac{1}{2N} \sum_{jlmn} J_{jl} e^{-i\vec{\pi}(\vec{x}_m - \vec{x}_n)} \Big[-i \langle [S_m^x , S_j^y S_e^z] \rangle \delta_{en}$$

$$- i \langle [S_m^x , S_j^z S_e^y] \rangle \delta_{jn} + i \langle [S_m^x , S_j^z S_e^y] \rangle \delta_{en}$$

$$+ i \langle [S_m^x , S_j^y S_e^z] \rangle \delta_{jn} \Big]$$

$$= \frac{1}{2N} \sum_{jlm} J_{jl} e^{-i\vec{\pi}\cdot(\vec{x}_m - \vec{x}_l)} \left(-i \langle [S_m^x , S_j^y S_e^z] \rangle \right)$$

$$+ \frac{1}{2N} \sum_{jlm} J_{jl} e^{-i\vec{\pi}\cdot(\vec{x}_m - \vec{x}_j)} \left(-i \langle [S_m^x , S_j^z S_e^y] \rangle \right)$$

$$+ \frac{1}{2N} \sum_{jlm} J_{jl} e^{-i\vec{\pi}\cdot(\vec{x}_m - \vec{x}_l)} \left(i \langle [S_m^x , S_j^z S_e^y] \rangle \right)$$

$$+ \frac{1}{2N} \sum_{jlm} J_{jl} e^{-i\vec{\pi}\cdot(\vec{x}_m - \vec{x}_j)} \left(i \langle [S_m^x , S_j^y S_e^z] \rangle \right)$$

- considering that $j \neq l$ (neighbours) and that 9

$$J_{jl} = J_{lj}$$

$$= \frac{i}{N} \sum_{jlm} J_{jl}\, e^{-i\vec{k}\cdot(\vec{x}_m - \vec{x}_j)} \langle [S^x_m, S^y_j S^z_l - S^z_j S^y_l] \rangle$$

work out commutators:

$$[S^x_m, S^y_j S^z_l] = S^y_j [S^x_m, S^z_l] + [S^x_m, S^y_j] S^z_l$$

$$= -i\, S^y_j S^y_l \delta_{lm} + i\, S^z_j S^z_l \delta_{jm}$$

$$[S^x_m, S^z_j S^y_l] = S^z_j [S^x_m, S^y_l] = \cancel{i\, S^z_j S^z_l \delta_{lm}}$$

$$+ [S^x_m, S^z_j] S^y_l$$

$$= i\, S^z_j S^z_l \delta_{lm} - i\, S^y_j S^y_l \delta_{jm}$$

$$= \frac{i}{N} \sum_{jlm} J_{jl}\, e^{-i\vec{k}\cdot(\vec{x}_m - \vec{x}_j)} \langle (i\delta_{lm} (S^z_j S^z_l + S^y_j S^y_l)$$

$$- i\delta_{jm} (S^z_j S^z_l + S^y_j S^y_l)) \rangle$$

$$= \frac{i}{N} \sum_{jl} J_{jl}\, e^{-i\vec{k}\cdot(\vec{x}_l - \vec{x}_j)}\, i \langle (S^z_j S^z_l + S^y_j S^y_l) \rangle$$

$$- \frac{i}{N} \sum_{jl} J_{jl} \quad i \langle (S^z_j S^z_l + S^y_j S^y_l) \rangle$$

$$= N^{-1} \sum_{jl} J_{jl} [\cos \vec{k}\cdot(\vec{x}_j - \vec{x}_l) - 1] \langle S^y_j S^y_l + S^z_j S^z_l \rangle$$

$$F(\vec{k}) = h m_{\vec{a}}^{z} + \frac{1}{N} \sum_{je} J_{je} \left[\cos[\vec{k}\cdot(\vec{r}_j - \vec{r}_e)] - 1\right] \langle S_j^y S_e^y + S_j^z S_e^z \rangle \quad (10)$$

$$\leq h m_{\vec{a}}^{z} + \frac{|\vec{k}|^2}{2N} \sum_{je} |J_{je}|\, |\vec{x}_j - \vec{x}_e|^2 \, |\langle S_j^y S_e^y + S_j^z S_e^z \rangle$$

$$|\langle S_j^y S_e^y + S_j^z S_e^z \rangle| \leq |\langle \vec{S}_j \cdot \vec{S}_j \rangle| \leq S(S+1)$$

$$F(\vec{k}) \leq h m_{\vec{a}}^{z} + S(S+1)\bar{J}|k|^2$$

$$\Downarrow$$

Bogoliubov inequality

$$m_{\vec{a}}^{z\,2} \leq \beta\, S^{yy}(\vec{k}+\vec{a}) \left[h m_{\vec{a}}^{z} + S(S+1)\bar{J} k^2\right]$$

$$\Rightarrow \quad S^{yy}(\vec{k}+\vec{a}) \geq \frac{m_{\vec{a}}^{z\,2}}{\beta\left[h m_{\vec{a}}^{z} + S(S+1)\bar{J} k^2\right]}$$

$$\frac{1}{N} \sum_{\vec{k}} S^{yy}(\vec{k}+\vec{a}) = \frac{1}{N} \sum_{\vec{k}} \langle S^y_{-\vec{k}-\vec{a}} S^y_{\vec{k}+\vec{a}} \rangle$$

$$= \frac{1}{N} \sum_{\vec{k}} \sum_{ij} \langle S_i^y S_j^y \rangle e^{i(\vec{k}+\vec{a})(\vec{x}_i - \vec{x}_j)}$$

$$= \frac{1}{N} \sum_i \langle S_i^{y\,2} \rangle \leq S(S+1)$$

$$\Rightarrow \quad S(S+1) \geq \frac{m_a^{z\,2}}{\beta\left[h m_{\vec{a}}^{z} + S(S+1)\bar{J} k^2\right]}$$

$$S(S+1) \geq \frac{m_{a}^{+2}}{\beta} \frac{1}{(2\pi)^d} \int_0^{\bar{k}} \frac{k^{d-1}\, dk}{[h m_a^+ + S(S+1) \bar{J} k^2]}$$

$\bar{k} \to$ smaller than Brillouin zone edge vector

$d = 2$

$$\int_0^{\bar{k}} \frac{k\, dk}{A + B k^2}$$

$$= \frac{1}{2} \int_0^{\bar{k}^2} \frac{dx}{A + Bx} = \frac{1}{2B} \ln(A + Bx) \Big|_0^{\bar{k}^2} = \frac{1}{2B} \ln\left(\frac{A + B\bar{k}^2}{A}\right)$$

$$\Rightarrow \quad S(S+1) \geq \frac{m_a^{+2}}{2\beta\, S(S+1) \bar{J} (2\pi)^2} \ln\left[\frac{h m_a^+ + S(S+1) \bar{J} \bar{k}^2}{h m_a^+}\right]$$

as $h \to 0^+$ right hand side diverges

$d = 1$

$$S(S+1) \geq \frac{m_a^{+2}}{\beta 2\pi \sqrt{\bar{J} S(S+1) h m_a^+}} \operatorname{atan}\left[\bar{k} \sqrt{\bar{J} S(S+1) / (h m_a^+)}\right]$$

$\Rightarrow$ generalization: ~~con~~ these results hold not only for the Heisenberg model, but for models with continuous symmetry

- continuous symmetry:

suppose you rotate all spins in system, if you get the same energy, model has continuous symmetry