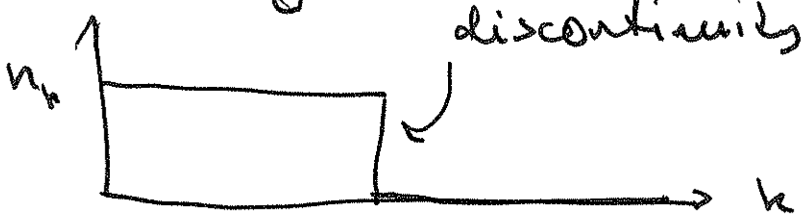
Peculiarities of 1D systems (from Giamarchi)

- before going into 1D review the theory valid for
  2D/3D (higher dimensions)
  $\Downarrow$
  Fermi liquid theory

Fermi liquid theory (on a qualitative level)

- in solids Coulomb interaction is comparable to kinetic energy => strictly speaking perturbation theory is not applicable
- Fermi liquid theory uses the system of non-interacting fermions as its starting point
  → non-interacting fermions at $T=0$
  n_k has a discontinuity
  n_k – momentum distribution
  $\Downarrow$
  discontinuity $Z=1$



excitations: adding particles with a well-defined momentum k at energy $\epsilon(k)$

energy levels: $\epsilon(k) = \dfrac{k^2}{2m}$

lifetime: infinite since they are eigenstates of the Hamiltonian

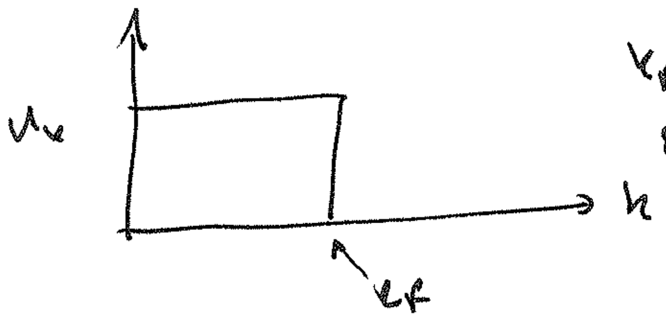
→ describe excitations by spectral function $A(k,\omega)$

→ probability to find a state with frequency ω and momentum k

for free electrons: $\delta(\omega - \xi(k)) = A(k,\omega)$

$$\xi(k) = \varepsilon(k) - \varepsilon_F \rightarrow \varepsilon_F = \frac{k_F^2}{2m}$$

k_F - where the discontinuity in the momentum distribution occurs



k_F - Fermi wave vector

ε_F = Fermi energy

- what happens when interactions are switched on?

Fermi liquid theory ⇒ properties of interacting system remain similar, with some crucial differences

- similarity: momentum distribution is discontinuous at k_F
- difference: near k_F the function itself is slightly different from non-interacting case

- another difference ⇒ distribution does not refer to particles anymore but so called quasi-particles ⇒ particles with density fluctuations around them
- electron surrounded by particle-hole excitations ⇒ quasi-particle

→ quasi-particles are considered essentially free
→ can formulate the theory using them in a manner similar to that of non-interacting particles

→ remark: this is only partially true even for Fermi liquid theory ⇒ interactions (residual) are considered in the complete theory, and contribute to quantities like susceptibilities

- still, many properties, and a general qualitative understanding can be gained by considering the one-particle distribution only

Z - discontinuity, corresponds to the fraction of electrons remaining in the quasi-particle state

- approximate description of energy levels

$$E(k) \simeq E(k_F) + \frac{k_F}{m^*}(k - k_F)$$

$m^* \rightarrow$ effective mass

$m^* = m_e \rightarrow$ non-interacting system

interactions can be represented by a change in the mass associated with single-particle states

- Luttinger theorem: Fermi momentum unchanged (volume enclosed by Fermi surface is an invariant)

- interactions can have another effect:

-> excitations will have a lifetime

time-propagation $e^{-iE_k t}$ $e^{-t/\tau}$

→ this is an approximate expression and it arises ~~as~~ as follows

if a particle is in an eigenstate of the Hamiltonian: time-development is described by $e^{-iE_k t}$

- if the state in which the particle is is not an eigenstate, then in general there will be matrix elements connecting the state with other states, and the particle can change states
- on average it will spend only some finite time in the original state, ~~→ this average~~ the more time passes, the more likely it jumps to another state → approximate all these processes by τ → relaxation time

$$e^{-iE_k t} \quad e^{-t/\tau}$$

- in Fermi liquid theory, one can prove (Landau) that the decay time becomes infinite as $E_k \to E_{k_F}$

⇒ near Fermi level τ can be assumed infinite

- in real systems Fermi level corresponds to ca. 10000K, so at temperatures of 300K the theory can be expected to be valid
  ⇒ quasi-particles are well defined excitations even if individual electrons are strongly coupled

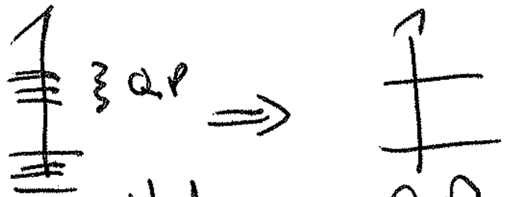
weight f → corresponds to discontinuity, and part of the system which is in quasi-particle states

$1-f$ ⇒ continuous background

- strength of Landau theory: not restricted to weak coupling
- two additional issues:
  - excitations can also be collective, e.g., zero sound, plasmon, etc.
  - quasiparticle states are not exact eigenstates
    - exact eigenstates are separated by smaller energy difference than $1/L$ ⇒
      ⇒ tend to cluster (bands)

    a quasiparticle state is usually made of a large number of exact eigenstates

(QP = quasi-particle)



real eigenstates   QP-eigenstates

1D: Failure of perturbation theory

- what about 1D?
- intuitively: when a particle moves in 1D, it will always disturb its neighbouring particles significantly (exception, if the potential is really short range, like Bethe ansatz solvable models)

→ excitations will not be nearly free QP excitations

→ they will always be collective and Fermi liquid theory will not work

formally: consider a perturbation $V(x,t)$

$$\Rightarrow \quad H' = \int dx \, V(x,t) \, \rho(x)$$

$\rho(x)$ - charge density

H' - corresponds to perturbation on Hamiltonian

susceptibility: $$\chi(q,\omega) = \frac{1}{\Omega} \sum_k \frac{f_F(\xi_k) - f_F(\xi_{k+q})}{\omega + \xi_k - \xi_{k+q} + i\delta}$$

$\delta = 0^+$ $\quad$ Ω - volume

static susceptibility $$\chi(q \to 0, \omega) = \frac{1}{\Omega} \sum_k \frac{f_F(\xi_k) - f_F(\xi_{k+q})}{\xi_k - \xi_{k+q}}$$

→ proportional to density of states at Fermi surface

- main contributions come from Fermi surface

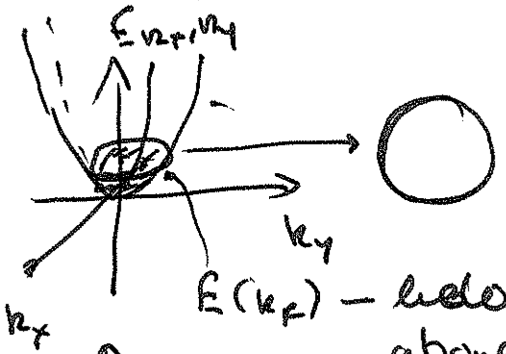
→ susceptibility depends on how energy levels vary with q

on Fermi surface: $\xi_k = 0$

can find Q s.t. $\xi_{k+Q} = 0$ also ⇒ ~~singu~~

χ becomes singular

in 2D: Fermi surface for a non-interacting system is a circle

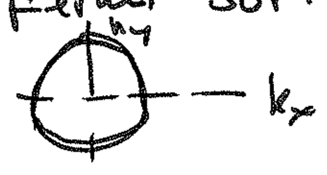


$\xi_{k+Q} = 0$ happens at a special set of points

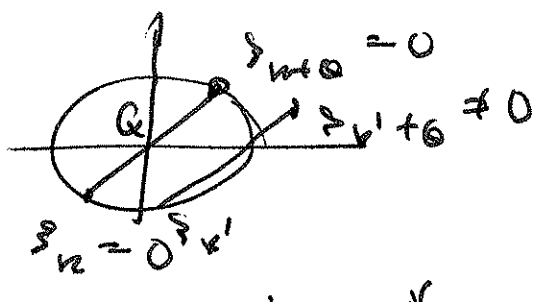
$E(k_F)$ – below – occupied states
above – unoccupied states

$E(k_x, k_y) = \frac{1}{2m}(k_x^2 + k_y^2)$

→ Fermi surface



example of Q



- but for any other pt. on Fermi surface

$$\xi_{k'} \to \xi_{k'+Q} \neq 0$$

⇒ $\chi(Q)$ will not be singular in 2D, since singularity occurs at ~~at~~ one point in the sum over k

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- stronger singularity occurs if <u>nesting</u> is satisfied

<u>nesting:</u> $\xi_{k+Q} = -\xi_k$

in this case

$$\chi(q) = \frac{1}{\Omega} \sum_k \frac{f_F(\xi_k) - f_F(\xi_{k+Q})}{\xi_k - \xi_{k+Q}}$$

$$= \frac{1}{\Omega} \sum_k \frac{f_F(\xi_k) - f_F(-\xi_k)}{2\xi_k}$$

$$f_F(\xi_k) = \frac{1}{1+e^{\beta\xi_k}} =$$

$$f_F(\xi_k) - f_F(-\xi_k) = \frac{1}{1+e^{\beta\xi_k}} - \frac{1}{1+e^{-\beta\xi_k}}$$

$$= -\frac{\sinh(\beta\xi_k)}{2+2\cosh(\beta\xi_k)} = -\frac{1}{2}\tanh\left(\frac{\beta\xi_k}{2}\right)$$

$$\chi(q) = -\frac{1}{2\Omega} \sum_k \frac{\tanh(\beta\xi_k/2)}{\xi_k}$$

$$\to -\int d\xi \, N(\xi) \frac{\tanh(\beta\xi/2)}{2\xi} \to -N(0)\log(E/T)$$

density of states

logarithmic singularity

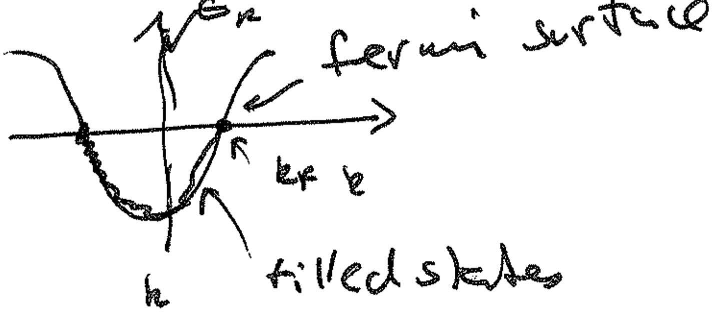
assumption $\Rightarrow$ $N(\xi)$ is constant near Fermi surface

temperature is low

in 1D nesting is the rule rather than exception

- fermi surface is two points

energy:



for $Q = 2k_F$ nesting property is satisfied, and since the fermi "surface" in 1D is just two points, the susceptibility will diverge

- linearize: $\xi(k) = v_F(k - k_F) \qquad k \sim k_F$

$\xi(k) = v_F(-k - k_F) \qquad k \sim -k_F$

$\xi(k + 2k_F) = -\xi(k)$

$\Rightarrow$ perturbation theory will diverge at $Q = 2k_F$

$\Downarrow$

ground state of the interacting system differs greatly from the ground state of the noninteracting system onto which perturbation theory is applied to obtain a theory of the interacting system

- BCS mean-field theory:

$$H_{pair} = \int dx\, \Psi(x,t)\, \Psi^{+}_{\uparrow}(x)\, \Psi^{+}_{\downarrow}(x) + H.c.$$

it arises by applying perturbation theory onto different sets of field operators than the usual mean-field theory:

example: Hubbard interaction:

$$H_U = U \int \rho_\uparrow(x)\, \rho_\downarrow(x)\, dx$$
$$= U \int c^{+}_{\uparrow}(x)\, c_{\uparrow}(x)\, c^{+}_{\downarrow}(x)\, c_{\downarrow}(x)\, dx$$

standard mean-field theory

$\bar{\rho}$ - average

$d\rho$ - deviation from average

$$\rho_\uparrow(x) = \bar{\rho}_\uparrow(x) + \delta\rho_\uparrow(x)$$
$$\rho_\downarrow(x) = \bar{\rho}_\downarrow(x) + \delta\rho_\downarrow(x)$$

$$\rho_\uparrow(x) = c^{+}_{\uparrow}(x)\, c_{\uparrow}(x)$$

BCS mean-field theory

write H_U as

$$H_U = U \int \underbrace{c^{+}_{\uparrow}(x)\, c^{+}_{\downarrow}(x)}\, c_{\downarrow}(x)\, c_{\uparrow}(x)\, dx$$

$\Downarrow$

~~write~~ apples mean-field ansatz to

$$c^{+}_{\uparrow}(x)\, c^{+}_{\downarrow}(x)$$

$$\Rightarrow \quad c^{+}_{\uparrow}(x)\, c^{+}_{\downarrow}(x) = \delta\left[c^{+}_{\uparrow}(x)\, c^{+}_{\downarrow}(x)\right] + \overline{c^{+}_{\uparrow}(x)\, c^{+}_{\downarrow}(x)}$$

results in a mean-field theory valid ~~for the~~ in which the Hamiltonian does not conserve particle number

⋮

susceptibility for BCS-type perturbation

$$\chi_{pair}(q=0,\omega) = \frac{1}{\Omega}\sum_k \frac{f(\xi_k) - f(-\xi_{-k})}{\omega - \xi_k - \xi_{-k} + i\delta}$$

$$\chi_{pair}(0,0) \sim N(\xi=0)\,\log(E/T)$$

logarithmic singularity still present, but sign of susceptibility changes

- mean-field treatment: makes assumption about ordered state $\Rightarrow$ anti-ferromagnetic, ferromagnetic, superconducting, etc.

- let's combine mean-field theory with standard perturbation theory, to obtain the effects on the susceptibility of both the perturbing field and the system interaction at the mean-field level

- mean-field theory (standard) applied to Hubbard term

$$H_U = U \int \rho_\uparrow(x) \rho_\downarrow(x)\, dx$$

$$\rightarrow \quad \rho_\uparrow(x) = \overline{\rho_\uparrow}(x) + \delta\rho_\uparrow(x)$$

$$H_U = U \int dx \, [\delta\rho_\uparrow(x) + \overline{\rho_\uparrow}(x)][\delta\rho_\downarrow(x) + \overline{\rho_\downarrow}(x)]$$

$$= U \int dx \, \delta\rho_\uparrow(x) \overline{\rho_\downarrow}(x) + U \int dx \, \overline{\rho_\uparrow}(x) \delta\rho_\downarrow(x)$$

$$+ U \int dx \, \overline{\rho_\uparrow}(x) \overline{\rho_\downarrow}(x)$$

substituting: $\delta\rho_\uparrow(x) = \rho_\uparrow(x) - \overline{\rho_\uparrow}(x)$

$$H_U = U \int dx \, \rho_\uparrow(x) \overline{\rho_\downarrow}(x) + U \int dx \, \rho_\downarrow(x) \overline{\rho_\uparrow}(x)$$

$$- U \int dx \, \overline{\rho_\uparrow}(x) \, \overline{\rho_\downarrow}(x)$$

mean-field potential experienced by up-spin electrons: $V^\uparrow_{MF}(x) = U \overline{\rho_\downarrow}(x)$

down-spin electrons: $V^\downarrow_{MF}(x) = U \overline{\rho_\uparrow}(x)$

- definition of susceptibility for non-interacting system

$$\overline{\rho_\uparrow(q,\omega)} = \chi^0(q,\omega) \, V^\uparrow_{ext}(q,\omega)$$

$V^\uparrow_{ext} \rightarrow$ ~~effective~~ external field acting on up-spin electrons

⇒ we can obtain a susceptibility which contains both the effects of the external field and the interaction at the mean-field level

$$\overline{\delta_\uparrow}(q,\omega) = \chi^0(q,\omega)\, V_{ext}^{\uparrow}(q,\omega)$$

effective potential: sum of external plus mean-field potentials

$$\overline{\delta_\uparrow}(q,\omega) = \chi^0(q,\omega)\left[V_{ext}^{\uparrow}(q,\omega) + U\,\overline{\delta_\downarrow(q,\omega)}\right]$$

$$\overline{\delta_\downarrow}(q,\omega) = \chi^0(q,\omega)\left[V_{ext}^{\downarrow}(q,\omega) + U\,\overline{\delta_\uparrow(q,\omega)}\right]$$

assume $V_{ext}^{\uparrow} = -V_{ext}^{\downarrow}$ (magnetic probe)

~~⇒ can be rearranged to~~

$$\overline{\delta_\uparrow} = \frac{\chi^0(q,\omega)\, V_{ext}^{\uparrow}(q,\omega)}{1 + U\chi^0(q,\omega)}$$

⇓

$$\chi(q,\omega) = \frac{\chi^0(q,\omega)}{1 + U\chi^0(q,\omega)}$$

susceptibility of the interacting system in terms of that of the non-interacting one can also apply same reasoning to pairing (BCS) mean-field theory

$$\chi_{pair} = \frac{\chi^0_{pair}}{1 + U\chi^0_{pair}}$$

but - recall that χ^0_{pair} has a different sign than χ^0 (of the particle-hole type theory)

~~when χ (either~~

<u>notation</u>: χ_{p-h} → refers to susceptibility from usual mean-field theory

χ_{p-p} → refers to susceptibility from BCS mean-field theory

(p-h → particle-hole / p-p → particle-particle)

compare:

$$\chi_{p-h} = \frac{\chi^0_{p-h}}{1 + U\chi^0_{p-h}}$$

$$\chi_{p-p} = \frac{\chi^0_{p-p}}{1 + U\chi^0_{p-p}}$$

χ^0_{p-h} and χ^0_{p-p} have opposite signs if $1+U\chi^0_{p-p}$ or $1+U\chi^0_{p-h}$ is zero then

χ_{p-p}, χ_{p-h} diverge, but since they have different signs this will only happen with one or the other

(note: BCS susceptibility is always divergent since $\varepsilon(k) = \varepsilon(-k)$ $\Rightarrow$ nesting due to time reversal symmetry)

$\cancel{\chi^0_{pair}}$ $\chi^0_{p-p} > 0$ $\Rightarrow$ transition for negative U (attractive) (χ_{p-p} divergence)

& in this case no transition in χ_{p-h}

$\chi^0_{p-p} < 0$ $\Rightarrow$ transition for positive U

results for high dimension

~~tran~~ transition for χ_{p-p} divergence $\Rightarrow$ BCS to normal state

transition for χ_{p-h} divergence $\Rightarrow$ paramagnet to ferro- or antiferromagnet

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nesting causes χ_{p-h} to always be divergent

1D: nesting always occurs $\Rightarrow$ we are in a sense always at a phase transition

also: RPA cannot be correct since in 1D there is usually no phase transition (for example for models with continuous symmetry, the Mermin-Wagner theorem holds $\Rightarrow$ no symmetry breaking at finite T

* this theorem will be demonstrated in the next lecture)

- quantum fluctuations only complicate the story

$\Rightarrow$ 1D systems exhibit behaviour which is like that of critical behaviour ~~and~~ in systems ~~with~~ in higher dimensions

- <u>nature of excitations in 1D:</u>

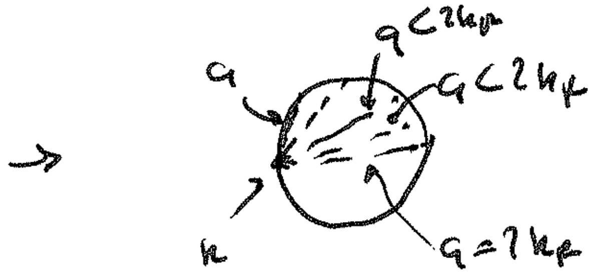
excitation: $\xi_{k+q} \leftrightarrow \xi_k \Rightarrow E_k(q) = \xi_{k+q} - \xi_k$

depends on both k and q

this is the form of a particle-hole excitation (particle removed from ξ_k promoted to ξ_{k+q})

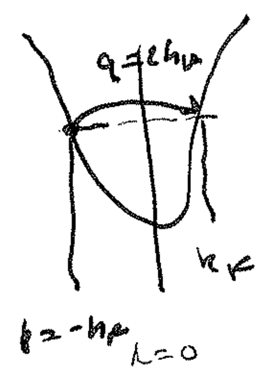
- consider $E_k(q)$ as a function of momentum (18)

in two-D can have zero energy excitations for any q $\quad 0 \le q \le 2k_F$

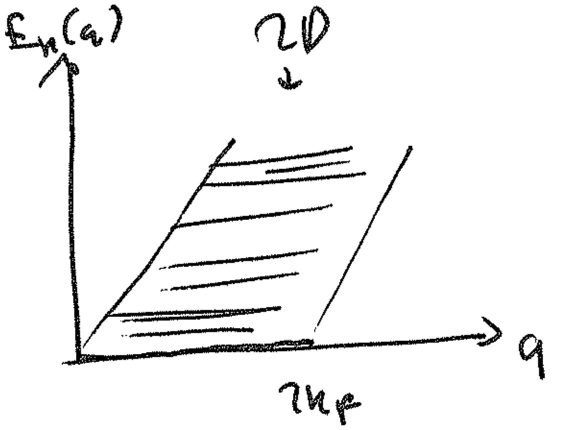


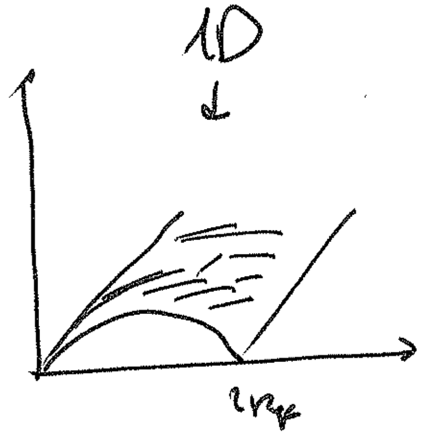
in 1D there are only two possible ~~zero~~ excitations with zero energy

$E_k(q) = \quad q=0$ or $2k_F$



plot excitations: $E_k(q)$





~~2D~~ ~~1D~~

can also calculate $E(q) \Rightarrow$ average excitation over all allowed k values for a particular q

$$E_k(q) = \frac{(k+q)^2}{2m} - \frac{k^2}{2m} = \frac{kq}{m} + \frac{q^2}{2m}$$

possible values of k for a particular q

$k \in [k_F - q, k_F]$

k has to be such that by adding q to k we reach at least k_F

largest value k_F

$$\rightarrow \quad E(q) = \frac{1}{q}\int_{k_F-q}^{k_F} E_k(q)\, dk = \frac{1}{\cancel{q}}\left[\frac{k^2 q}{2m} + \frac{kq^2}{2m}\right]_{k_F-q}^{k_F}$$

$$\frac{k_F^2}{2m} + \frac{k_F q}{2m} - \frac{(k_F-q)^2}{2m} - \frac{(k_F-q)q}{2m}$$

$$\cancel{\frac{k_F^2}{2m}} + \frac{k_F q}{\cancel{2}m} - \cancel{\frac{k_F^2}{2m}} + \frac{k_F q}{m} - \cancel{\frac{q^2}{2m}}$$

$$-\frac{k_F q}{\cancel{2}m} + \cancel{\frac{q^2}{2m}}$$

$E(q) = V_F\, q \qquad V_F = \frac{k_F}{m} \rightarrow$ fermi velocity

can calculate also the difference between maximum $E_k(q)$ and minimum $E_k(q)$

$$\delta E(q) = \frac{(k_F+q)^2}{2m} - \frac{k_F^2}{2m} - \frac{k_F^2}{2m} + \frac{(k_F-q)^2}{2m}$$

$$= \cancel{\frac{k_F q}{m}} + \frac{q^2}{2m} - \cancel{\frac{k_F q}{m}} + \frac{q^2}{2m} = \frac{q^2}{m} = \frac{E(q)^2}{V_F^2\, m}$$

→ in other words $v_F q$ is sharply peaked

for $q \to 0$

⇒ this property will be exploited in the theory

of bosonization