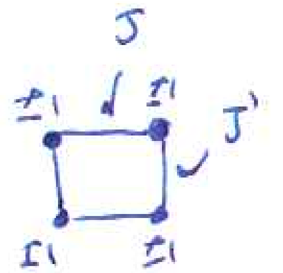
General Comments: 2D Models

- some can be solved: Ising type models without external field
- contain essential element for a physical model to describe phase transitions
  → short range interaction: vary temperature and encounter a phase transition (major change in macroscopic behaviour of system)
- insight into critical behaviour
- testing ground for scaling/universality hypotheses
  - square lattice Ising model ⇒ example for universality → $C \sim \ln|T-T_c|$ independent of ratio J/J'
    J - horizontal couplings
    J' - vertical couplings
  - exception to universality → 8-vertex model

Duality for square lattice Ising model

- can locate critical temperature of Ising model without actually solving it

- Ising model on square lattice $\mathcal{L}$
  - each site: spin $\sigma_i = \pm 1$



$$\mathcal{H} = \underbrace{- J \sum_{\langle ij \rangle} \sigma_i \sigma_j}_{\text{horizontal bonds}} \underbrace{- J' \sum_{\langle ij \rangle} \sigma_i \sigma_j}_{\text{vertical bonds}}$$

- no external field in this case
- partition function: $Z_N = \sum_{\sigma_1} \cdots \sum_{\sigma_N} \exp\left[\underbrace{K \sum \sigma_i \sigma_j}_{\text{horizontal}} + \underbrace{L \sum \sigma_i \sigma_j}_{\text{vertical}} \right]$

- critical pt. can be located by considering two equivalent representations of the partition function:
  - low-temperature representation
  - high-temperature representation

Low-Temperature Representation

N - lattice sites

also N horizontal bonds

N vertical bonds

let r be the number of <u>vertical</u> bonds on which the spins are anti-parallel

let s be the number of <u>horizontal</u> bonds on which the spins are anti-parallel

→ if all spins parallel (ground state) contribution to partition function: $2\exp[N(K+L)]$

$2 \rightarrow$ all up or all down

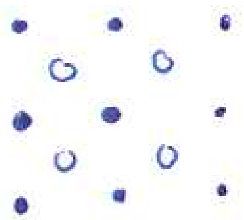
in this case $r = s = 0$

→ ~~to~~ in general $r \neq 0$ $s \neq 0$ $r \neq s$

⇒ contribution to partition function

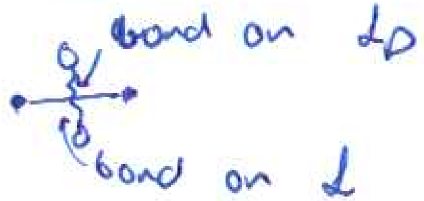
$$\exp[K(N-2r)+L(N-2s)]$$

⇒ introduce concept of <u>dual lattice</u>: $\mathcal{L}_D$



- sites of $\mathcal{L}_D$ are on faces formed by bonds of regular lattice

⇒ ~~if~~ consider the bond on $\mathcal{L}_D$ which crosses a particular bond on $\mathcal{L}$

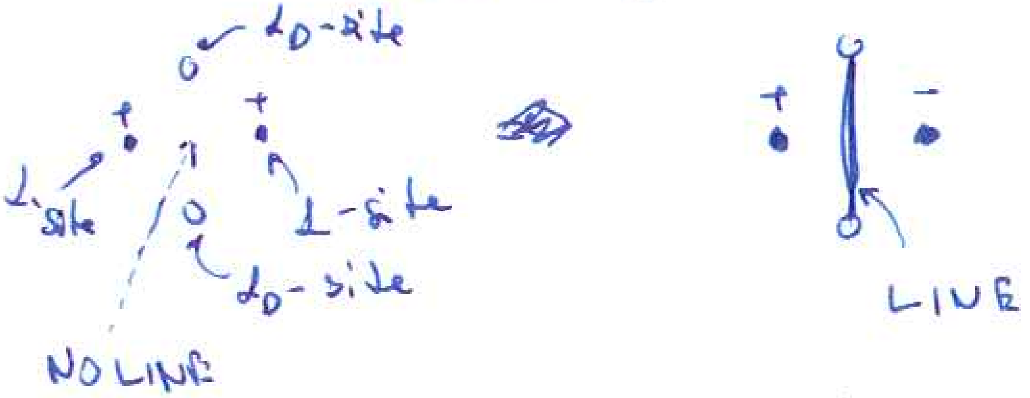
⇒ 

if spins on each end of bond are <u>equal</u>

→ <u>do not draw a line</u> on $\mathcal{L}_D$ bond crossing bond in $\mathcal{L}$

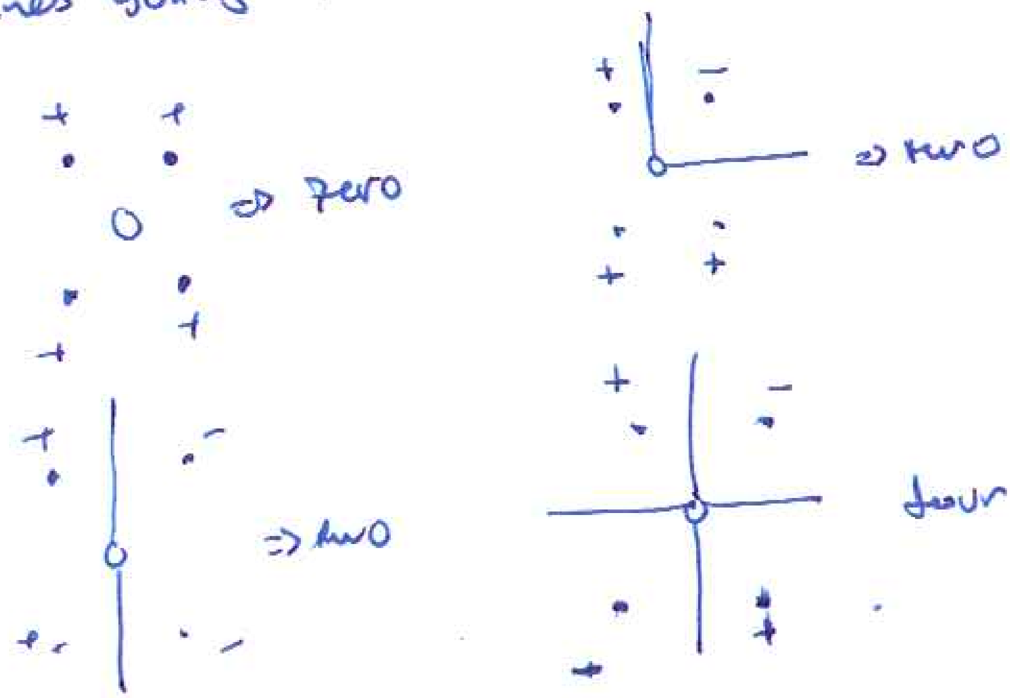
if spins on ends are <u>not equal</u>

→ draw a line on $\mathcal{L}_D$ bond crossing bond in $\mathcal{L}$



- do this for entire lattice!
  → there will be r- horizontal lines on $\mathcal{L}_D$
  s- vertical lines on $\mathcal{L}_D$

- into each site there must be an even number of lines going in [site here means $\mathcal{L}_D$]

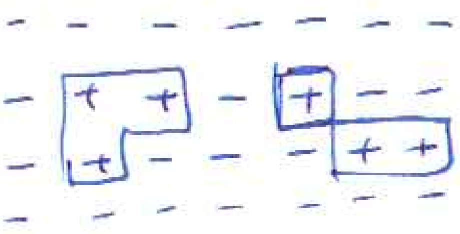


→ configurations consist of <u>closed polygons</u>

− polygons divide plane into up domains and down domains

partition function: $$\boxed{Z_N = 2\exp[N(K+L)] \sum_P \exp(-2Ks - 2Lr)}$$

$\sum_P$ − sum over all closed polygons



−, + − spins on $\mathcal{L}$

corners of polygons are sites of $\mathcal{L}_D$

− low-T representation (dominant term: $2\exp[N(K+L)]$)

− if all $\sum_P$ -sum is carried out, representation is exact

High-temperature representation

use: $\exp[K\sigma_i\sigma_j] = \cosh K + \sinh K\,\sigma_i\sigma_j$

$$Z_N = \cosh K^N \cosh L^N \sum_{\sigma_1}\cdots\sum_{\sigma_N} \underbrace{\prod_{\langle ij\rangle}(1+v\sigma_i\sigma_j)}_{\text{horizontal edges}} \underbrace{\prod_{\langle ij\rangle}(1+w\sigma_i\sigma_j)}_{\text{vertical edges}}$$

$v = \tanh K$

$w = \tanh L$

− expansion of product will lead to 2^{2N} terms

- terms in expansion will have either 1 or $v\sigma_i\sigma_j$ for horizontal bonds, either 1 or $w\sigma_i\sigma_j$ for vertical bonds

$\Downarrow$

graphical representation on $\mathcal{L}$

if $1 \to$ no line
if v or w term $\Rightarrow$ draw line

- term in expansion will look like:

$$v^r \sigma_1^{n_1} \dots \; w^s \sigma_1^{m_1} \dots$$

powers of σ's must be even numbers
if even one of the $n_1 \dots \; m_1 \dots$ is odd
then term is zero, since in partition function
it is summed over

$$\sum_{\sigma_j=-1}^{+1} \sigma_j^{n_j} = 0$$

$\Downarrow \qquad \Downarrow$

number of lines going into each lattice site is even $\Rightarrow$ again we get a sum over closed polygons

$$Z_N = 2^N \cosh^N K \cosh^N L \sum_P v^r w^s$$

$$v = \tanh K \qquad w = \tanh L$$

Duality → guessing critical point

free energy: $\Psi = -\frac{\ln Z_N}{N}$

in low-T $\quad Z_N = 2e^{N(K+L)} \sum_P e^{-2Ks-2Lr}$

in high-T $\quad Z_N = 2^N \cosh^N K \cosh^N L \sum_P v^r w^s$

$$\Psi(K,L) = -\frac{1}{N}\left(\ln 2 + K + L + \Phi(e^{-2L}, e^{-2K})\right)$$

$$= -\frac{1}{N}\left(N \ln 2 \cosh K \cosh L + \Phi(v,w)\right)$$

as $N \to \infty$ $\quad \frac{\ln 2}{N} \to 0$

define K^*, L^* ⇒ $\tanh K^* = e^{-2L} \quad \tanh L^* = e^{-2K}$

$$\Psi(K^*,L^*) = -\ln(2\cosh K^* \cosh L^*) + \Phi(e^{-2L}, e^{-2K})$$

$$\Psi(K^*,L^*) = -\ln(2\cosh K^* \cosh L^*) + \Psi(K,L) + K + L$$

$K, L \to$ large ⇒ $K^* L^* \to$ small ⇒ relates free energy between low-T and high-T

- write in more symmetric form

$$\sinh 2L = \frac{e^{2L} - e^{-2L}}{2} = \frac{1}{2}\left[\frac{1}{\tanh K^*} - \tanh K^*\right]$$

$$= \frac{1}{2}\left[\frac{e^{K^*}+e^{-K^*}}{e^{K^*}-e^{-K^*}} - \frac{e^{K^*}-e^{-K^*}}{e^{K^*}+e^{-K^*}}\right]$$

$$= \frac{1}{2}\left[\frac{4}{e^{2K^*}-e^{-2K^*}}\right] = \frac{1}{\sinh 2K^*}$$

⇒ $\sinh 2L \sinh 2K^* = 1 \;/\; \sinh 2K \sinh 2L^* = 1$

rewrite also free energies:

$$\Psi(K^*, L^*) = \Psi(K,L) + \ln \frac{e^{(K+L)}}{2\cosh K^* \cosh L^*}$$

$$= \Psi(K,L) + \ln \frac{1}{2\sqrt{\tanh K^* \tanh L^*}\,\cosh K^* \cosh L^*}$$

$$= \Psi(K,L) + \ln \frac{1}{2\sqrt{\sinh K^* \cosh K^* \sinh L^* \cosh L^*}}$$

$$= \Psi(K,L) + \frac{1}{2}\ln \sinh 2K \sinh 2L$$

- suppose $K = L$ / $K^* = L^*$

$$\Psi(K^*) = \Psi(K) + \frac{1}{2}\ln \sinh 2K$$

assuming there is one critical point K_c

at $K = K_c \Rightarrow \Psi(K)$ becomes nonanalytic

also at $K^* = K_c \Rightarrow \Psi(K^*)$ becomes nonanalytic

$$\Rightarrow \boxed{\sinh 2K_c = 1}$$

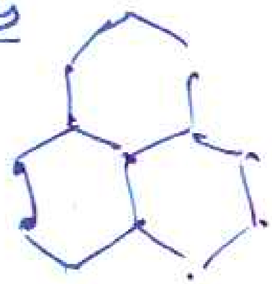
- for anisotropic case:

$$\boxed{\frac{1}{2}\ln \sinh 2K \sinh 2L = 0} \Rightarrow \text{line of critical points}$$

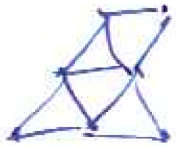
$$\boxed{\sinh 2K \sinh 2L = 1}$$

Honeycomb - triangular duality

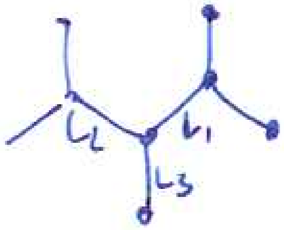
- honeycomb lattice



- triangular lattice



- consider honeycomb lattice on N sites
  - group edges into three types.



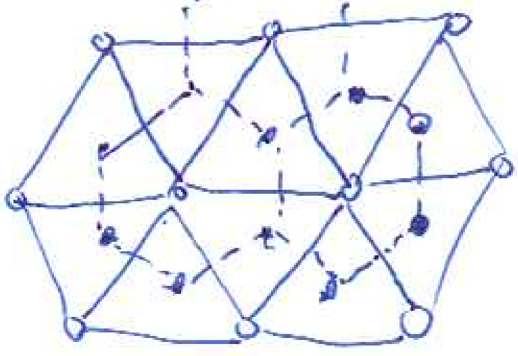
$$\Rightarrow Z_N^H(L) = \sum_{\sigma_1} \dots \sum_{\sigma_N} e^{L_1 \sum \sigma_i \sigma_j + L_2 \sum \sigma_i \sigma_j + L_3 \sum \sigma_i \sigma_j}$$

- consider triangular lattice on N sites
  - group edges into three types

$$\Rightarrow Z_N^T(K) = \sum_{\sigma_1} \dots \sum_{\sigma_N} e^{K_1 \sum_{\omega} \sigma_i \sigma_j + K_2 \sum_{\omega} \sigma_i \sigma_j + K_3 \sum_{\omega} \sigma_i \sigma_j}$$

- duality .. dual lattice of the honeycomb lattice is the hexagonal lattice
  - also dual in this case $\frac{1}{2}$ sites of original

honeycomb lattice



o - triangular lattice

• - hexagonal lattice

$$Z_{2N}^{H}(L) = \exp N(L_1+L_2+L_3) \sum_P \exp(-2r_1L_1 - 2r_2L_2 - 2r_3L_3)$$ 10

(result of low-temperature procedure on hexagonal lattice)

$$Z_N^T(K) = (2\cosh K_1 \cosh K_2 \cosh K_3)^N \sum_P v_1^{r_1} v_2^{r_2} v_3^{r_3}$$

$$v_i = \tanh K_i$$

$$\tanh K_i = e^{-2L_i}$$

$$\frac{e^{L_1}}{\cosh K_1} = \frac{1}{\sqrt{\sinh K_1 \cosh K_1}} = \frac{1}{\sqrt{\sinh 2K_1/2}}$$

$$Z_{2N}^{H}(L) = A^N Z_N^T(K)$$

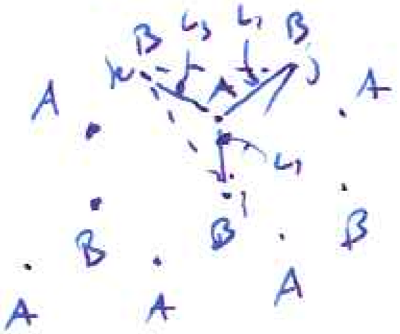
$$A = \frac{1}{2}\frac{e^{L_1}}{\cosh K_1}\frac{e^{L_2}}{\cosh K_2}\frac{e^{L_3}}{\cosh K_3} = \frac{1}{2}\frac{\sqrt{2}}{\sqrt{\sinh 2K_1 \sinh 2K_2 \sinh 2K_3}}$$

$$= \sqrt{2}(\sinh 2L_1 \sinh 2L_2 \sinh 2L_3)^{1/2}$$

- duality relation: not sufficient to guess critical temperature ⇒ must go between equivalent lattices

Star-Triangle Relation

- another relation between partition functions of the hexagonal and honeycomb Ising models
- honeycomb lattice is bi-partite: sites can be divided into A and B categories such that all A sites have only nearest neighbors in B and vice versa



$\Rightarrow$ two triangular lattices A, B

- can trace out one of the sublattices and end up with a triangular lattice
  - consider summand in the partition function of hexagonal lattice

$$Z_N^H(L) = \sum_{\sigma_1}\dots\sum_{\sigma_N}\prod_\ell e^{\sum^A \sigma_\ell (L_1\sigma_i + L_2\sigma_j + L_3\sigma_k)}$$

$$= \sum_{\sigma_1}\dots\sum_{\sigma_N}\prod_0^A 2\cosh(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k)$$

!! spins are all on B-sublattice

- can it be mapped onto triangular lattice?

$$w(\sigma_i\,\sigma_j\,\sigma_k) = 2\cosh(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k)$$

$$w'(\sigma_i\,\sigma_j\,\sigma_k) = R\,e^{K_1\sigma_j\sigma_k + K_2\sigma_i\sigma_k + K_3\sigma_i\sigma_j}$$

can they be set equal for particular values of K_1, K_2, K_3 given the values of L_1, L_2, L_3?

configurations

σ_i	σ_j	σ_k
1	1	1
1	1	-1
1	-1	1
-1	1	1

$\Rightarrow$ Flip all spins results in the same values for both w and w'

$\Downarrow$

four equations $\longleftrightarrow$ four unknowns (R, K_1, K_2, K_3)

$\Downarrow$

$$2\cosh(L_1+L_2+L_3) = R\,e^{K_1+K_2+K_3}$$

$$2\cosh(-L_1+L_2+L_3) = R\,e^{K_1-K_2-K_3}$$

$$2\cosh(L_1-L_2+L_3) = R\,e^{-K_1+K_2-K_3}$$

$$2\cosh(L_1+L_2-L_3) = R\,e^{-K_1-K_2+K_3}$$

$$c = \cosh(L_1+L_2+L_3)$$

$$c_i = \cosh(-L_i+L_j+L_k)$$

$$\frac{c\,c_1}{c_2c_3} = e^{4K_1}$$

$$\rightarrow \frac{c\,c_1 - c_2c_3}{c_2c_3} = e^{4K_1} - 1$$

$$\left[\frac{c\,c_1 - c_2c_3}{c_2c_3}\right] e^{-2K_1} = 2\sinh 2K_1$$

$$cc_1 = \frac{1}{4}\left[e^{L_1+L_2+L_3} + e^{-L_1-L_2-L_3}\right]\left[e^{-L_1+L_2+L_3} + e^{L_1-L_2-L_3}\right]$$

$$= \frac{1}{4}\left[e^{2L_2+2L_3} + e^{2L_1} + e^{-2L_1} + e^{-2L_2-2L_3}\right]$$

$$= \frac{1}{2}\left[\cosh(2L_2+2L_3) + \cosh 2L_1\right]$$

$$c_2c_3 = \frac{1}{4}\left[e^{-L_2+L_1+L_3} + e^{L_2-L_1-L_3}\right]\left[e^{-L_3+L_1+L_2} + e^{L_3-L_1-L_2}\right]$$

$$= \frac{1}{4}\left[e^{2L_1} + e^{2L_3-2L_2} + e^{2L_2-2L_3} + e^{-2L_1}\right]$$

$$= \frac{1}{2}\left[\cosh 2L_1 + \cosh(2L_2-2L_3)\right]$$

$$cc_1 - c_2c_3 = \frac{1}{2}\left[\cosh(2L_2+2L_3) - \cosh(2L_2-2L_3)\right]$$

$$= \frac{1}{2}\left[\cosh 2L_2 \cosh 2L_3 + \sinh 2L_2 \sinh 2L_3 - \cosh 2L_2 \cosh 2L_3 + \sinh 2L_2 \sinh 2L_3\right]$$

$$= \sinh 2L_2 \sinh 2L_3$$

$$\frac{\sinh 2L_2 \sinh 2L_3}{2\, c_2c_3}\, e^{-2K_1} = \sinh 2K_1$$

$$\frac{cc_1}{c_2c_3} = e^{4K_1} \qquad \sqrt{\frac{cc_1}{c_2c_3}} = e^{2K_1}$$

$$\rightarrow \boxed{\frac{\sinh 2L_1 \sinh 2L_2 \sinh 2L_3}{2\sqrt{cc_1c_2c_3}} = \sinh 2K_1 \sinh 2L_1}$$

$$\boxed{\sinh 2K_1 \sinh 2L_1 = \frac{1}{k}}$$

- can also permute K_1, K_2, K_3 and still obtain the same result $\Rightarrow$ since $\underline{k}$ is symmetric in indices 1, 2, 3

$$\sinh 2K_i \sinh 2L_i = \frac{1}{k} \qquad i = 1, 2, 3$$

- can also determine R

$$16 c c_1 c_2 c_3 = R^4 \qquad R^2 = 4\sqrt{c c_1 c_2 c_3}$$

$$R^2 = 2k \sinh 2L_1 \sinh 2L_2 \sinh 2L_3$$
$$= \frac{2}{k^2 \sinh 2K_1 \sinh 2K_2 \sinh 2K_3}$$