Quiz 1

1.) orthogonality theorem

$$a_j = \frac{1}{N} \sum_k N_k \chi_j(C_k) \chi(C_k)$$

A_1: $\frac{1}{24}[4\cdot1\cdot1 + 8\cdot1\cdot1 - 3\cdot1\cdot1 - 6\cdot1\cdot1 - 6\cdot1\cdot1] = 0$

A_2: $\frac{1}{24}[7\cdot1\cdot1 + 8\cdot1\cdot1 - 3\cdot1\cdot1 + 6\cdot1\cdot1 + 6\cdot1\cdot1] = 1$

E: $\frac{1}{24}[4\cdot2\cdot1 + 8\cdot1\cdot(-1) - 3\cdot2\cdot1 + 6\cdot0\cdot1 + 0\cdot1\cdot1] = 0$

T_1: $\frac{1}{24}[21 + 3] = 1$

T_2: $\cancel{0}\ \frac{1}{24}[21 + 3] = 1$

$\boxed{A_2 \text{ and } T_1, T_2}$

2.) two up-spins: Hamiltonian matrix elements

$\psi(n_1, n_2)$: satisfies

$n_1 + 1 < n_2 \Rightarrow \quad \psi(n_1+1, n_2) + \psi(n_1-1, n_2)$
$+ \psi(n_1, n_2+1) + \psi(n_1, n_2-1) = (E - 2\Delta)\psi(n_1, n_2)$

$$H \to \delta_{n_1' n_1+1}\delta_{n_2' n_2} + \delta_{n_1' n_1-1}\delta_{n_2' n_2} + \delta_{n_1' n_1}\delta_{n_2' n_2+1} + \delta_{n_1' n_1}\delta_{n_2' n_2-1} + 2\Delta\,\delta_{n_1' n_1}\delta_{n_2' n_2}$$

$n_1 + 1 = n_2 \qquad u(n_1, n_2+1) + u(n_1 - 1, n_2) + 2J\,u(n_1, n_1)$

$= E\,u(n_1, n_2)$

$$\delta_{n_1' n_1}\, \delta_{n_2' n_2+1} + \delta_{n_1' n_1 - 1}\, \delta_{n_2' n_2} + 2J\,\delta_{n_1' n_1}\, \delta_{n_2' n_2}$$

3. N sites twisted BC's

$$S^{\pm}_{N+1} = e^{\pm i\Phi} S_1^{\pm}$$

$$\frac{J}{2}\left[S_1^+ S_2^- + S_1^- S_2^+ + \dots e^{-i\Phi} S_N^+ S_1^- + e^{i\Phi} S_N^- S_1^+ \right]\psi$$

$$+ \cancel{2}J\left[S_1^z S_2^z + \dots\right]\psi = 2E\psi$$

1-up spin

solution assumed to be of form: $\cancel{e^{i(\Phi_n + k)n}}$ $a(n) = e^{ikn} e^{i\Phi_n}$

equation for each $a(n)$

$$a(2) + e^{i\Phi} a(N) = (2E - \cancel{2}J)\, a(1)$$

$$a(3) + a(1) = (2E - 2J)\, a(2)$$

$$\vdots$$

$$e^{-i\Phi} a(1) + a(N-1) = (2E - 2J)\, a(N)$$

$\cancel{e^{i(k+\Phi_n)}}$

$$e^{-i\Phi_2} e^{i2k} \cancel{e^{i\Phi}} + e^{i\Phi} e^{ikN} e^{-i\Phi_N} = (2E - 2J)\, e^{ik} e^{-i\Phi_1}$$

$$e^{i3k} e^{-i\Phi_3} + e^{ik} e^{-i\Phi_1} = (2E - 2J)\, e^{i2k} e^{-i\Phi_2}$$

$$e^{-i\Phi} e^{ik} e^{-i\Phi_1} + e^{ik(N-1)} e^{-i\Phi_{N-1}} = (2E - 2J)\, e^{iNk} e^{-i\Phi_N}$$

$$\Rightarrow \quad e^{i2k} e^{-i(\Phi_2 - \Phi_1)} + e^{ik(N-1)} e^{i\Phi} e^{-i(\Phi_N - \Phi_1)} = (2E - 2J)\cancel{e^{ik}}$$

$$e^{ik} e^{-i(\Phi_3-\Phi_2)} + e^{-ik} e^{-i(\Phi_2-\Phi_1)} = (2E-2J)$$

$$\vdots$$

$$e^{-i\Phi} e^{ik(1-N)} e^{i(\Phi_N-\Phi_1)} + e^{-ik} e^{-i(\Phi_{N-1}-\Phi_N)} = (2E-2J)$$

consistent iff $k = \frac{2\pi}{N} j$ j - integer

$$\Phi_n = \frac{\Phi}{N} n$$

$$\underbrace{e^{ik} e^{-i\frac{\Phi}{N}} + e^{-ik} e^{i\frac{\Phi}{N}} = 2E - 2J}_{\text{all three equations give this}}$$

$k \Rightarrow k - \frac{\Phi}{N}$ (momentum shifted!)

1)

$$\psi(n_1, n_2)$$

$$n_1 < n_2 \qquad n_1 + 1 \neq n_2$$

$$\psi(n_1-1, n_2) + \psi(n_1+1, n_2) + \psi(n_1, n_2+1) + \psi(n_1, n_2-1) + 4\Delta\psi(n_1, n_2) = 2E\,\psi(n_1, n_2)$$

$$n_1 + 1 = n_2$$

$$\psi(n_1-1, n_1+1) + \psi(n_1, n_1+2) + 2\Delta\psi(n_1, n_1+1) = 2E\,\psi(n_1, n_1+1)$$

continuity condition

$$\psi(n_1+1, n_1+1) + \psi(n_1, n_1) + 2\Delta\psi(n_1, n_1+1) = 0$$

From twisted B.C.'s we have $a(n) = e^{i(k+\frac{\varphi}{\nu})n}$

$$\psi(n_1, n_2) = A_{12}\, e^{i(k_1+\frac{\varphi}{\nu})n_1 + i(k_2+\frac{\varphi}{\nu})n_2} \cancel{+A_{21}}$$
$$+ A_{21}\, e^{i(k_1+\frac{\varphi}{\nu})n_2 + i(k_2+\frac{\varphi}{\nu})n_1}$$

$$A_{12}\left[e^{i(k_1+\frac{\varphi}{\nu}) + i(k_2+\frac{\varphi}{\nu})} + 1 + 2\Delta e^{i(k_2+\frac{\varphi}{\nu})}\right]$$
$$+ A_{21}\left[e^{i(k_1+\frac{\varphi}{\nu}) + i(k_2+\frac{\varphi}{\nu})} + 1 + 2\Delta e^{i(k_1+\frac{\varphi}{\nu})}\right] = 0$$

$$\cancel{2\left[e^{i\left[\frac{k_1+k_2}{2}+\frac{\varphi}{\nu}\right]} + e^{-i\left[\frac{k_1+k_2}{2}+\frac{\varphi}{\nu}\right]}\right] + 2\Delta\left[e^{i\frac{k_1-k_2}{2}} + e^{-i\frac{(k_1-k_2)}{2}}\right] = 0}$$

cos

$$\frac{A_{21}}{A_{12}} = -\frac{\cos\left(\frac{k_1+k_2}{2}+\frac{\varphi}{\nu}\right) + \Delta e^{i\frac{(k_2-k_1)}{2}}}{\cos\left(\frac{k_1+k_2}{2}+\frac{\varphi}{\nu}\right) + \Delta e^{-i\frac{(k_2-k_1)}{2}}}$$

$\Delta = 1$

$$\frac{A_{21}}{A_{12}} = -\frac{\cos\left(\frac{k_1+k_2}{2}+\frac{\Phi}{N}\right)+\cos\left(\frac{k_2-k_1}{2}\right)+i\sin\left(\frac{k_2-k_1}{2}\right)}{\cos\left(\frac{k_1+k_2}{2}+\frac{\Phi}{N}\right)+\cos\left(\frac{k_2-k_1}{2}\right)-i\sin\left(\frac{k_1-k_1}{2}\right)}$$

$$= -\frac{2\cos\left(\frac{k_1}{2}+\frac{\Phi}{2N}\right)\cos\left(\frac{k_2}{2}+\frac{\Phi}{2N}\right)\cancel{-\sin\left(\frac{k_1}{2}+\frac{\Phi}{2N}\right)\sin\left(\frac{k_2}{2}+\frac{\Phi}{2N}\right)} + i\sin\left(\frac{k_2}{2}+\frac{\Phi}{2N}\right)\cos\left(\frac{k_1}{2}+\frac{\Phi}{2N}\right)-i\sin\left(\frac{k_1}{2}+\frac{\Phi}{2N}\right)\cdot\cos\left(\frac{k_2}{2}+\frac{\Phi}{2N}\right)}{}$$

$$= -\frac{2+i\left[\tan\left(\frac{k_2}{2}+\frac{\Phi}{2N}\right)-\tan\left(\frac{k_1}{2}+\frac{\Phi}{2N}\right)\right]}{2+i\left[\tan\left(\frac{k_1}{2}+\frac{\Phi}{2N}\right)-\tan\left(\frac{k_2}{2}+\frac{\Phi}{2N}\right)\right]}$$

$$= -\frac{1+i(v_2-v_1)}{1+i(v_1-v_2)} \qquad v_i = \frac{1}{2}\tan\left(\frac{k_i}{2}+\frac{\Phi}{2N}\right)$$

P.B.C.: $\quad \psi(n_1,n_2) = \psi(n_2, n_1+N)$

$$A_{12}\, e^{i(k_1+\frac{\Phi}{N})n_1} e^{i(k_2+\frac{\Phi}{N})n_2} + A_{21}\, e^{i(k_1+\frac{\Phi}{N})n_2} e^{i(k_2+\frac{\Phi}{N})n_1}$$

$$= A_{12}\, e^{i(k_1+\frac{\Phi}{N})n_2} e^{i(k_2+\frac{\Phi}{N})n_1} e^{i(k_2+\frac{\Phi}{N})N} + A_{21}\, e^{i(k_1+\frac{\Phi}{N})n_1} e^{i(k_2+\frac{\Phi}{N})n_2} e^{i(k_2+\frac{\Phi}{N})N}$$

$$A_{12} = A_{21}\, e^{i(k_2+\frac{\Phi}{N})N}$$

$$\Rightarrow \quad e^{i(k_2+\frac{\Phi}{N})N}$$

express $e^{i(h_i+\frac{\Phi}{N})}$ in terms of rapidities

$$[i v_i] = \frac{e^{i(\frac{h_i}{2}+\frac{\Phi}{N})} - e^{-i(\frac{h_i}{2}+\frac{\Phi}{N})}}{e^{i(\frac{h_i}{2}+\frac{\Phi}{N})} + e^{-i(\frac{h_i}{2}+\frac{\Phi}{N})}}$$

$$i v_i = \frac{e^{i(h_i+\frac{\Phi}{N})} - 1}{e^{i(h_i+\frac{\Phi}{N})} + 1}$$

$$i v_i \left(e^{i(h_i+\frac{\Phi}{N})} + 1\right) = e^{i(h_i+\frac{\Phi}{N})} - 1$$

$$e^{i(h_i+\frac{\Phi}{N})}(i v_i - 1) = -i v_i - 1$$

$$e^{i(h_i+\frac{\Phi}{N})} = \frac{1+i v_i}{1-i v_i}$$

Bethe ansatz equations:

$$\left(\frac{1+i v_i}{1-i v_i}\right)^N = (-1)^{N-1} \prod_{j\neq i} \frac{1+i(v_j-v_i)}{1-i(v_j-v_i)}$$

$$\left(\frac{v_i-i}{v_i+i}\right)^N = \prod_{j\neq i}\left(\frac{v_i-v_j+i}{v_i-v_j-i}\right)$$

$$v_i = \frac{1}{2}\tan\left(\frac{h_i}{2}+\frac{\Phi}{2N}\right)$$

2.)

$$\langle \psi | X_j^{d\leftarrow d} | \psi \rangle$$

$$= \langle \psi | e^{-iS} e^{iS} X_j^{d\leftarrow d} e^{-iS} e^{iS} | \psi \rangle$$

$$= \langle \tilde{\psi} | e^{iS} X_j^{d\leftarrow d} e^{-iS} | \tilde{\psi} \rangle$$

$$= \langle \tilde{\psi} | \left[1 + iS - \frac{S^2}{2} \right] X_j^{d\leftarrow d} \left[1 - iS - \frac{S^2}{2} \right] | \tilde{\psi} \rangle$$

$$= \cancel{\langle \tilde{\psi} | X_j^{d\leftarrow d} | \tilde{\psi} \rangle}^{\,0} + i \cancel{\langle \tilde{\psi} | [S, X_j^{d\leftarrow d}] | \tilde{\psi} \rangle}^{\,0}$$

$$+ \underline{\langle \tilde{\psi} | S X_j^{d\leftarrow d} S | \tilde{\psi} \rangle}$$

$$-\frac{1}{u^2} \langle \tilde{\psi} | (\mathcal{H}_t^+ - \mathcal{H}_t^-) X_j^{d\leftarrow d} (\mathcal{H}_t^+ - \mathcal{H}_t^-) | \tilde{\psi} \rangle$$

$$= + \frac{1}{u^2} \langle \tilde{\psi} | \mathcal{H}_t^- X_j^{d\leftarrow d} \mathcal{H}_t^+ | \tilde{\psi} \rangle$$

$$= + \frac{1}{u^2} \sum_l \langle \tilde{\psi} | \left(X_j^{\downarrow\leftarrow d} X_l^{\uparrow\leftarrow 0} - X_j^{\uparrow\leftarrow d} X_l^{\downarrow\leftarrow 0} \right) X_j^{d\leftarrow d} \left(X_j^{d\leftarrow\downarrow} X_l^{0\leftarrow\uparrow} - X_j^{d\leftarrow\uparrow} X_l^{0\leftarrow\downarrow} \right) | \tilde{\psi} \rangle$$

$$= \frac{1}{u^2} \sum_l \langle \tilde{\psi} | \left[X_j^{\downarrow\leftarrow d} X_j^{d\leftarrow\downarrow} X_l^{\uparrow\leftarrow 0} X_l^{0\leftarrow\uparrow} + X_j^{\uparrow\leftarrow d} X_j^{d\leftarrow\uparrow} X_l^{\downarrow\leftarrow 0} X_l^{0\leftarrow\downarrow} - X_j^{\downarrow\leftarrow d} X_j^{d\leftarrow\uparrow} X_l^{\uparrow\leftarrow 0} X_l^{0\leftarrow\downarrow} - X_j^{\uparrow\leftarrow d} X_j^{d\leftarrow\downarrow} X_l^{\downarrow\leftarrow 0} X_l^{0\leftarrow\uparrow} \right] | \tilde{\psi} \rangle$$

$$= \frac{1}{u^2} \sum_l \langle \tilde{\psi} | X_j^{\downarrow\leftarrow\downarrow} X_l^{\uparrow\leftarrow\uparrow} + X_j^{\uparrow\leftarrow\uparrow} X_l^{\downarrow\leftarrow\downarrow} - X_j^{\downarrow\leftarrow\uparrow} X_l^{\uparrow\leftarrow\downarrow} - X_j^{\uparrow\leftarrow\downarrow} X_l^{\downarrow\leftarrow\uparrow}] | \tilde{\psi} \rangle$$

$$= \frac{2t^2}{u^2} \sum_l \langle \tilde{\psi} | \left(\frac{1}{4} - \vec{S}_j \cdot \vec{S}_l \right) | \tilde{\psi} \rangle$$

Quiz 3:

$$H = -t \sum_{\langle i,j\rangle \sigma} c_{i\sigma}^{\dagger} c_{j\sigma} + \text{H.c.} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

$$= -t \sum_i \sum_j^{(i)} \left[c_{i\sigma}^{\dagger} d_{j\sigma} + d_{j\sigma}^{\dagger} c_{i\sigma}\right]$$

$$+ U \sum_i m_{i\uparrow} m_{i\downarrow} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

antiferromagnetism:

interaction written as:

$$\cancel{U \sum_i (m_{i\uparrow} - m_{i\uparrow}) m}$$

$$U \sum_i m_{i\uparrow} \bar{m}_{i\downarrow} + U \sum_i m_{i\downarrow} \bar{m}_{i\uparrow} - U \sum_i \bar{m}_{i\uparrow} \bar{m}_{i\downarrow}$$

$$+ U \sum_i n_{i\uparrow} \bar{n}_{i\downarrow} + U \sum_i n_{i\downarrow} \bar{n}_{i\uparrow} - U \sum_i \bar{n}_{i\uparrow} \bar{n}_{i\downarrow}$$

average:

$$\bar{m}_{i\uparrow} = \frac{\bar{n}}{2}(1+\mu) \qquad \bar{n}_{i\uparrow} = \frac{\bar{n}}{2}(1-\mu)$$

$$\bar{m}_{i\downarrow} = \frac{\bar{n}}{2}(1-\mu) \qquad \bar{n}_{i\downarrow} = \frac{\bar{n}}{2}(1+\mu)$$

μ - order parameter

$$H = -t \sum_i \sum_j^{(i)} \left[c_{i\sigma}^{\dagger} d_{j\sigma} + d_{j\sigma}^{\dagger} c_{i\sigma}\right]$$

$$+ U \sum_i m_{i\uparrow} \frac{\bar{n}}{2}(1-\mu) + U \sum_i m_{i\downarrow} \frac{\bar{n}}{2}(1+\mu)$$

$$+ U \sum_i n_{i\uparrow} \frac{\bar{n}}{2}(1+\mu) + U \sum_i m_{i\downarrow} \frac{\bar{n}}{2}(1-\mu)$$

up spin part

$$-t \sum_i \sum_j^{(3)} \left(c_{i\uparrow}^{\dagger} d_{j\uparrow} + d_{j\uparrow}^{\dagger} c_{i\uparrow}\right]$$

$$+ U \sum_i m_{i\uparrow} \frac{\bar{n}}{2}(1-\mu) + U \sum_i m_{i\downarrow} \frac{\bar{n}}{2}(1+\mu$$

$$= \sum_{k\sigma} \left(c_{k\sigma}^{\dagger} \; d_{k\sigma}^{\dagger} \right) \begin{pmatrix} -\frac{U\mu\bar{n}}{2} & F(k) \\ F^{*}(k) & \frac{U\mu\bar{n}}{2} \end{pmatrix} \begin{pmatrix} c_{k\sigma} \\ d_{k\sigma} \end{pmatrix}$$

$$\left(-\frac{U\mu\bar{n}}{2} - \lambda\right)\left(\frac{U\mu\bar{n}}{2} - \lambda\right) - |F(k)|^2 = 0$$

$$-\left(\frac{(U\mu\bar{n})^2}{4} - \lambda^2\right) - |F|^2 = 0$$

$$\lambda^2 = \pm\sqrt{|F|^2 + \left(\frac{U\mu\bar{n}}{2}\right)^2}$$

~~ga~~ total energy for half filled system

$$E = -2\sum_{k} \sqrt{|F(k)|^2 + \frac{U^2\mu^2\bar{n}^2}{4}} + \cancel{t} \; 2UN\bar{n}$$

$$-UN\frac{\bar{n}^2}{2}(1-\mu^2)$$

gap equation:

$$\frac{\partial E}{\partial \mu} = 0 \Rightarrow -\sum_{k} \frac{1}{\sqrt{|F|^2 + \frac{U^2\mu^2\bar{n}^2}{4}}} \left(\frac{U^2\bar{n}^2}{2}\right)\cancel{\mu}$$

$$+ \frac{UN\bar{n}}{2}\cancel{\mu} = 0$$