Problem 5: Solutions

Problem 1: 5.1

$$a_1 = \frac{\sum y \sum x^2 - \sum x \sum xy}{S \sum x^2 - (\sum x)^2} \qquad a_2 = \frac{S \sum xy - \sum x \sum y}{S \sum x^2 - (\sum x)^2}$$

$$\frac{\partial a_1}{\partial y_i} = \frac{1}{\sigma_i^2} \, \frac{\sum x^2 - (\sum x) x_i}{S \sum x^2 - (\sum x)^2}$$

$$\left(\frac{\partial a_1}{\partial y_i}\right)^2 = \frac{1}{\sigma_i^4} \left(\frac{(\sum x^2) - (\sum x) x_i}{S \sum x^2 - (\sum x)^2}\right)^2$$

$$\sigma_{a_1}^2 = \sum_i \frac{1}{\sigma_i^2} \, \frac{(\sum x^2)^2 - 2 \sum x^2 (\sum x) x_i + (\sum x)^2 x_i^2}{(S^2 \sum x^2 - (\sum x)^2)^2}$$

$$= \cancel{S}\left[\frac{(\sum x^2)^2 S - 2(\sum x^2)(\sum x)^2 + (\sum x)^2 (\sum x^2)}{(S^2 \sum x^2 - (\sum x)^2)^2}\right]$$

$$\cancel{= S\left[(\sum x^2)\right](\sum x^2) -}$$

$$= \frac{(\sum x^2)\left[S(\sum x^2) - (\sum x)^2\right]}{\left[S(\sum x^2) - (\sum x)^2\right]^2} = \frac{\sum x^2}{\left[S(\sum x^2) - (\sum x)^2\right]}$$

$$\sigma_{a_1} = \sqrt{\frac{\sum x^2}{S \sum x^2 - (\sum x)^2}} \quad \checkmark$$

$$\frac{\partial a_2}{\partial y_i} = \frac{\frac{S x_i}{\sigma_i^2} - \frac{\sum x}{\sigma_i^2}}{S \sum x^2 - (\sum x)^2} = \frac{\frac{1}{\sigma_i^2}(S x_i - \sum x)}{S \sum x^2 - (\sum x)^2}$$

$$\left(\frac{\partial a_2}{\partial y_i}\right)^2 = \frac{1}{\sigma_i^4} \, \frac{(S x_i - \sum x)^2}{(S \sum x^2 - (\sum x)^2)^2} \quad \checkmark$$

$$\sigma_{a_2}^2 = \frac{S^2 \sum x^2 - 2S(\sum x)^2 + S(\sum x)^2}{(S \sum x^2 - (\sum x)^2)^2} = \frac{S}{S \sum x^2 - (\sum x)^2}$$

Problem 7: 8.2

$R(T) = a_1 + a_2 T$

for constant error

$$\sigma_{a_1} = \frac{\sigma_0}{\sqrt{N}} \sqrt{\frac{\langle x^2 \rangle}{\langle x^2 \rangle - \langle x \rangle^2}} \qquad \sigma_{a_2} = \frac{\sigma_0}{N} \sqrt{\frac{1}{\langle x^2 \rangle - \langle x \rangle^2}}$$

$a_1 \to \sigma_{a_1}$ smaller for ~~second~~ first scientist

$a_2 \to \sigma_{a_2}$ smaller for second scientist

~~1 is behind two, but near where~~ on second frame

1 is behind where 2 was on the first frame, but closer to 2 on the first frame than 2 on the second frame

5.16 $\quad$ 0 1 2 3

$y = [0, 1, 0, -1]$ $\qquad w = e^{\frac{i2\pi}{N}} \quad N = 4$

direct = $\qquad e^{i\frac{\pi}{2}}$

$$\tilde{y}_k = \frac{1}{\sqrt{N}} \sum_j w^{jk} y_j = \cancel{\frac{1}{\sqrt{4}}\left(e^{i\pi/2} - e^{i3\pi/2}\right)} = =$$

$$= \frac{1}{\sqrt{4}}\left(w^k - w^{3k}\right) = \frac{1}{\sqrt{4}}\left(e^{\frac{i2\pi k}{4}} - e^{\frac{i6\pi k}{4}}\right)$$

$$\tilde{y}_0 = 0$$

$$\tilde{y}_1 = \frac{1}{\sqrt{4}}\left(e^{i\pi/2} - e^{+i3\pi/2}\right) = \cancel{\frac{1}{\sqrt{4}}(-i)\left(e^{i\pi} - e\right)} \quad i$$

$$\tilde{y}_2 = \frac{1}{\sqrt{4}}\left(e^{i\pi} - e^{i3\pi}\right) = 0$$

$$\tilde{y}_3 = \frac{1}{\sqrt{4}}\left(e^{i\frac{3\pi}{2}} - e^{i\frac{4\pi}{2}}\right) = -i$$

FFT:

	j_1	j_0	Y_j
0	0	0	0
1	0	1	1
2	1	0	0
3	1	1	-1

$$\tilde{Y}(k_1, k_0) = \sum_{j_1 j_0} W^{(2j_1+j_0)(2k_1+k_0)} Y(j_1, j_0)$$

$$= \sum_{j_1 j_0} W^{2j_1k_0 + 2k_1j_0 + j_0k_0} Y(j_1, j_0)$$

$$= \sum_{j_1 j_0} W^{2k_0j_1 + (2k_1+k_0)j_0} Y(j_1, j_0)$$

$$= \sum_{j_0} W^{(2k_1+k_0)j_0} \underbrace{\sum_{j_1} W^{2k_0j_1} Y(j_1, j_0)}_{Y_1(k_0, j_0)}$$

$$W^2 = e^{\frac{i2\pi \cdot 2}{4}} = -1$$

$$Y_1(0,0) = 0 + 0 = 0$$

$$Y_1(0,1) = 1 + -1 = 0$$

$$Y_1(1,0) = 0 + W^2 \cdot 0 = 0$$

$$Y_1(1,1) = 1 + W^2(-1) = 2$$

$$\cancel{2\, W^{(2k_1+1)}\, 2}$$

$$\tilde{Y}(k_1, k_0) = \frac{1}{2} W^{(2k_1+k_0)} Y_1(k_0, 1)$$

$$= \frac{1}{2}$$

$$\boxed{\begin{aligned} \tilde{Y}(0,1) &= i \\ \tilde{Y}(1,1) &= -i \\ \tilde{Y}(0,0) &= 0 \\ \tilde{Y}(1,0) &= 0 \end{aligned}}$$