Problem Set 9: Solutions

1.)

energy: depends on B-field

B-field is finite in between two shells

$$\oint \vec{B}\cdot d\vec{l} = 2\pi r B(r) = \mu_0 I$$

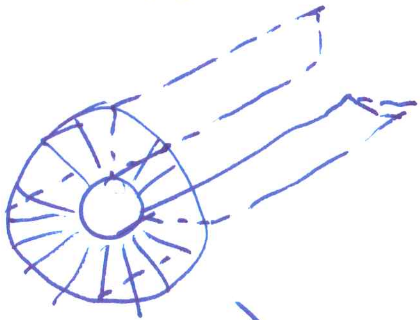
$$B(r) = \frac{\mu_0 I}{2\pi r}$$

energy density: $u(r) = \frac{\mu_0^2 I^2}{4\pi^2 r^2 \cdot 2\mu_0} \quad \left(= \frac{B^2(r)}{2\mu_0} \right)$

$$= \frac{\mu_0 I^2}{8\pi^2 r^2}$$

$$U/L = \cancel{2\pi} \int_R^{R'} \frac{\mu_0 I^2}{4\,8\pi^2 r} dr = \frac{\mu_0 I^2}{4\pi} \int_R^{R'} \frac{dr}{r} = \frac{\mu_0 I^2}{4\pi} \ln R'/R$$

2.) self-inductance



$$N\Phi = LI$$

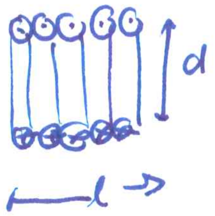
N remains constant (large $\to \infty$)

Φ increases

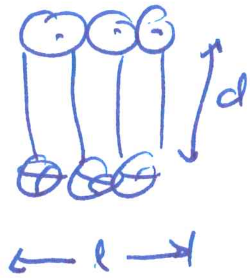
$\Downarrow$

L decreases

3.) a.) ①



②



③

$N\Phi = LI$

$NBA = LI$ $\qquad$ $Bl = \mu_0 N I$

$B = \frac{\mu_0 N I}{l}$

$$\boxed{\frac{N^2 \mu_0 \pi d^2}{l} = L}$$

① will have larger self-inductance

b.) time constant

L $\qquad$ $R = ?$ $\qquad$ $R = \cancel{\frac{\rho l}{A}}\ \frac{\rho d'}{A}$ $\qquad$ d' - length of wire in solenoid

A = area of wire

~~N windings => d =~~

for N windings: $d' = \frac{2\pi d}{2} \cdot N = N\pi d$

$A = \pi r^2 \Rightarrow r = \frac{l}{2N}$

$A = \frac{\pi l^2}{4N^2}$

$$R = 4\rho \frac{N^3 \pi d}{\pi l^2}$$

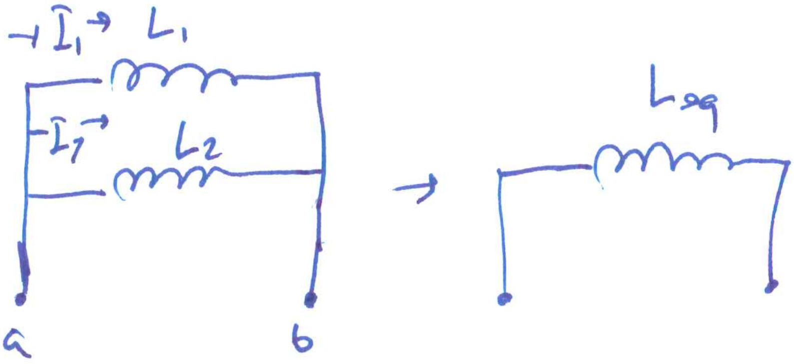
time constant

$\tau_L = L/R \Rightarrow$ proportional to $1/N$

larger time constant → fewer N

4.) series $L_{eq} = L_1 + L_2$ $\left(V_{ab} = L_1 \frac{dI}{dt} + L_2 \frac{dI}{dt} = (L_1 + L_2)\frac{dI}{dt} \right)$ 3

parallel



$$V_{ab} = L_{eq}\frac{dI}{dt} = L_1 \frac{dI_1}{dt} = L_2 \frac{dI_2}{dt} \Rightarrow \frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

~~$L_{eq} = L_1$~~

$$L_{eq}\frac{dI}{dt} = L_1\left(\frac{dI}{dt} - \frac{dI_2}{dt}\right) = L_2 \frac{dI_2}{dt}$$

$$(L_{eq} - L_1)\frac{dI}{dt} = -L_1 \frac{dI_2}{dt} = -\frac{L_1 L_{eq}}{L_2}\frac{dI}{dt}$$

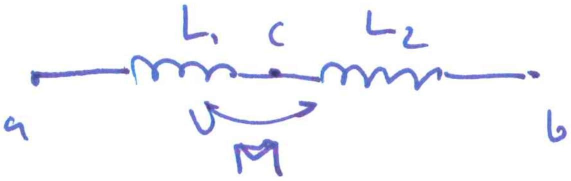
$$(L_{eq} - L_1)L_2 = -L_1 L_{eq}$$

$$L_{eq}(L_1 + L_2) = L_1 L_2$$

$$\Rightarrow L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$$

5.) mutual inductance not negligible:

series

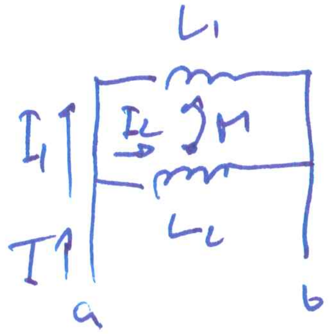


$$V_{ac} = L_1 \frac{dI}{dt} + M \frac{dI}{dt}$$

$$V_{cb} = L_2 \frac{dI}{dt} + M \frac{dI}{dt} \Rightarrow V_{ab} = (L_1 + L_2 + 2M)\frac{dI}{dt}$$

$$L_{eq} = L_1 + L_2 + 2M$$

paralel



4

$$V_{ab} = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

$$= L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = L_{eq} \frac{dI}{dt}$$

$$\frac{dI}{dt} = \frac{dI_1}{dt} + \frac{dI_2}{dt}$$

$$L_{eq} \frac{dI}{dt} = L_1 \frac{dI_1}{dt} + M \left(\frac{dI}{dt} - \frac{dI_1}{dt} \right)$$

$$L_{eq} \frac{dI}{dt} = L_2 \left(\frac{dI}{dt} - \frac{dI_1}{dt} \right) + M \frac{dI_1}{dt}$$

$$(L_{eq} - M) \frac{dI}{dt} = (L_1 \mp M) \frac{dI_1}{dt}$$

$$(L_{eq} - L_2) \frac{dI}{dt} = (M - L_2) \frac{dI_1}{dt}$$

$$\Rightarrow (L_{eq} - M) = \frac{(L_1 - M)(L_{eq} - L_2)}{(M - L_2)}$$

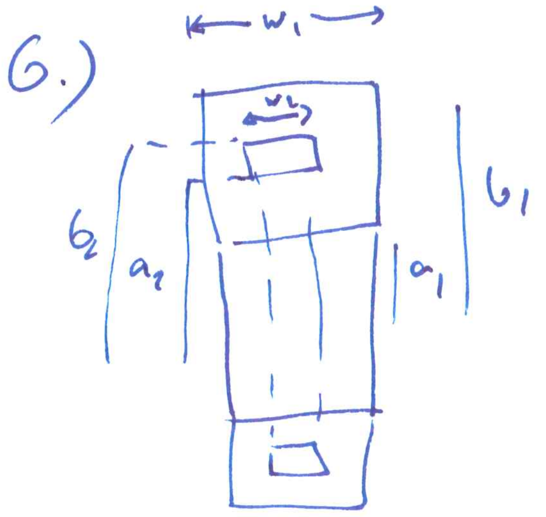
$$(M - L_2)(L_{eq} - M) = (L_1 - M)(L_{eq} - L_2)$$

$$L_{eq}(M - L_2) - M(M - L_2) = L_{eq}(L_1 - M) - L_2(L_1 - M)$$

$$L_{eq}(2M - L_2 - L_1) = M(M - \cancel{L_2}) - L_2(L_1 - \cancel{M})$$

$$L_{eq} = \frac{M^2 - L_1 L_2}{2M - L_1 - L_2}$$

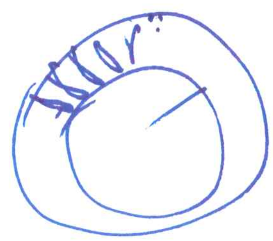
6.)



N_2, N_1

8

magnetic field inside rectangular toroidal solenoid



$\cancel{B_r 2\pi r} =$

$B(r)\, 2\pi r = \mu_0 N I$

$$B(r) = \frac{\mu_0 N I}{2\pi r}$$

inductance of solenoid 1

$$N_1 \Phi = L_1 I \qquad \Phi = w_1 \int_{a_1}^{b_1} \frac{\mu_0 N_1 I \, dr}{2\pi r} = \frac{\mu_0 N_1 w_1 I}{2\pi} \ln b_1/a_1$$

$$L_1 = \frac{N_1^2 \mu_0}{2\pi} w_1 \ln b_1/a_1$$

inductance of solenoid 2:

$$L_2 = \frac{N_2^2 \mu_0}{2\pi} w_2 \ln b_2/a_2$$

mutual inductance: $N_1 \Phi_2 = M I_2$

$$\frac{N_2 N_1 \mu_0 w_2 \ln b_2/a_2}{2\pi}$$

<u>for serial connection</u>: field inside adds in a simple way

for example inductance in solenoid 1

$$N_1 \Phi = N_1 (\underbrace{\Phi_{11}}_{\text{flux in 1 due to current in 1}} + \underbrace{\Phi_{12}}_{\text{flux in 1 due to current in 2}}) = L_1 \cancel{\tfrac{dI}{dt}} I$$

$$= N_1 \left(\frac{\mu_0 N_1}{2\pi} w_1 \ln b_1/a_1 \, I + \frac{\mu_0 N_2 w_2 \ln b_2/a_2 \, I}{2\pi} \right)$$

$$= (L_1 + M) I$$

inductance in solenoid 2

$N_2(\Phi_{22} + \Phi_{21}) = ?$ Φ_{22} - flux in 2 due to current in 2

Φ_{21} - flux in 2 due to current in 1

$$N_2 \frac{N_2 \mu_0 w_2 \ln b_2/a_2 \, I}{2\pi} + N_2 \frac{N_1 \mu_0 w_2 \ln b_2/a_2 \, I}{2\pi}$$

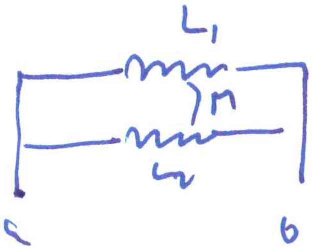
$$= (L_2 + M) I$$

parallel connection:

$$N_1(\Phi_{11} + \Phi_{12}) = \frac{N_1 \mu_0 N_1 w_1 \ln b_1/a_1}{2\pi} I_1 + \frac{N_1 \mu_0 N_2 w_1 \ln b_2/a_2}{2\pi} I_2$$

$$= L_1 I_1 + \cancel{L_2} M I_2$$

$$N_2(\Phi_{22} + \Phi_{21}) = L_2 I_2 + M I_1$$



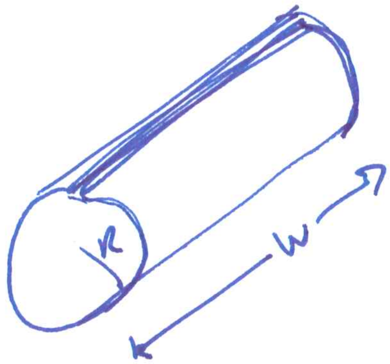
$$\rightarrow V_{ab} = \cancel{d}\, L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt}$$

$$= L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = L_{eq} \frac{dI}{dt}$$

$$I = I_1 + I_2$$

- subsequently derivation is the same as in problem 5

7.) sheet:



inductance

$N\Phi = LI$

$\Phi = ?$

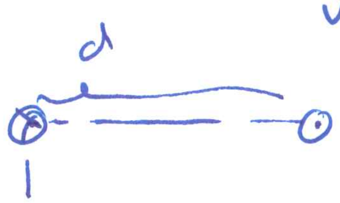
$B \cdot W = \mu_0 I$

$B = \frac{\mu_0 I}{W}$

flux = $\frac{\mu_0 I}{w}\pi R^2$

$L = \frac{\mu_0 \pi R^2}{w}$

8.)



$B(r)2\pi r = \mu_0 I$

$B(r) = \frac{\mu_0 I}{2\pi r}$

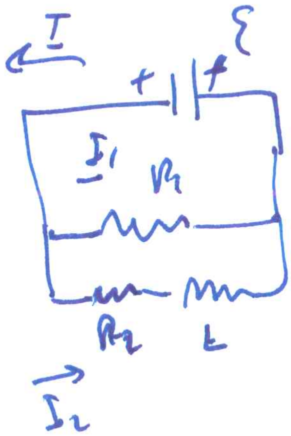
assume length ℓ

$$\Phi = 2L\int_{a/2}^{d-a/2} dr\,\frac{\mu_0 I}{2\pi r} = \frac{\cancel{2}L\mu_0 I}{\cancel{2}\pi}\ln\frac{d-a/2}{a/2} = \frac{\ell\mu_0 I}{\pi}\ln\left(\frac{2d-a}{a}\right)$$

$$\Phi = LI \Rightarrow L = \frac{\ell\mu_0 I}{\pi}\ln\left(\frac{2d-a}{a}\right)$$

inductance per unit length $\frac{L}{\ell} = \frac{\mu_0 I}{\pi}\ln\left(\frac{2d-a}{a}\right)$

9.)



$I = I_1 + I_2$

$\varepsilon - I_1R_1 = 0$

$\varepsilon - I_2R_2 - L\frac{dI_2}{dt} = 0$

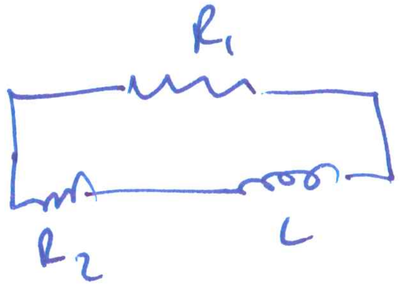
$\Downarrow$

$\frac{\varepsilon}{R_2} - I_2 - \frac{L}{R_2}\frac{dI_2}{dt} = 0$

$\frac{\varepsilon - I_2R_2}{\cancel{R_2}} = \frac{L}{R_2}\frac{dI_2}{dt} \Rightarrow \frac{R_2\,dt}{L} = \frac{dI_2}{\varepsilon/R_2 - I_2}$

answers to questions: (a) $\mathcal{E}$ (b) a (c) $\mathcal{E}$ ~~ab~~ c

open switch: only one loop



$$-IR_1 - L\frac{dI}{dt} - IR_2 = 0$$

$$-I(R_1+R_2) - L\frac{dI}{dt} = 0$$

$$-IR - L\frac{dI}{dt} = 0$$

$I(0) = \mathcal{E}/R_2$

$$I(t) = \frac{\mathcal{E}}{R_2} \cancel{e^{-tL/(R_1+R_2)}}\, e^{-t\frac{(R_1+R_2)}{L}}$$

answers to questions:

(e) $-\frac{\mathcal{E}R_1}{R_2}$ (f) b (g) $\frac{\mathcal{E}(R_1+R_2)}{R_2}$ (h) d

10.) assume coil is shorter

$N' < N$

mutual inductance $N'\Phi = MI$

$$\Phi = B\pi R^2$$

$$BL = \mu_0 NI$$

$$B = \frac{\mu_0 NI}{L}$$

$$M = \frac{N'N\mu_0}{L}$$

11.) current is I $V = IR$ $\frac{d\rho}{A} = R$ $E = \frac{I\rho}{A}$

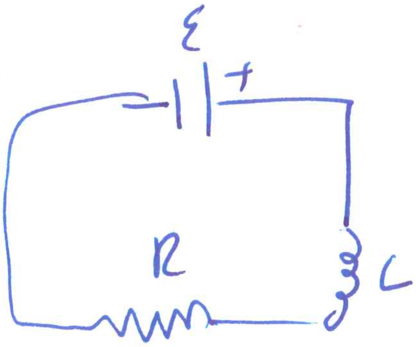
$$Ed = JAR$$

$$E = J\frac{AR}{d} = J\rho \qquad R = \frac{\rho d}{A} \qquad B = \frac{\mu_0 I}{2\pi a}$$

energy density electric: $\frac{E^2}{2\epsilon_0} = \frac{I^2\rho^2}{2\epsilon_0 A^2}$

energy density magnetic: $\frac{B^2}{2\mu_0} = \frac{\mu_0^2 I^2}{8\pi^2 a^2 \mu_0} = \frac{\mu_0 I^2}{8\pi^2 a^2}$

12.)



$$\mathcal{E} - L\frac{dI}{dt} - IR = 0$$

current:

$$\mathcal{E} - IR = L\frac{dI}{dt}$$

$$\frac{\mathcal{E}}{R} - I = \frac{L}{R}\frac{dI}{dt}$$

$$\frac{1}{\frac{\mathcal{E}}{R} - I} = \frac{dt}{dI}\frac{R}{L}$$

$$\frac{dI}{\mathcal{E}/R - I} = dt\left(\frac{R}{L}\right)$$

$$\frac{dI}{I - \mathcal{E}/R} = -dt\left(\frac{R}{L}\right)$$

$$\ln\frac{I - \mathcal{E}/R}{-\mathcal{E}/R} = -tR/L \Rightarrow \frac{I - \mathcal{E}/R}{-\mathcal{E}/R} = e^{-tR/L}$$

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-tR/L}\right)$$

energy stored in inductor $\frac{LI^2}{2}$

$$L\frac{\mathcal{E}^2}{R^2}\left(1 - e^{-tR/L}\right)^2$$