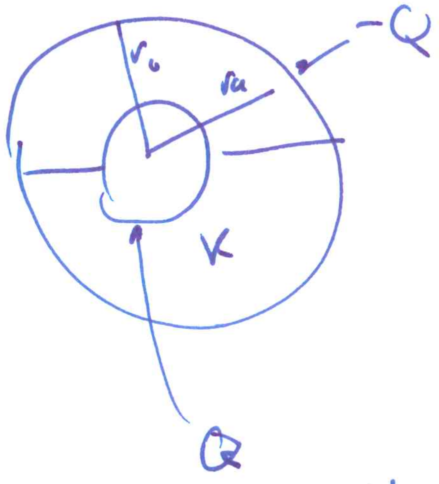
Problem Set 7: Solutions

1.)



a.) capacitance without dielectric:

$$C = \frac{Q}{V_{ab}} \qquad V_{ab} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$C = 4\pi\varepsilon_0\left(\frac{r_a r_b}{r_b - r_a}\right)$$

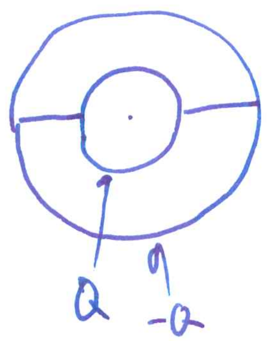
capacitance with dielectric:

$$C = \frac{Q}{V_{ab}} \qquad V_{ab} = \frac{Q}{4\pi\varepsilon} \cancel{\frac{r_a r_b}{r_b - r_a}} \left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$C = K 4\pi\varepsilon_0 \frac{r_a r_b}{r_b - r_a} =$$

half-filled $\Rightarrow$ $C = \dfrac{4\pi\varepsilon_0 r_a r_b}{(r_b - r_a)} \left(\dfrac{1+K}{2}\right)$

b.)



e-field: Q divides on surface so that overall σ is constant

$\sigma_i = (1 - \frac{1}{K})\sigma$

upper half $\Rightarrow \sigma$

cont upper half: σ (only free charges) ②
lower half: σ (free + bound charges)

~~Bom~~

$$\sigma_{free}^{upper} = \sigma_{free}^{lower} + \sigma_i^{lower}$$

$$\sigma_i^{lower} = \left(1 - \frac{1}{k}\right)\sigma_{free}^{lower}$$

$$\sigma_{free}^{upper} = \cancel{\sigma_{free}^{lower}} - \left(\cancel{1} - \frac{1}{k}\right)\sigma_{free}^{lower} = \frac{\sigma_{free}^{lower}}{K}$$

$$\sigma_{free}^{upper} + \sigma_{free}^{lower} = \sigma = \frac{Q 3}{4\pi\varepsilon_0 r_a^2}$$

$$\sigma_{free}^{upper} + \frac{K\sigma_{free}^{upper}}{\cancel{K}} = \sigma_{free}^{upper}\left(1 + \cancel{\frac{1}{K}}K\right) = \sigma = \frac{3Q}{4\pi\varepsilon_0 r_a^2}$$

$$\sigma_{free}^{upper} = \frac{3Q}{4\pi\varepsilon_0 r_a^2} \frac{\cancel{K}}{K+1}$$

$$Q' = \frac{Q\cancel{K}}{K+1}$$

electric field: $E(r) = \dfrac{Q\cancel{K}}{4\pi\varepsilon_0 r^2 (K+1)}$

c.) $\sigma_{free}^{upper} = \dfrac{3Q}{4\pi\varepsilon_0 r_a^2} \dfrac{1}{(K+1)}$

$\sigma_{free}^{lower} = \dfrac{3Q}{K 4\pi\varepsilon_0 r_a^2} \dfrac{K}{(K+1)}$

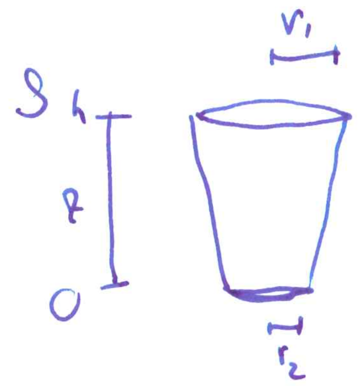
d.) inner surface: $\sigma_i = \left(1 - \dfrac{1}{K}\right) \dfrac{3Q}{4\pi\varepsilon_0 r_a^2 \cancel{K}} \dfrac{K}{(K+1)}$

outer surface: $\sigma_i = \left(1 - \dfrac{1}{K}\right) \dfrac{3Q}{4\pi\varepsilon_0 r_b^2} \dfrac{K}{K+1}$

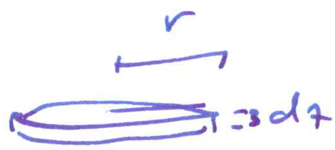
e) zero: $\vec{E}$-field is parallel to the surface, so it will not polarize in the direction perpendicular to the surface

3

2.) ρ



thin slice:



$A = \pi r^2$ r depends on z

$r = r(z)$

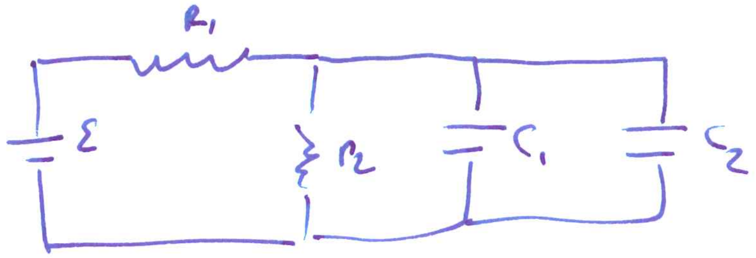
$$R = \frac{\rho d}{A} = \frac{\rho\, dz}{\pi\, r(z)^2}$$

$$r(z) = \left(\frac{r_1 - r_2}{h}\right) z + r_2 = Az + r_2 \qquad A = \frac{r_1 - r_2}{h}$$

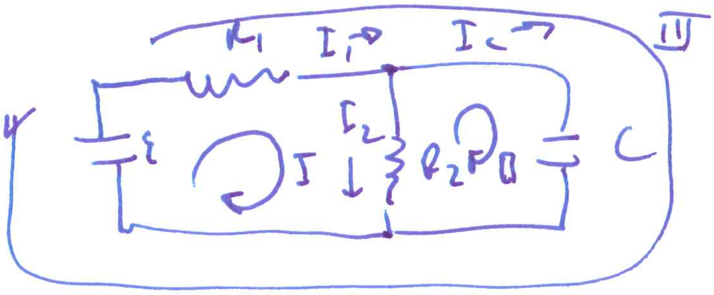
$$R = \int_0^h \frac{\rho\, dz}{\pi\,(Az + r_2)^2} = \frac{\rho}{\pi}\left(-\frac{1}{(Az + r_2)}\right)_0^h$$

$$= \frac{\rho}{\pi}\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

3.)



$\Downarrow$ parallel capacitors



$C = C_1 + C_2$

Kirchhoff: $I_1 = I_2 + I_C$

I) $\varepsilon - I_1 R_1 - I_2 R_2 = 0$

II) $\cancel{-R_2 I_2 + I_2 R_2 = 0} \rightarrow -\frac{Q}{C} + I_2 R_2 = 0$

III) $\varepsilon - I_1 R_1 - \frac{Q}{C} = 0$

from III: and from junction rule: (4)

$$\mathcal{E} - I_1 R_1 - \frac{Q}{C} = 0 \longrightarrow \mathcal{E} - I_2 R_1 - I_c R_1 - \frac{Q}{C} = 0$$

from II: $-\frac{Q}{C} + I_2 R_2 = 0 \Rightarrow I_2 = +\frac{Q}{R_2 C}$

$$\mathcal{E} - \frac{Q R_1}{C R_2} - I_c R_1 - \frac{Q}{C} = 0$$

$$\mathcal{E} - \frac{Q}{C}\left(\frac{R_1}{R_2} + 1\right) - I_c R_1 = 0$$

define $C' = \frac{C R_2}{R_2 + R_1}$

$$\mathcal{E} - \frac{Q}{C'} - I_c R_1 = 0 \qquad I_c = \frac{dQ}{dt}$$

$$\mathcal{E} - \frac{Q}{C'} - \frac{dQ}{dt} R_1 = 0$$

$$\mathcal{E} C' - Q - \frac{dQ}{dt} R_1 C' = 0$$

$$-\frac{dQ\, R_1 C'}{dt} = Q - \mathcal{E} C'$$

$$\frac{dQ}{Q - \mathcal{E} C'} = -\frac{dt}{R_1 C'}$$

$$\left|\frac{Q(t) - \mathcal{E} C'}{Q(0) - \mathcal{E} C'}\right| = e^{-t/R_1 C'}$$

$\mathcal{E} C'$ final charge on combined capacitor

$$Q(\infty) = \mathcal{E} C'$$

$$Q(\infty) = \frac{\mathcal{E}\, C\, R_2}{R_2 + R_1} = \frac{\mathcal{E}(C_1 + C_2) R_2}{R_2 + R_1}$$

Q_1 - charge on capacitor 1

$$Q_1 = \frac{Q}{1 + \frac{C_1}{C_2}} \qquad Q_2 = \frac{Q}{1 + \frac{C_2}{C_1}}$$

$$Q_1 = 18\,\mu C \Rightarrow$$

$$18 \times 10^{-6}\, C = \frac{Q}{1 + \frac{3}{6}} = \frac{2Q}{3}$$

$$\frac{3}{2} \times 18 \cancel{\times 10}\, \mu C = Q \qquad Q = 27\,\mu C$$

$$Q_2 = \frac{27\,\mu C}{1 + \frac{6}{3}} = 9\,\mu C$$

6.) $R_2 = ?$

$$Q(\infty) = \frac{\mathcal{E}(C_1 + C_2) R_2}{R_2 - R_1}$$

$$(27\,\mu C) = \frac{(60.0\,V)(9.00\,\mu F)(2.00\,\Omega)}{(2.00\,\Omega - R_1)}$$

$$(2.00\,\Omega - R_1) = \frac{(60.0)(9.00)(2.00)}{27}$$

$$2 + R_1 = 20 \cdot 2 = 40$$

$$R_1 = 38\,\Omega$$

25.87

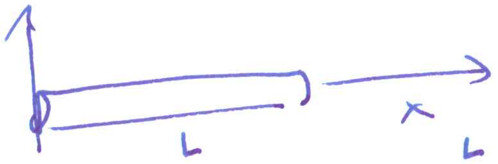
$\rho(x) = \rho_0 e^{-x/L}$

$E = \rho J$

$V = Ed = \rho J d$

$= \rho \frac{I d}{A} \Rightarrow R = \frac{\rho d}{A}$

a.)



total resistance: $R = \frac{\rho_0}{A}\int_0^L e^{-x/L}\,dx$

$R = \frac{\rho_0}{A} e^{-x/L}(-L)\Big|_0^L = \frac{\rho_0}{A} L\,(1 - e^{-1})$

current $\Rightarrow$ $I = \frac{V}{R} = \frac{VA}{\rho_0 L (1 - \frac{1}{e})}$

b.) $J = \frac{I}{A} = \frac{V}{\rho_0 L (1 - \frac{1}{e})}$

$\rho(x) = \rho_0 e^{-x/L}$

$\Rightarrow \quad E(x) = \rho(x) J = \frac{V}{\rho_0 L (1 - \frac{1}{e})} e^{-x/L}$

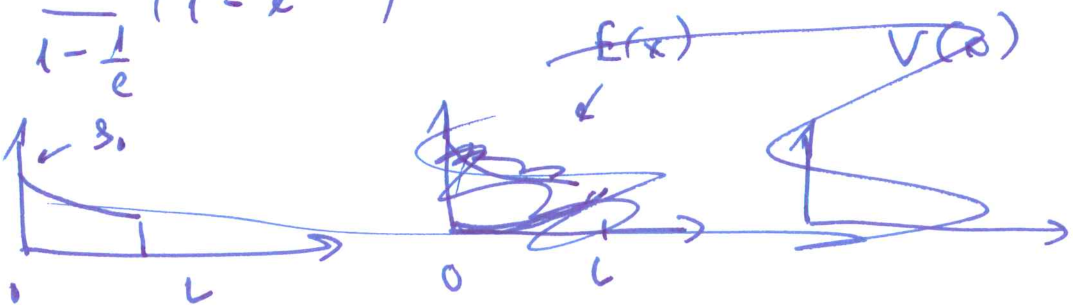
c.) $V(x) = -\int_0^x dx' E(x')$

$= -\int_0^x dx' \frac{V e^{-x/L}}{L(1-\frac{1}{e})} = \frac{-V}{L(1-\frac{1}{e})}\int_0^x dx' e^{-x'/L}$

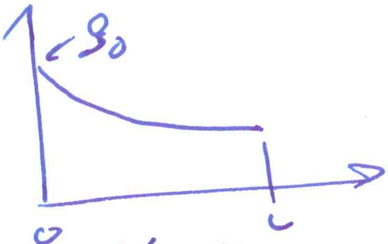
$= \frac{+V}{L(1-\frac{1}{e})} e^{-x'/L}(+L)\Big|_0^x$

$= \frac{V}{1-\frac{1}{e}}\left(1 - e^{-x/L}\right)$

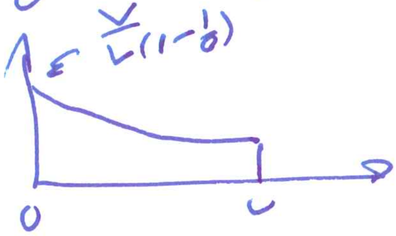
d.) $\rho(x)$



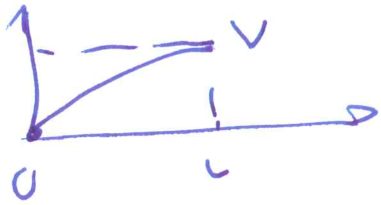
d.) $\rho(x)$



$E(x)$



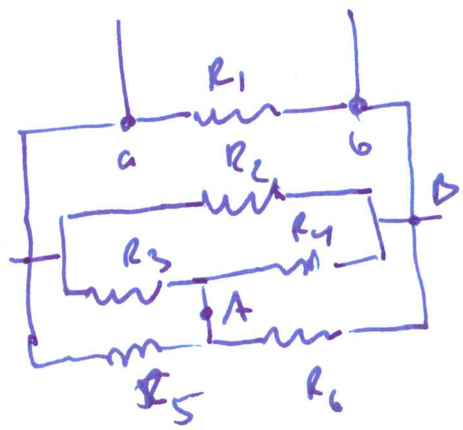
$V(x)$



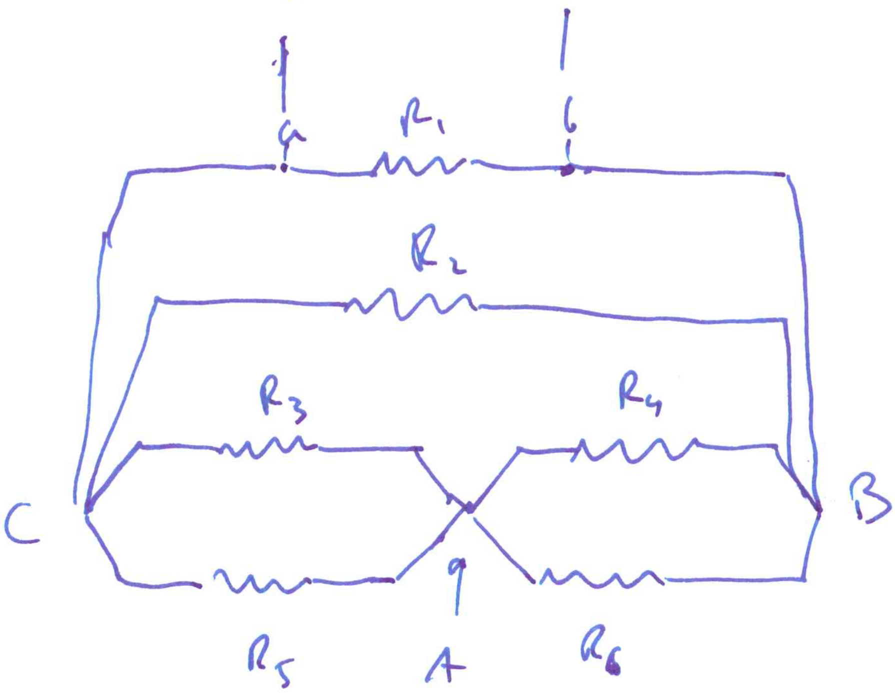
5. <u>26.61</u>

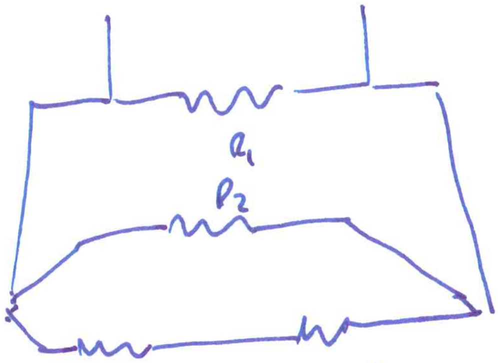
(a)

$R_1 = 100.0\,\Omega$
$R_2 = 50.0\,\Omega$
$R_3 = 75.0\,\Omega$
$R_4 = 25.0\,\Omega$
$R_5 = 40.0\,\Omega$
$R_6 = 50.0\,\Omega$



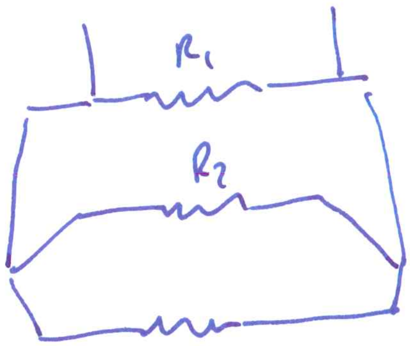
equivalent to





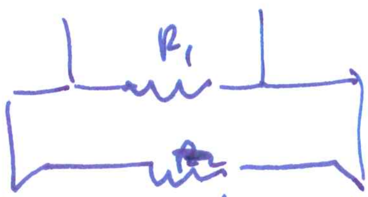
$$\frac{R_3 R_5}{R_3 + R_5} \qquad \frac{R_4 R_6}{R_4 + R_6} \rightarrow 26.1\,\Omega, \; 16.7\,\Omega$$

$\downarrow$



$$\frac{R_3 R_5}{R_3 + R_5} + \frac{R_4 R_6}{R_4 + R_6} = R' = 42.8\,\Omega$$

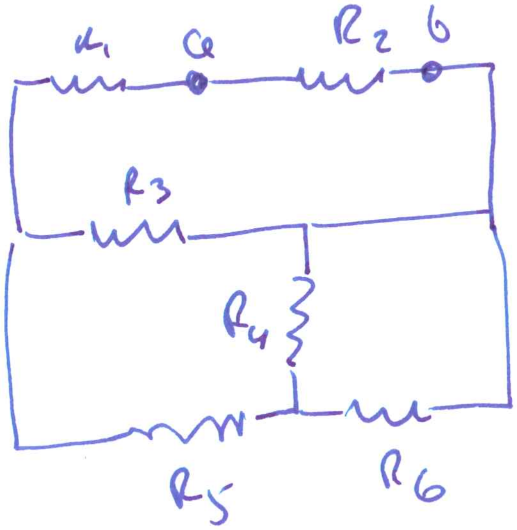
$\downarrow$



$$\frac{R_2 R'}{R_2 + R'} = R'' = 23.1$$

equivalent resistance: $\frac{R'' R_1}{R_1 + R''} = 18.7\,\Omega$

(b)



$R_1 = 7.00\ \Omega$

$R_2 = 10.00\ \Omega$

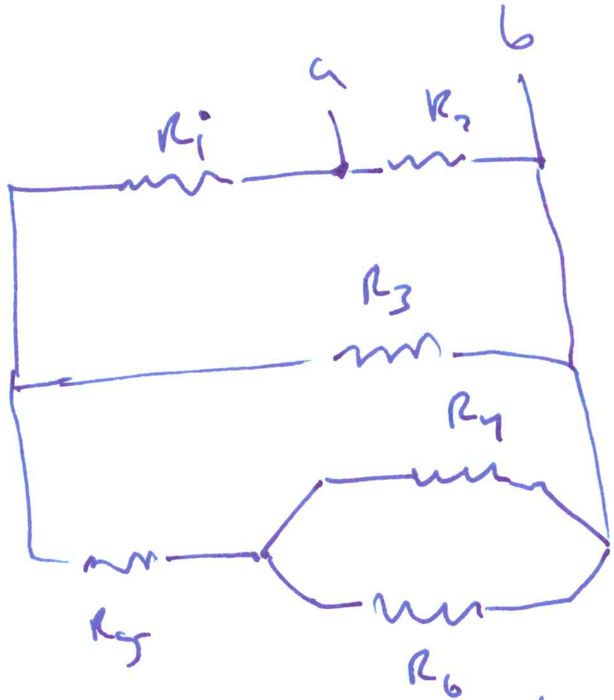
$R_3 = 60.00\ \Omega$

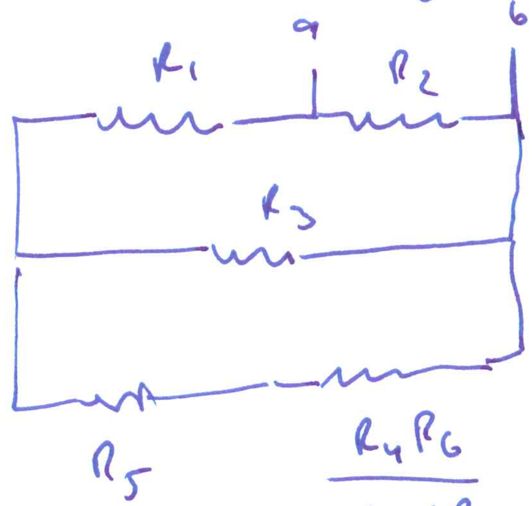
$R_4 = 30.00\ \Omega$

$R_5 = 20.0\ \Omega$

$R_6 = 45.0\ \Omega$

redraw



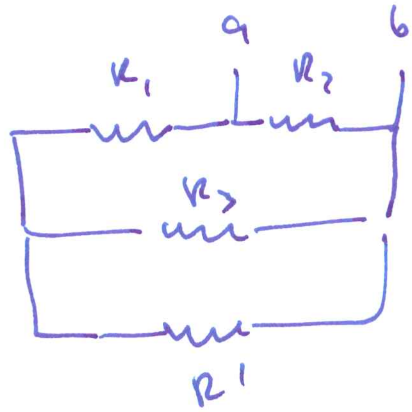


$$\frac{R_4 R_6}{R_4 + R_6}$$

$\downarrow$

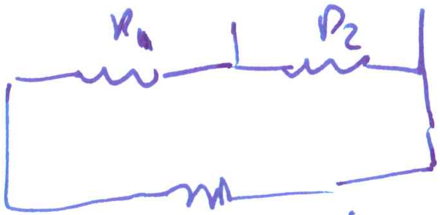
$18\ \Omega$

$\rightarrow$

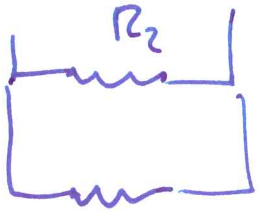


$$R' = R_5 + \frac{R_4 R_6}{R_4 + R_6}$$

$R' = 38.0\ \Omega$

→ 

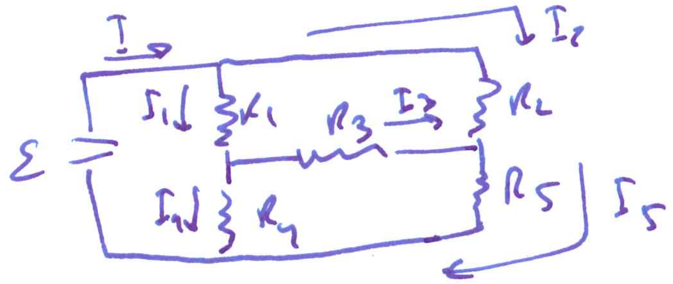
$$R'' = \frac{R'_1 R_3}{R' + R_3} = 23.3\ \Omega$$

→ 

$$R''' = R'' + R_1 = 30.3\ \Omega$$

equivalent resistance $\dfrac{R_2 R'''}{R_2 + R'''} = 7.5\ \Omega$

6.) 26.66



$R_1 = 1.00\ \Omega$
$R_2 = 2.00\ \Omega$
$R_3 = 1.00\ \Omega$
$R_4 = 2.00\ \Omega$
$R_5 = 1.00\ \Omega$
$\varepsilon = 14.0\ V$

Kirchhoff:
$I = I_1 + I_2$
$I_1 = I_3 + I_4$
$I_2 + I_3 = I_5$
$I_4 + I_5 = I$

Loops:
$\varepsilon - I_1 R_1 - I_4 R_4 = 0$
$\varepsilon - I_2 R_2 - I_5 R_5 = 0$
$-I_2 R_2 + I_3 R_3 + I_1 R_1 = 0$
$-I_3 R_3 - I_5 R_5 + I_4 R_4 = 0$
$\cancel{-I_2 R_2 - I_5 R_5 + I_4 R_4 + I_1 R_1 = 0}$ $\varepsilon - I_1 R_1 - I_3 R_3 - I_5 R_5 = 0$

$$\varepsilon - I_1 R_1 - (I_1 - I_3) R_4 = 0$$

$$\varepsilon - I_1 (R_1 + R_4) + I_3 R_4 = 0$$

$$\varepsilon - I_2 R_2 - I_2 R_5 - I_3 R_5 = 0$$

$$\varepsilon - I_2 (R_2 + R_5) - I_3 R_5 = 0$$

$$\varepsilon - I_1 (R_1 + R_4) + I_3 R_4 = 0$$

$$\varepsilon - I_2 (R_2 + R_5) - I_3 R_5 = 0$$

$$I_2 = \frac{I_1 R_1 + I_3 R_3}{R_2}$$

$$\varepsilon - I_1 (R_1 + R_4) + I_3 R_4 = 0$$

$$\varepsilon - I_1 \frac{R_1}{R_2} (R_2 + R_5) - \frac{I_3 R_3}{R_2} (R_2 + R_5) - I_3 R_5 = 0$$

$$\varepsilon - I_1 \frac{R_1}{R_2} (R_2 + R_5) - I_3 \left[\frac{R_3}{R_2} (R_2 + R_5) + R_5 \right] = 0$$

$$\varepsilon - A I_1 - B I_3 = 0$$

$$\varepsilon - C I_1 - D I_3 = 0$$

$$A = R_1 + R_4 = 3.00\,\Omega$$

$$B = R_4 = 2.00\,\Omega$$

$$C = \frac{R_1}{R_2} (R_2 + R_5) = 1.5\,\Omega$$

$$D = \left[\frac{R_3}{R_2} (R_2 + R_5) + R_5 \right] = 2.5\,\Omega$$

$$I_1 = \frac{\mathcal{E} - B I_3}{A} =$$

$$A\mathcal{E} - C\frac{\mathcal{E}}{\cancel{A}} + \frac{BC}{\cancel{A}} I_3 - AD I_3 = 0$$

$$(AD - BC) I_3 = \mathcal{E}(A - C)$$

$$I_3 = \mathcal{E}\frac{(A-C)}{AD-BC} = \frac{14.0\times(1.5)}{4.5} = \frac{14.0}{3.0} = 4.7\,A$$

$$I_1 = \frac{\mathcal{E}}{A} - \frac{\mathcal{E}B}{A}\frac{(A-C)}{(AD-BC)} = \frac{14}{3} - \frac{2}{3}\times\frac{14}{3} = \frac{14}{9}\,A$$

$$\cancel{\frac{7}{3}A = 2.3\,A}$$

$$I_4 = I_1 - I_3$$

$$I_4 = \cancel{-2.4\,A}\quad \frac{14}{9} - \frac{14}{3} = -\frac{28}{9}\,A$$

$$I_2 = \frac{I_1 R_1 + I_2 R_3}{R_2} = \frac{\frac{14}{9} + \frac{14}{3}}{2} = \frac{7}{9} + \frac{7}{3} = \frac{28}{9}\,A$$

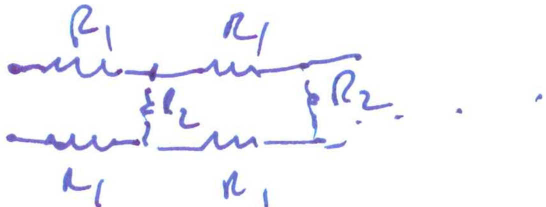
$$I_5 = \frac{28}{9} + \frac{14}{3} = \frac{28}{9} + \frac{42}{9} = \frac{70}{9}\,A$$

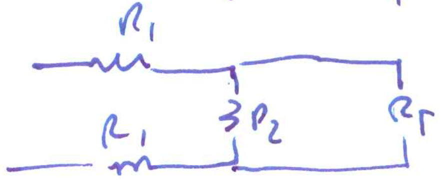
$$I = \frac{42}{9}\,A$$

(b) $\mathcal{E} = IR \Rightarrow \frac{14}{42}\cdot 9 = \frac{2\cdot 9}{6} = \frac{9}{3} = 3\,\Omega$

1

26.91





$$R_T = 2R_1 + \frac{R_2 R_T}{R_2 + R_T}$$

$$\Rightarrow R_T^2 + \cancel{R_2 R_T} = 2R_1(R_2 + R_T)$$

$$R_T^2 - 2R_1R_T - 2R_1R_2 = 0$$

$$R_T = \frac{2R_1 \pm \sqrt{4R_1^2 + 8R_1R_2}}{2}$$

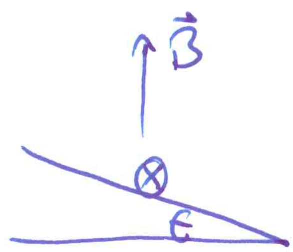
$$= R_1 \pm \sqrt{R_1^2 + 2R_1R_2}$$

cannot be minus since resistance cannot be negative!

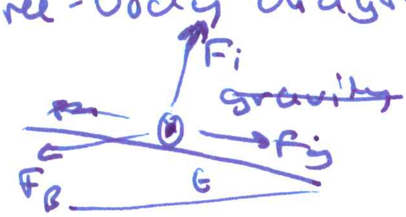
$$R_T = R_1 + \sqrt{R_1^2 + 2R_1R_2}$$

8.

27.69



free-body diagram



mass M

length L

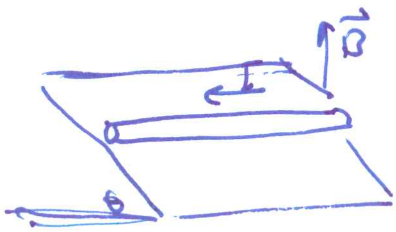
$$F_B = ILB$$

$$F_B\cos\theta = F_g = Mg$$

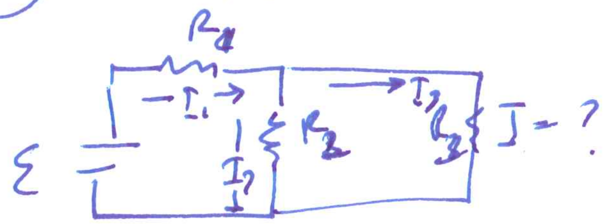
$$\Rightarrow ILB\cos\theta = Mg$$

$$I = \frac{\cancel{LB}Mg}{LB\cos\theta}$$

direction of current



9. 2M.70



$R_1 = 25.0\,\Omega$

$R_2 = 10.0\,\Omega$

$R_3 = 10.0\,\Omega$

$\mathcal{E} = 120.0\,V$

$$I_1 = I_2 + I_3$$

$$\mathcal{E} - I_1 R_1 - I_2 R_2 = 0$$

$$\mathcal{E} - I_1 R_1 - I_3 R_3 = 0$$

$$\mathcal{E} - I_1 R_1 - I_1 R_2 + I_3 R_2 = 0$$

$$\mathcal{E} - I_1 R_1 - I_3 R_3 = 0$$

$$\mathcal{E} - A I_1 - B I_3 = 0$$

$$\mathcal{E} - C I_1 - D I_3 = 0$$

$A = R_1 + R_2 = 35.0\,\Omega$

$B = -R_2 = -10.0\,\Omega$

$C = R_1 = 25.0\,\Omega$

$D = R_3 = 10.0\,\Omega$

$$I_1 = \frac{\mathcal{E} - D I_3}{C}$$

$$\mathcal{E} - \frac{A\mathcal{E}}{C} + \frac{AD I_3}{C} - B I_3 = 0$$

$$\mathcal{E}C - A\mathcal{E} + (AD - BC) I_3 = 0$$

$$\boxed{I_3 = \cancel{AD}\ \frac{(A-C)\mathcal{E}}{AD - BC}}$$

$$I_3 = \frac{120.0 \times 10.0}{350 + 250} = \frac{12\cancel{0} \times 1\cancel{0}}{6\cancel{0}\cancel{0}} = 2\,A$$

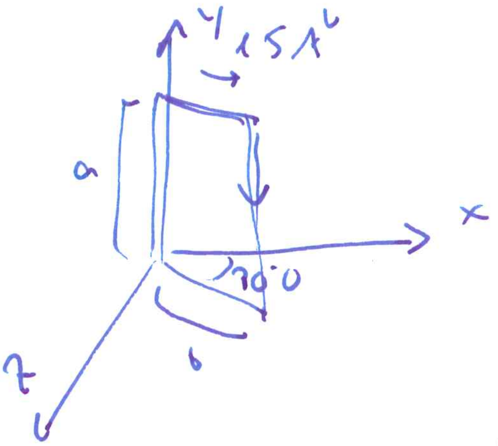
$$|\vec{F}_B| = |I\,\vec{L} \times \vec{B}| = \cancel{ILB}\ ILB$$

$$= 2 \times 1.50 \times 1.60 = ma$$

$m = \frac{2.60}{g} = \frac{2.60}{9.8} = .26$

$a = \frac{4.8}{.26} = 18.5 \frac{m}{s^2}$

10.) 27.78



$I = 15.0 A$

$a = .08 m$

$b = .06 m$

(a) $B = .48T$ $\vec{B} = (B_0, 0, 0)$

torque originates from outer vertical segment

$\vec{F} = I \vec{L} \times \vec{B}$

$F = I L B = 15.0 \cdot (.08)(.48)$

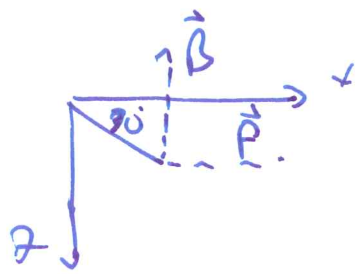
$\vec{\tau} = \vec{r} \times \vec{F} \Rightarrow \tau = rF \sin\theta$

$\tau = bF \sin 60$

$\tau = .03 \, Nm$

direction $\otimes$

(b) Field in $-z$ direction



$\tau = IaBb \sin 30 = .0178 \, Nm$

direction $\odot$

~~28~~

11.) 28.60

$q_1 = 4.80 \times 10^{-6}$ C

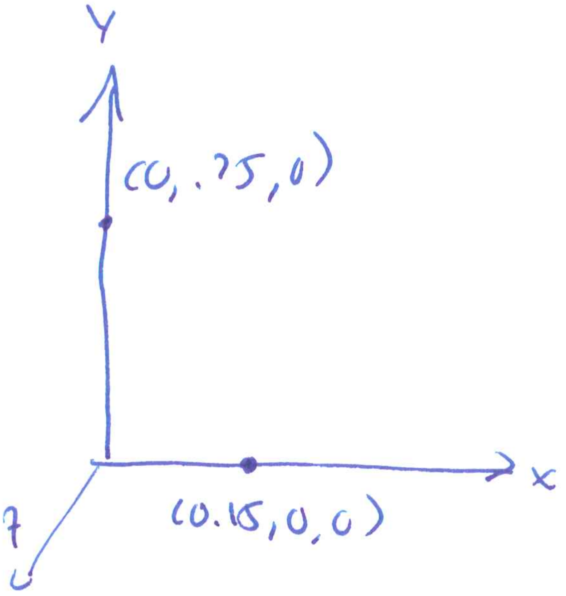
$\vec{r}_1 = (0, .0.250, 0)$

$\vec{v}_1 = 9.20 \times 10^5 \frac{m}{s} \hat{i}$

$q_2 = -2.90 \times 10^{-6}$ C

$\vec{r}_2 = (0.150 m, 0, 0)$ m

$\vec{v}_2 = (-5.30 \times 10^5 \frac{m}{s}) \hat{j}$



force on q_2

$$\vec{F} = q_2(\vec{E} + \vec{v}_2 \times \vec{B})$$

$$\vec{E}_1 = \frac{q_1}{4\pi\varepsilon_0 r^3}(x\hat{i} + y\hat{j})$$

$$r = \sqrt{(.15)^2 + (.25)^2}$$

$$\vec{F}_B = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{x}{r^3}\hat{i} + y\hat{j})$$

electric force:

$$F_x = \frac{q_1 q_2 (.150)}{4\pi\varepsilon_0 (.292)} \qquad F_y = -\frac{q_1 q_2 (.25)}{4\pi\varepsilon_0 (.292)}$$

magnetic force

need $\vec{B}$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q_1 \vec{v}_1 \times \vec{r}_{21}}{r_{21}^3}$$

$$\vec{B} = \frac{q_1 \mu_0}{4\pi} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ |v_1| & 0 & 0 \\ .15 & -.25 & 0 \end{vmatrix} = \qquad \hat{k}$$

$$\vec{B} \implies \frac{q_1 \mu_0}{4\pi} (-.25) v_1 \quad \hat{k}$$

$$\vec{F}_B = q_2 \vec{v}_2 \times \vec{B} = q_2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -v_2 & 0 \\ 0 & 0 & B_z \end{vmatrix}$$

$$= q_2 \; (-v_2 B_z) \hat{i}$$

$$F_x = -q_2 v_2 B_z = \frac{\mu_0}{4\pi} q_1 q_2 v_2 v_1 \; (.25)$$

$$F_y = 0$$

using $\frac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9 \ Nm^2/C^2$

$$\frac{\mu_0}{4\pi} = 10^{-7} \ \frac{Tm}{A}$$

---

electric: $F_x^E = -9.65 \times 10^{-3}$ N

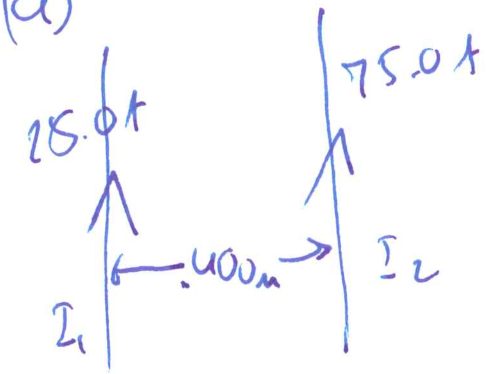
$F_y^E = +16.09 \times 10^{-3}$ N

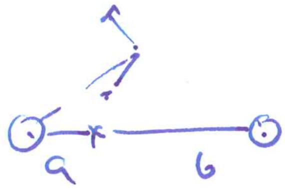
magnetic: $F_x^B = +1.69 \times 10^{-7}$ N

$F_y^B = 0$

(12) 28.61

(a)





$a + b = .400\,m$

B-field of current 1

$B_1 = \frac{\mu_0 I}{2\pi r} \Rightarrow \frac{25\mu_0}{2\pi r}$ $\qquad \frac{25\cancel{\mu_0}}{\cancel{2\pi} a} = \frac{75\cancel{\mu_0}}{\cancel{2\pi} b}$

$B_2 = \frac{75\mu_0}{2\pi r}$

$B_1 = B_2$ if $\frac{25\mu_0}{2\pi a} = \frac{75\mu_0}{2\pi b}$

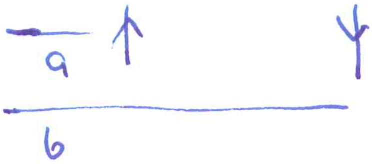
$25b = 75a$

$b = 3a$

$a = .1\,m$

$b = .3\,m$

(b)



$\frac{\mu_0 25}{4\pi a} = \frac{\mu_0 75}{4\pi b}$

$25b = 75a$

$b = 3a$

$a = .20\,m$

$b = .60\,m$

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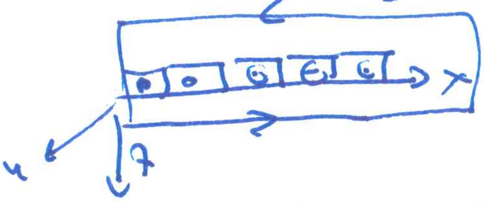
Ampère's law

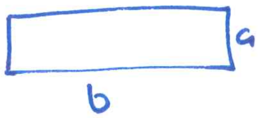


$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enclosed}$

below direction of $\vec{B}$-field $\longrightarrow$

above direction of $\vec{B}$-field $\longleftarrow$

take Ampère rectangle to be infinite in horizontal direction



$b \gg a$

$\oint \vec{B} \cdot d\vec{l} = 2Bb = \mu_0 n I b$

$\Rightarrow B = \frac{\mu_0 n I}{2}$