Problem Set 12: Solutions

1.) (a) two regions $r<R$ (R radius of sphere), $r>R$

Gauss' law $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$

$r<R$: $E\,4\pi r^2 = \frac{1}{\varepsilon_0}\frac{4\pi r^3}{3}\rho$

$$E = \frac{\rho r}{3\varepsilon_0}$$

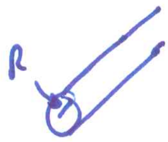
$r>R$: $E\,4\pi r^2 = \frac{1}{\varepsilon_0}\frac{4\pi R^3}{3}\rho$

$$E = \frac{\rho R^3}{3\varepsilon_0 r^2}$$

(b) cylinder: radius R

$r<R$

$r>R$



gaussian surface: $\Rightarrow$

associate length L with cylinder

for $r<r$: $\oint E\cdot d\vec{A} = E\,2\pi r L$

$Q_{enclosed} = \rho\pi r^2 L$

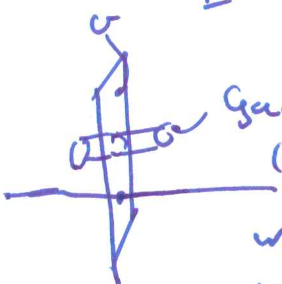
$\Rightarrow E\,2\pi r L = \frac{\rho\pi r^2 L}{\varepsilon_0} \Rightarrow E = \frac{\rho r}{2\varepsilon_0}$

for $r>R$

$E\,2\pi r L = \rho\pi R^2 L$

$$E = \frac{\rho\pi R^2}{2r}$$

(c)



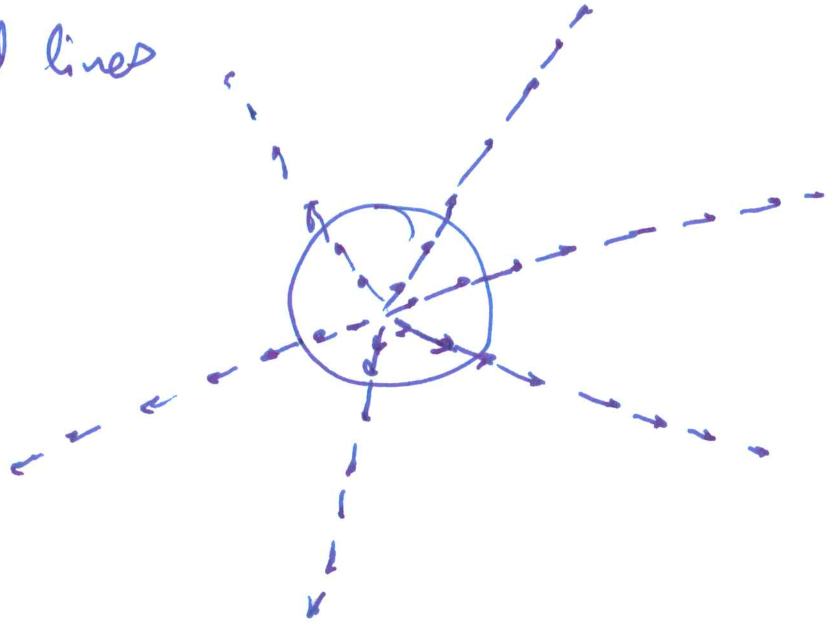
Gaussian surface: (cylinder w/ radius r, length l

$\int \vec{B}\cdot d\vec{A} = 2E\pi r^2 = \frac{\sigma\pi r^2}{\varepsilon_0}$

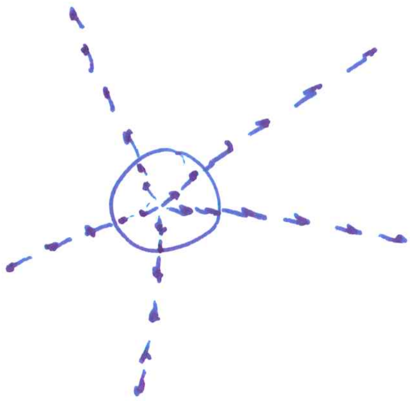
$\Rightarrow E = \frac{\sigma}{2\varepsilon_0}$

electric field lines

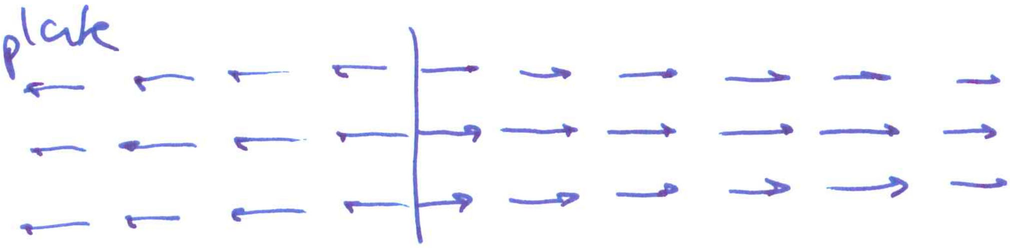
sphere



cylinder



plate



2.) 1.1(a) use reference point $r = \infty$

$r > R \quad V(r) = -\int_{\infty}^{r} E(r')\,dr' = -\int_{\infty}^{r} \frac{\rho R^3}{3\epsilon_0 r^2}\,dr$

$= -\frac{\rho R^3}{3\epsilon_0} \int_{\infty}^{r} \frac{dr}{r^2} = +\frac{\rho R^3}{3\epsilon_0 r}$

$r < R \quad V(r) = -\int_{R}^{r} E(r')\,dr' + \frac{\rho R^2}{3\epsilon_0}$

$= -\int_{R}^{r} \frac{\rho r'}{3\epsilon_0}\,dr' + \frac{\rho R^2}{3\epsilon_0} = \frac{\rho}{3\epsilon_0} \frac{r^2}{2}\Big|_{R}^{r} + \frac{\rho R^2}{3\epsilon_0}$

$= \frac{\rho}{6\epsilon_0}(r^2 - R^2) + \frac{\rho R^2}{3\epsilon_0} = \frac{\rho}{6\epsilon_0}(r^2 + R^2)$

1.) b.) choose reference potential to be at radius of cylinder $(r = R)$

for $r > R$

$$V(r) = -\int_R^r E(r')\,dr'$$

$$= -\frac{\rho R^2}{2\varepsilon_0}\int_R^r \frac{dr'}{r'}$$

$$= -\frac{\rho R^2}{2\varepsilon_0} \ln r' \Big|_R^r$$

$$= -\frac{\rho R^2}{2\varepsilon_0} \ln r/R = \frac{\rho R^2}{2\varepsilon_0} \ln R/r$$

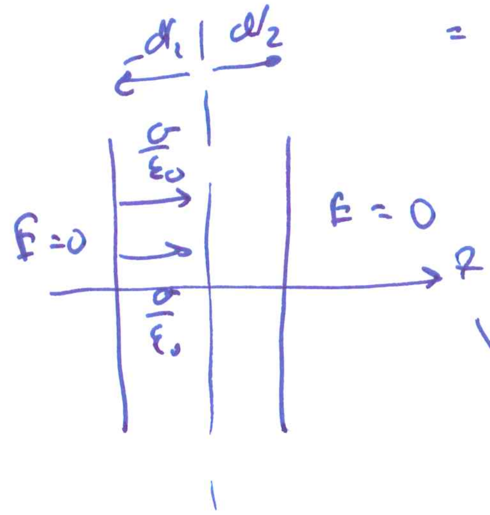
for $r < R$

$$V(r) = -\int_R^r E(r')\,dr'$$

$$= -\frac{\rho}{2\varepsilon_0}\int_R^r r'\,dr' = -\frac{\rho}{2\varepsilon_0}\frac{r'^2}{2}\Big|_R^r$$

$$= -\frac{\rho}{4\varepsilon_0}(r^2 - R^2) = \frac{\rho}{4\varepsilon_0}(R^2 - r^2)$$

1. c.)



$V(-d/2) = 0$ (reference point)

$$V(z) = -\int_{-d/2}^{z} E \cdot dz' \quad (\text{for } -d/2 < z < d/2)$$

$$= -\frac{\sigma}{\varepsilon_0} z' \Big|_{-d/2}^{z}$$

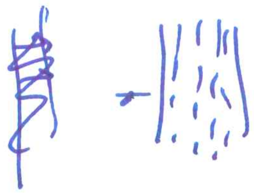
$$V(z) = -\frac{\sigma}{\varepsilon_0}(z + d/2)$$

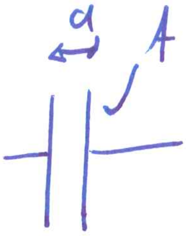
equipotential surfaces

sphere: → spheres

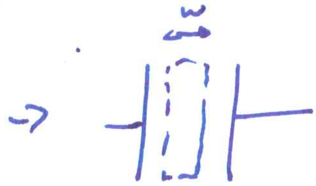
cylinder → cylinders

parallel plate capacitor: planes



3.) 

Slab (conducting



$$V = \frac{Q}{C}$$

$$Ed = \frac{\sigma A}{C} = \frac{\sigma d}{\varepsilon_0} \Rightarrow C = \frac{\varepsilon_0 A}{d}$$

$C = ?$

new arrangement is like two capacitors in series

$$a + b = d - w \qquad C_1 = \frac{\varepsilon_0 A}{a} \quad C_2 = \frac{\varepsilon_0 A}{b}$$

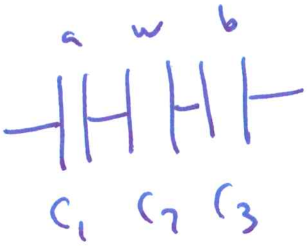
capacitors in series: $V_{\alpha\gamma} = V_{\alpha\beta} + V_{\beta\gamma}$

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} \Rightarrow \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$\cancel{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$

$$\frac{1}{C_{eq}} = \frac{a}{\varepsilon_0 A} + \frac{b}{\varepsilon_0 A} = \frac{a+b}{\varepsilon_0 A} = \frac{d-w}{\varepsilon_0 A}$$

- for slab with dielectric: three capacitors in series



$$a + w + b = d$$

$$C_1 = \frac{\varepsilon_0 A}{a}$$

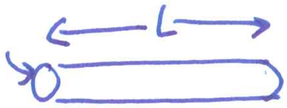
$$C_2 = \frac{K\varepsilon_0 A}{w}$$

$$C_3 = \frac{\varepsilon_0 A}{b}$$

$$\frac{1}{C_{eq}} = \frac{a + w/K + b}{\varepsilon_0 A} \Rightarrow \boxed{C_{eq} = \frac{\varepsilon_0 A}{a + \frac{w}{K} + b}}$$

(5)

4.) A



$E = \rho J$ $\qquad R = \frac{\rho L}{A}$

$EL = \rho J L$

$V = \rho J L = \frac{\rho I L}{A} \Rightarrow I = \frac{VA}{\rho L}$

power dissipated

$P = I^2 R = \frac{V^2 A^{\cancel{2}}}{\rho^2 L^{\cancel{2}}} \frac{\rho \cancel{L}}{\cancel{A}} = \frac{V^2 A \cancel{\rho}}{\rho^{\cancel{2}} L} = \frac{V^2 A}{\rho L}$

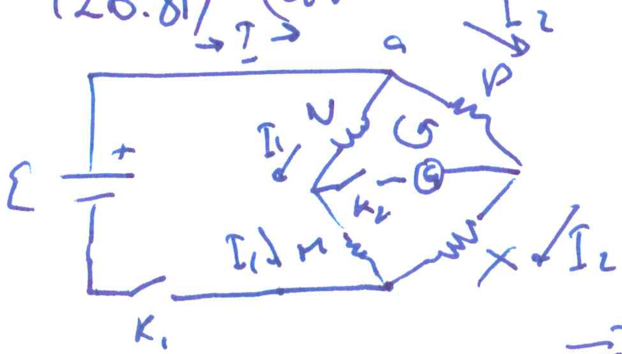
$P = \frac{V^2 A}{\rho L}$ $\qquad I = \frac{VA}{\rho L}$

~~$P' = 30P$~~

$P = \frac{E^2 L A}{\rho}$ $\qquad I = \frac{EA}{\rho}$

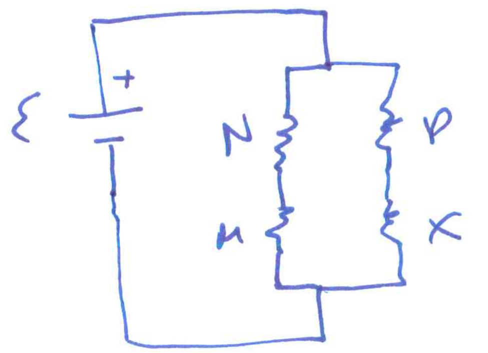
$A \to 4A, \quad L \to 7.5L \Rightarrow P \to 30P, \quad I \to 4I$

5.) (26.81) (a)



(a) when balanced (current in G is zero)

$\Rightarrow$



(b) $\rightarrow$

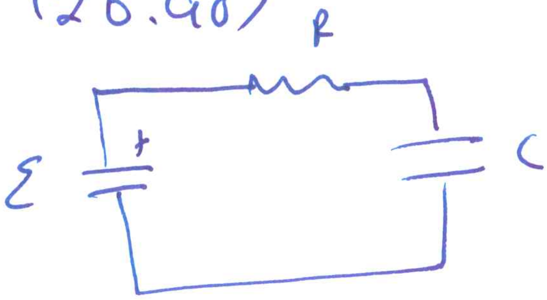
equivalent resistance of bridge

$$\frac{1}{R_{eq}} = \frac{1}{M+N} + \frac{1}{P+X}$$

we also have: loops $\Rightarrow NI_1 = PI_2 \,/\, MI_1 = XI_2 \Rightarrow X = \frac{MN}{P}$ ✓

(b) 1847 Ω

(b) (26.90)



$$\mathcal{E} - IR - \frac{Q}{C} = 0$$

$$\mathcal{E} - \frac{dQ}{dt}R - \frac{Q}{C} = 0$$

$$\mathcal{E}C - \frac{dQ}{dt}(RC) - Q = 0$$

$$-\frac{dQ}{dt}(RC) = Q - \mathcal{E}C$$

$$-\frac{dt}{RC} = \frac{dQ}{Q - \mathcal{E}C} \qquad Q(0) = 0$$

$$\Downarrow$$

$$-\frac{t}{RC} = \ln\left(\frac{Q(t) - \mathcal{E}C}{Q(0) - \mathcal{E}C}\right)$$

$$-\mathcal{E}C\, e^{-t/RC} = Q(t) - \mathcal{E}C$$

$$Q(t) = \mathcal{E}C(1 - e^{-t/RC})$$

(a) $Q(\infty) = 1000V \times 10.0\,pF = 1000V \times 10.0 \times 10^{-12}\,F$

$= 10.0 \times 10^{-9}\,C = 10.0\,nC$

(b) $Q(0) = \mathcal{E}C$ ($\mathcal{E} = 1000\,V$, $C = 10.0\,pF \rightarrow$ original capacitance)

~~$Q(t) = \mathcal{E}C'(1-$~~

in general: $Q(t) = (Q(0) - \mathcal{E}C')e^{-t/RC'} + \mathcal{E}C'$

$= Q(0) + \mathcal{E}C'(1 - e^{-t/RC'})$

where C' is the new capacitance

$Q(0) = \mathcal{E}C$

$\Downarrow$

$Q(t) = \mathcal{E}C + \mathcal{E}C'(1 - e^{-t/RC'}) \Rightarrow I(t) = \frac{\mathcal{E}}{R}e^{-t/RC'}$

$$I(t) = \frac{\varepsilon}{R} e^{-t/RC'}$$

if R too large → I(t) to small to be detected

if R too small → I(t) decays to fast to zero and will not be detected

6.) mass m charge q potential difference V

energy = qV

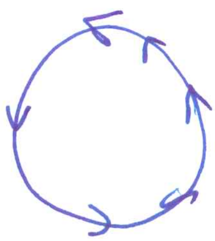
$$\frac{mv^2}{2} = qV \Rightarrow \text{velocity } \sqrt{\frac{2qV}{m}} = v$$

radius of cyclotron motion

$$qvB = \frac{mv^2}{r} \Rightarrow r = \frac{mv}{qB} = \frac{m}{qB}\sqrt{\frac{2qV}{m}}$$

$$= \sqrt{\frac{2mV}{qB^2}}$$

7.)

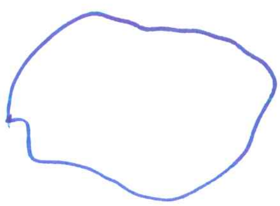


$$\vec{F} = I \oint d\vec{l} \times \vec{B}$$

$$= I \oint dL\, B \sin\theta$$

$$= IB \oint dL \sin\theta$$

in general (loop does not need to be circular)



$$F_x = I \oint (dL_y B_z - dL_z B_y)$$

$$= I B_z \oint dL_y - I B_y \oint dL_z$$

$\oint dL_y = 0$ loop integral which goes from some point back to the same point

$\Rightarrow F_x = 0$ $F_z = 0$ (same proof)

7. (con't): for inhomogeneous magnetic field result does not hold

8.) (a) $j(r) = j_0\left(1-\left(\frac{r}{r_0}\right)^3\right)$

total current: $I = \int \vec{j}\cdot d\vec{A}$

$$= 2\pi j_0 \int_0^{r_0} r\,dr\left(1-\frac{r^3}{r_0^3}\right)$$

$$= \cancel{2\pi}\ 2\pi j_0\left(\frac{r_0^2}{2} - \frac{r_0^5}{5 r_0^3}\right)$$

$$= 2\pi j_0 r_0^2\left(\frac{3}{10}\right) = \frac{3\pi}{5} j_0 r_0^2$$

(b) magnetic field (Ampère's law)

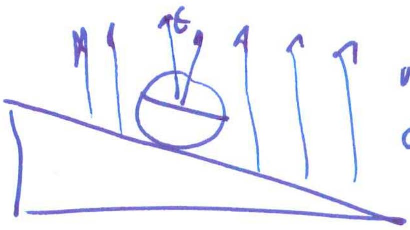
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$$

$$B 2\pi r = 2\pi j_0 \int_0^r r'dr'\left(1-\frac{r'^3}{r_0^3}\right)$$

$$= 2\pi j_0\left(\frac{r^2}{2} - \frac{r^5}{5r_0^3}\right)$$

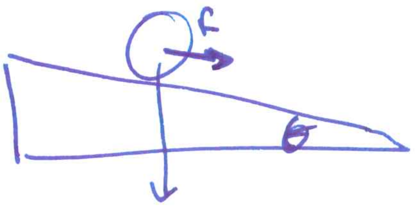
$$B = j_0\left(\frac{r}{2} - \frac{r^4}{5r_0^3}\right)$$

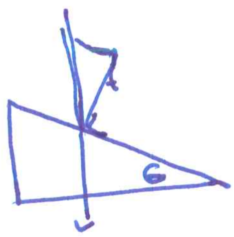
9.)



magn. torque $\vec{\tau}_B = \vec{\mu}\times\vec{B} \Rightarrow I A B \sin\theta$

grav. torque $= I L 2R B \sin\theta$





$f = mg\sin\theta$

$a = g\sin\theta \qquad r\alpha = g\sin\theta$

$\alpha = \frac{g\sin\theta}{R} \qquad \tau_g = I_r\alpha =$

$\cancel{I_r} =$

$I_r = 2\pi \int \rho\, r^2\, r\,dr\, L$

$= 2\pi L \rho \int_0^R r^3 dr = \frac{2\pi L \rho R^4}{4} =$

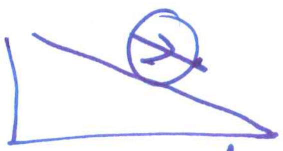
total mass $2\pi \rho \int r\,dr\, L = \frac{2\pi \rho L R^2}{2}$

$I_r = \frac{MR^2}{2}$

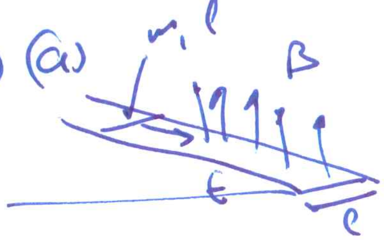
$\tau_g = \tau_B \qquad \frac{m\cancel{R}^{2}}{2}\frac{g\cancel{\sin\theta}}{\cancel{R}} = IL2\cancel{R}B\cancel{\sin\theta}$

~~B=~~ $I = \frac{mg}{4LB}$

direction of current:



(10.) (a)



gravitational force

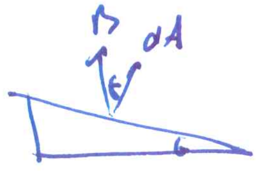
$f_g = mg\sin\theta$

should be balanced by magnetic force

$f_B = IlB\cos\theta$

$IlB\cos\theta = mg\sin\theta$

$I = \frac{mg\tan\theta}{lB}$



$\varepsilon = -\frac{d\Phi_B}{dt} = IR$

$\Phi_B = \int B\cdot dA = Blv\cos\theta = IR$

$I = \frac{Blv\cos\theta}{R} = \frac{mg\sin\theta}{lB\cos\theta} \Rightarrow v = \frac{mgR\sin\theta}{B^2 l^2 \cos^2\theta}$

(b) dissipated power

$$P_B = I^2 R = \frac{B^2 \ell^2 v^2 \cos^2 t}{R^2} R = B^2 \ell^2$$

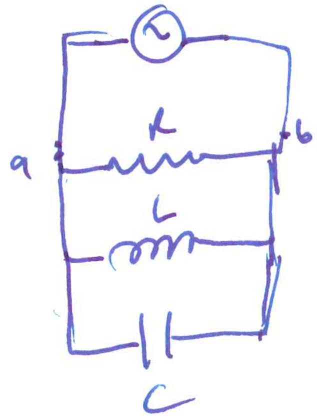
$$= m^2 g^2$$

~~mgh~~ =
potential energy change / unit time

$$P_g = mg v \sin t = \frac{mg\, mg R \sin^2 t}{B^2 \ell^2 \cos^2 t}$$

$$= \frac{m^2 g^2 \sin^2 t}{B^2 \ell^2 \cos^2 t} R = I^2 R \checkmark$$

11. (31.5~~6~~)



$v = V \cos \omega t$

(a) they are all equal since one can take path from a to b
across resistor $\to V_{ab}$
across inductor $\to V_{ab}$
across capacitor $\to V_{ab}$
all give same potential difference

$i = i_R + i_L + i_C$

$v = i_R R$

$V \cos \omega t = I_R R \cos \omega t \qquad V = I_R R$

inductor:

$$V \cos \omega t = v = L \frac{di_L}{dt}$$

$$V \cos \omega t = L \frac{di_L}{dt}$$

$$\frac{V \sin \omega t}{\omega} = L i_L(t) \Rightarrow i_L = \frac{V}{\omega L} \sin \omega t$$

$$i_L = I_L \sin \omega t$$

$\omega L I_L = V \Rightarrow V = X_L I_L \Rightarrow v = V\cos\omega t$

$i_R = I_R \cos\omega t$

$i_L = I_L \sin\omega t = I_L \cos(\omega t - \frac{\pi}{2})$

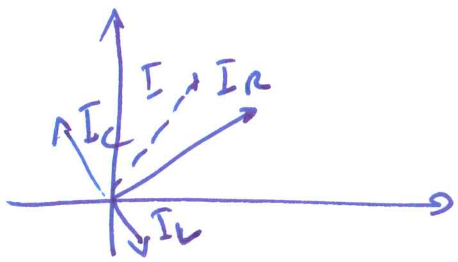
capacitor:

$$V\cos\omega t = \frac{q}{C}$$

$$-V\omega\sin\omega t = \frac{i_C}{C} \Rightarrow i_C = -\omega C V \sin\omega t = I_C \cos(\omega t + \frac{\pi}{2})$$

$$= -\frac{V}{X_C}\sin(\omega t + \frac{\pi}{2})$$

current phasor diagram:



$$I = \sqrt{(I_C - I_L)^2 + I_R^2}$$

$$I = \sqrt{V^2\left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2 + \frac{1}{R^2}}$$

$$I = V\sqrt{\left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2 + \frac{1}{R^2}}$$

$$I = V\sqrt{\left(\omega C - \frac{1}{\omega L}\right)^2 + \frac{1}{R^2}}$$

$$Z = \frac{1}{\sqrt{\left(\omega C - \frac{1}{\omega L}\right)^2 + \frac{1}{R^2}}}$$

$$\vec{E} = E_{max}\,\hat{j}\cos(kx - \omega t)$$

$$\vec{B} = B_{max}\,\hat{k}\cos(kx - \omega t + \varphi)$$

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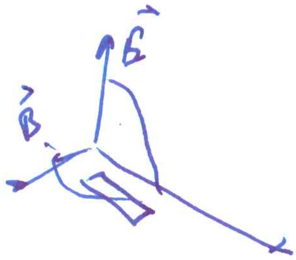
use Ampère's or Faraday's law

Ampère:



$$\oint \vec{B}\cdot d\vec{l} = \epsilon_0 \mu_0 \frac{\partial \Phi_E}{\partial t}$$

$$\Downarrow$$

$$\left[B(x+\Delta x, t) - B(x,t)\right] a = -\epsilon_0\mu_0 \frac{\partial E(x,t)}{\partial t}\, a\,\Delta x$$

$$\frac{\partial B(x,t)}{\partial x} = -\epsilon_0\mu_0 \frac{\partial E(x,t)}{\partial t}$$

$$\Downarrow$$

$$-k B_{max}\cos(kx - \omega t + \varphi) = -\epsilon_0\mu_0\,\omega\, E_{max}\cos(kx - \omega t)$$

satisfied if $E_{max} = c B_{max}$ $\qquad c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = \frac{\omega}{k}$

$\varphi = 0$