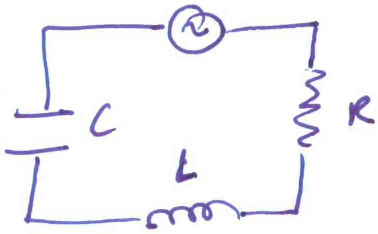
Problem Set 11: Solutions

1.) (31.53)

a.)



current: $i = I\cos\omega t$

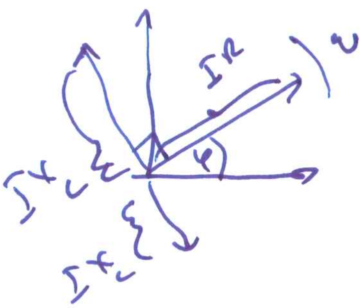
$$v_R = IR\cos\omega t$$

$$v_L = L\frac{di}{dt} = LI(-\omega\sin\omega t)$$

$$= LI\omega\cos(\omega t + \frac{\pi}{2}) = IX_L\cos(\omega t + \frac{\pi}{2})$$

$$v_X = \frac{q}{C} = \frac{I\sin\omega t}{\omega C} = X_C I\cos(\omega t - \frac{\pi}{2})$$

phasor representation



sum of three vectors will have a magnitude: $I\sqrt{(X_L - X_C)^2 + R^2} = V$

$$I = \frac{V}{\sqrt{(X_L - X_C)^2 + R^2}}$$

b.) average dissipated power:

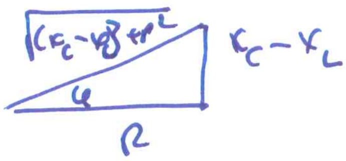
$$p = iv = I\cos\omega t\, V\cos(\omega t + \varphi)$$

$$\tan\varphi = \frac{X_C - X_L}{R}$$

$$p = iv = IV\cos^2\omega t\cos\varphi - IV\cos\omega t\sin\omega t\sin\varphi$$

average: $P_{av} = \int_0^{2\pi/\omega} p\,dt = \frac{IV}{2}\cos\varphi = I_{rms}V_{rms}\cos\varphi$

P_{av} $\cos\varphi = \frac{R}{\sqrt{(X_L - X_C)^2 + R^2}}$



$$P_{av} = \frac{V}{2}\left(\frac{V}{\sqrt{(X_L - X_C)^2 + R^2}}\right)\left(\frac{R}{\sqrt{(X_L - X_C)^2 + R^2}}\right)$$

$$= \frac{V^2 R}{2((X_L - X_C)^2 + R^2)}$$

c.) $I = \dfrac{V}{\sqrt{(\omega L - \frac{1}{\omega C})^2 + R^2}}$ max. when $(\omega L - \frac{1}{\omega C}) = 0$ (2)

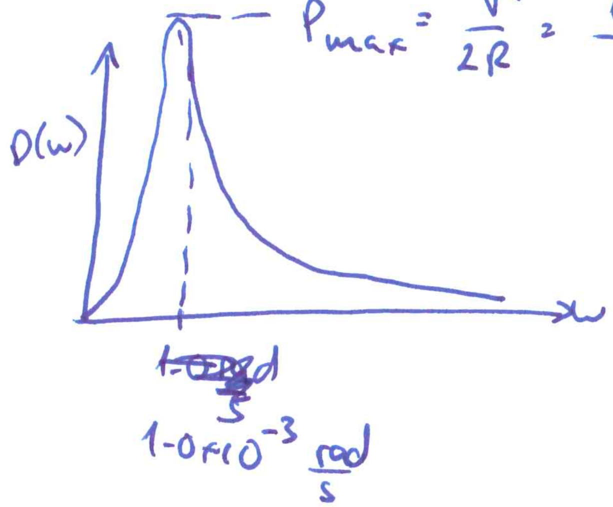
$\omega = \dfrac{1}{\sqrt{LC}}$

P same

d.) $\lim_{\omega \to 0} P(\omega) = 0$ since $X_L \to 0$, $X_C \to \infty$

$\lim_{\omega \to \infty} P(\omega) = 0$ since $X_L \to \infty$, $X_C \to 0$

graph



$P_{max} = \dfrac{V^2}{2R} = \dfrac{10000}{400} = 25\,W$

(2.) (31.65)

$I_{av} = 0$ (easy to see => oscillating function around $I = 0$)

$I_{rms} = \sqrt{\frac{1}{\tau} \int_0^\tau I^2(t)\,dt}$

$= \sqrt{\frac{1}{\tau} \left[2 \int_0^{\tau/2} (2 I_0 t')^2 dt' \right]}$

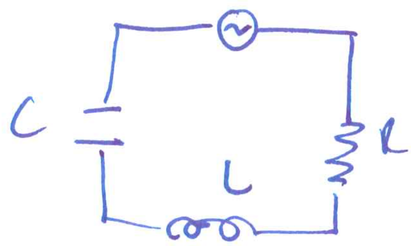
$= \sqrt{\frac{1}{\tau} \times 4 \times 4 \times I_0^2 \int_0^{\tau/2} t'^2 dt'}$

$= 4 I_0 \sqrt{\frac{1}{\tau} \int_0^{\tau/2} t'^2 dt}$

$= 4 I_0 \sqrt{\frac{1}{\tau} \frac{t'^3}{3} \Big|_0^{\tau/2}} = 4 I_0 \sqrt{\frac{1}{\tau} \times \frac{\tau^3}{24}} = \dfrac{4 I_0 \tau}{2\sqrt{6}} = \dfrac{2 I_0 \tau}{\sqrt{6}}$

3.) (31.67)

$L' = 2L \quad C' = 2C$



(a) resonance frequency

$$\omega' \overset{!}{=} \frac{1}{\sqrt{L'C'}} \rightarrow \frac{1}{2\sqrt{LC}} = \frac{\omega}{2}$$

is halved

(b) $X_L' = \omega L' = 2\omega L \Rightarrow$ doubles $(2X_L)$

(c) $X_C' = \frac{1}{\omega C'} = \frac{1}{2\omega C} \Rightarrow$ is halved $\left(\frac{X_C}{2}\right)$

(d) $Z' = \sqrt{(X_L' - X_C')^2 + R'^2}$

$$= \sqrt{\left(2X_L - \frac{X_C}{2}\right)^2 + 4R^2}$$

no Z' does not double

4.) (31.74)

(a) voltage amplitude across resistor

$V_R = IR \rightarrow$ independent of ω

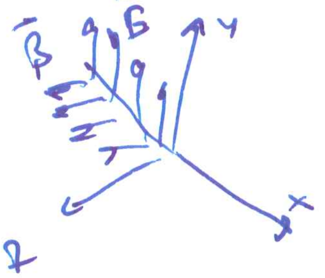
(b) voltage amplitude across inductor

$V_L = I X_L = I\omega L \rightarrow$ maximum for $\omega \rightarrow \infty$

(c) voltage amplitude across capacitor

$V_C = I X_C = \frac{I}{\omega C} \rightarrow$ maximum for $\omega \rightarrow 0$

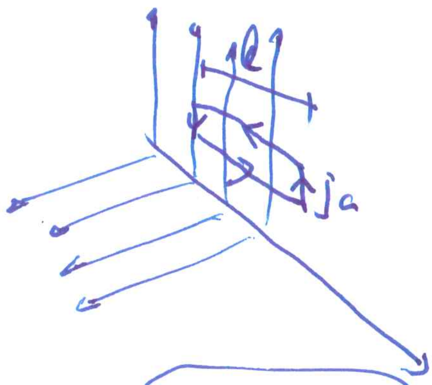
5a.)



ratios of two fields

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

bigger picture: ->



$$\oint \vec{E}\cdot d\vec{l} = -Ea$$

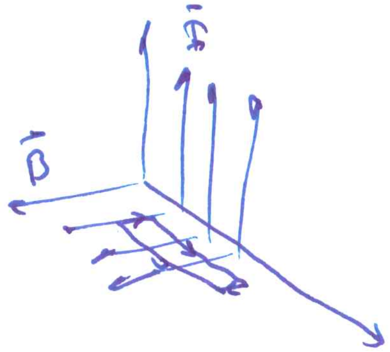
$$-\frac{d\Phi_B}{dt} = -\frac{d\int \vec{B}\cdot d\vec{A}}{dt} = -Bac$$

$$-aE = -Bac \Rightarrow \boxed{E = cB}$$

c -> propagation velocity

$$\oint \vec{B}\cdot d\vec{l} = +\mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

picture ->



$$\oint \vec{B}\cdot d\vec{l} = -Ba$$

$$\frac{d\Phi_E}{dt} = -Eac$$

$$\Rightarrow \boxed{B = \mu_0 \varepsilon_0 E c} \Rightarrow \boxed{c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}$$

place of boundary after time T

$$@\ t = 0 \quad x(t) = 0 \qquad x(0) = 0$$

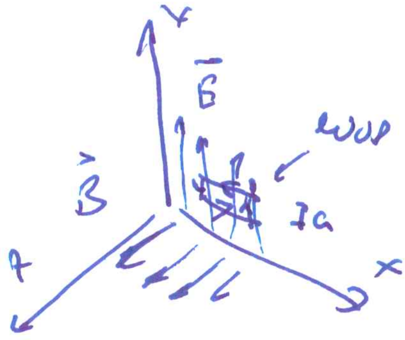
$$x(T) = x(0) + cT = \frac{T}{\sqrt{\mu_0 \varepsilon_0}}$$

b.) $\varepsilon_0 \longrightarrow K\varepsilon_0 = \varepsilon \qquad K > 1$

$$E = c'B \qquad c' = \frac{1}{\sqrt{K\mu_0 \varepsilon_0}} \qquad x(T) = \frac{T}{\sqrt{K\mu_0\varepsilon_0}}$$

c.) $$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} \longrightarrow \frac{EB}{\mu_0} = \frac{E^2}{c\mu_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E^2$$

6.)

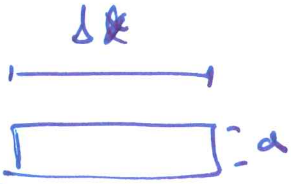


$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$

E and B vary with x, t

$E(x,t)$

$B(x,t)$

(loop dimensions: )

Faraday's law

$$\oint \vec{E}\cdot d\vec{l} = aE(x+\Delta x,t) - aE(x,t)$$

$$\int \vec{B}\cdot d\vec{A} = B(x,t)\,a\Delta x$$

⇓

$$a\frac{\partial E(x,t)}{\partial x}\Delta x = -\frac{\partial B(x,t)}{\partial t}a\Delta x \Rightarrow -\frac{\partial E(x,t)}{\partial x} = \frac{\partial B(x,t)}{\partial t}$$

Ampère's law (loop on x-z plane)

$$\oint \vec{B}\cdot d\vec{l} = \epsilon_0\mu_0\frac{\partial \Phi_E}{\partial t}$$

$$\rightarrow \frac{\partial B(x,t)}{\partial x} = -\epsilon_0\mu_0\frac{\partial E(x,t)}{\partial t}$$

$$\frac{\partial^2 E(x,t)}{\partial x^2} = -\frac{\partial^2 B(x,t)}{\partial x\,\partial t}$$

$$\frac{\partial^2 B(x,t)}{\partial x\,\partial t} = -\epsilon_0\mu_0\frac{\partial^2 E(x,t)}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 E(x,t)}{\partial x^2} = \epsilon_0\mu_0\frac{\partial^2 E(x,t)}{\partial t^2}$$

similarly: $$\frac{\partial^2 B(x,t)}{\partial x^2} = \epsilon_0\mu_0\frac{\partial^2 B(x,t)}{\partial t^2}$$

(b) ~~$E(x,t) \rightarrow \partial^2 E$~~

$$E(x,t) = E_{max} \cos(kx - \omega t)$$

$$B(x,t) = B_{max} \cos(kx - \omega t \qquad E_{max} = c B_{max}$$

$$\frac{\partial^2 E(x,t)}{\partial x^2} = -k^2 E_{max} \cos(kx - \omega t)$$

$$\frac{\partial^2 E(x,t)}{\partial t^2} = -\omega^2 E_{max} \cos(kx - \omega t)$$

wave equation is satisfied if

$$k^2 = \frac{\omega^2}{c^2} \qquad c = \pm\sqrt{k/\omega}$$

(c) other functions

for example: $E_{max} \sin(kx - \omega t)$

$B_{max} \sin(kx - \omega t)$

or sums thereof: ~~E_{max}~~

$$E_c \cos(kx - \omega t) + E_s \sin(kx - \omega t)$$

$$B_c \cos(kx - \omega t) + B_s \sin(kx - \omega t)$$

7.) intensity: $I = \langle S \rangle = \frac{E_{max}^2}{2\mu_0 c}$

$$E_{max} = 2\mu_0 c I$$

$$\vec{E} = \hat{j} E_{max} \cos(kx - \omega t) \qquad c B_{max} = E_{max}$$

$$\vec{B} = \hat{k} B_{max} \cos(kx - \omega t)$$

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{E_{max} B_{max}}{\mu_0} \cos^2(kx - \omega t) \hat{h}$$

$$E = \frac{1}{T}\int_0^T\int d\vec{A}\cdot\vec{S}\,dt = \frac{1}{T}\int_0^T dt\int d\vec{A}\cdot\vec{S}$$

$$= \frac{1}{T}\int_0^T dt\,\frac{E_{max}B_{max}}{\mu_0}\cos^2(kx-\omega t)\times a\cdot b$$

assume $x = 0$

$$E = \frac{E_{max}B_{max}\,a\cdot b}{T\mu_0}\int_0^T dt\cos^2(\omega t)$$

$$\cos^2(\omega t) = \frac{1}{2} + \frac{1}{2}\cos 2\omega t$$

$$= \frac{E_{max}B_{max}\,a\cdot b}{T\mu_0}\left(\frac{T}{2} + \left(\frac{1}{2}\frac{\sin 2\omega t}{2\omega}\right)\Big|_0^T\right)$$

$$= \frac{E_{max}B_{max}\,a\cdot b}{\mu_0}\left(\frac{1}{2} + \frac{\sin 2\omega T}{4\omega T}\right)$$

(b) same as (a) times $\cos\theta$

8.) (32.44)

$$E_y = E_{max}\cos(kx-\omega t)$$

$\omega = 3.02\times10^{15}\ \frac{rad}{s}$

$k = 1.39\times10^{7}$ rad/m

(a) frequency: $\nu = \frac{\omega}{2\pi}$ $\qquad \nu = \frac{\omega}{2\pi} = \frac{3.02}{2\pi}\times10^{15}\ \frac{\text{cycles}}{\text{sec}}$

(b) wavelength: $\lambda = \frac{2\pi}{k} = \frac{2\pi}{1.39}\times10^{-7}$ m

speed: $v = \frac{\omega}{k} = \frac{3.02\times10^{15}}{1.39\times10^{7}} = \frac{3.02\times10^{8}}{1.39}\ \frac{m}{s} = 2.17\times10^{8}\ \frac{m}{s}$

(b) index of refraction: $\frac{v}{c} = \frac{3.02}{1.39\times3} = .72$

(c) for example:

change λ (or k) so that $\frac{\omega}{k} = c$

$$\frac{\omega}{k} = 2.17 \times 10^8 \frac{m}{s}$$

$$\frac{\omega}{k'} = 3 \times 10^8 \frac{m}{s} \Rightarrow \frac{\omega}{Ak} = \frac{1}{A}\left(2.17 \times 10^8 \frac{m}{s}\right) = 3.0 \times 10^8 \frac{m}{s}$$

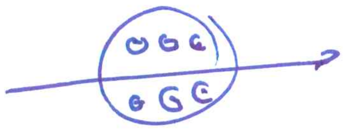
$$A = ~~\frac{3.0}{2.17}~~ \quad \frac{2.17}{3.0}$$

$$k' = A(1.39) \times 10^7 \frac{rad}{m} = \frac{3.0}{2.17} \times 1.39 \times 10^7 \frac{rad}{m}$$

~~$= 1.92 \times 10^7 \frac{rad}{s}$~~

$$= 1.92 \times 10^7 \frac{rad}{m}$$

8.) (32.48) air ≅ vacuum



intensity: $I = S_{av} = \int \frac{(\vec{E} \times \vec{B})}{\mu_0} dA$

induced emf: $\mathcal{E} = -\frac{d\Phi_B}{dt}$

$$\Phi_B = \int \vec{B} \cdot d\vec{A}$$

~~mag~~ magnetic field can be taken to be constant since $r \ll \lambda$

$$\Rightarrow \Phi_B = B(t) \pi r^2$$

($r = 7.50$ cm $= .075$ m, $\lambda = 6.9$ m)

$$I = \frac{E_{max}^2}{2\mu_0 c} =$$

$I = .0195 \text{ W/m}^2$

$$E_{max} = \sqrt{I 2 \mu_0 c}$$

$$B_{max} = \sqrt{\frac{2I\mu_0}{c}}$$

9

$$\mathcal{E} = -\frac{dB(x,t)}{dt}\pi r^2$$

$$B(x,t) = B_{max}\cos(kx - \omega t)$$

$$-\frac{dB}{dt} = -B_{max}\frac{d\cos(kx-\omega t)}{dt}$$

$$= -\omega B_{max}\frac{\sin(kx-\omega t)}{dt}$$

maximum: $\omega B_{max}\pi r^2 = \mathcal{E}_{max}$

$$\mathcal{E}_{max} = \omega\pi r^2\sqrt{\frac{2I\mu_0}{c}}$$

$$\frac{\omega}{k} = c \qquad \omega = kc = \frac{2\pi c}{\lambda} = \frac{(2\pi)\,3.0\times10^8\,m/s}{6.90m} = ~~2.4~~\frac{rad}{sec}$$

$$\omega = 2.73\times10^8\ \frac{rad}{sec}$$

$$B_{max} = \sqrt{\frac{2(.0195)\,4\pi\times10^{-7}}{3\times10^8}}$$

$\mathcal{E}_{max} =$ ~~3.49~~ $.0062\ V$

10. (32.56)

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = \frac{\mu}{\rho}\frac{\partial E_y(x,t)}{\partial t}$$

(a) Solution: $E_y(x,t) = E_{max}\,e^{-k_c x}\cos(k_c x - \omega t)$

verify: $$\frac{\partial E_y(x,t)}{\partial x} = -k_c E_{max}e^{-k_c x}\cos(k_c x - \omega t)$$
$$- k_c E_{max}e^{-k_c x}\sin(k_c x - \omega t)$$

(10)

$$\frac{\partial^2 E_y(x,t)}{\partial x^2} = \cancel{k_c^2 E_{max} e^{-k_c x} \cos(k_c x - \omega t)}$$
$$- 2k_c^2 E_{max} e^{-k_c x} \sin(k_c x - \omega t)$$
$$\cancel{+ k_c^2 E_{max} e^{-k_c x} \sin(k_c x - \omega t)}$$
$$\cancel{- k_c^2 E_{max} e^{-k_c x} \cos(k_c x - \omega t)}$$

$$\frac{\partial E_y(x,t)}{\partial t} = -E_{max} e^{-k_c x} \sin(k_c x - \omega t)\omega$$

$$\Downarrow \qquad \Downarrow$$

$$-2k_c^2 E_{max} \cancel{e^{-k_c x}} \cancel{\sin(k_c x - \omega t)} = -\frac{k_c \mu}{\rho} E_{max} \cancel{e^{-k_c x}} \cancel{\sin(k_c x - \omega t)}$$

$$\Rightarrow 2k_c^2 = \frac{\mu\omega}{\rho} \Rightarrow k_c = \sqrt{\frac{\mu\omega}{2\rho}}$$

(b) as the $\vec{E}$-field propagates through the medium its energy is dissipated, resulting in a decay of the electric field

(c) decreased $1/e \rightarrow e^{-1}$

$$k_c x = 1 \qquad x = \frac{1}{k_c} \qquad k_c = \sqrt{\frac{\mu\omega}{2\rho}}$$

radio wave:

$f = 1MHz = 1\times10^6 \frac{cycles}{sec} = 2\pi\times10^6 \frac{rad}{sec}$

$\rho = 1.72\times10^{-8}\ \Omega m$

$\mu = \mu_0 = 4\pi\times10^{-7}$

$$k_c = \sqrt{\frac{(4\pi)(2\pi)\times10^{-1}}{2\times1.72}\times10^8} = \sqrt{\frac{8\pi^2}{1.172}\times10^7}$$

$$\frac{1}{k_c} \sim 6.6\times10^{-5}\ m$$

11. (37.54)

(a) $\frac{dE}{dt} = \frac{J}{s}$ (units) $= \frac{kg\,m^2}{s^3}$

units of ϵ_0

$\frac{dE}{dt} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} = \frac{C^2 m^2}{~~(6\pi)~~}$

$F = \frac{Q^2}{4\pi\epsilon_0 r^2}$

$\epsilon_0 = \frac{Q^2}{4\pi F r^2} = \frac{C^2}{N m^2}$

~~$\frac{dE}{dt}$ = units: $C^2 m^2$~~

$\epsilon_0 \rightarrow \frac{C^2 s^2}{kg\,m^3}$

$\frac{q^2 a^2}{6\pi\epsilon_0 c^3} \rightarrow \frac{C^2 m^2 s^3\, kg\, m^3}{s^4 m^3 C^2 s^2} = \frac{kg\,m^2}{s^3}$ ✓

(b) $a = \frac{v^2}{r}$ $v^2 = ?$ $K_p = \frac{m v^2}{2}$

$v^2 = \frac{2K_p}{m}$

$a_p = \frac{2K_p}{m_p r}$

$K_p = 6.0 \times 10^6$ eV

$= 1.6 \times 10^{-19} \cdot 6.0 \times 10^6 = 9.6 \times 10^{-13}$ J

$a = \frac{19.2 \times 10^{-13}}{1.67 \times 10^{-27} (.75)} \frac{m}{s^2} = 15.3 \times 10^{14} \frac{m}{s^2}$

$\alpha_p = \frac{1}{E}\frac{dE}{dt} = \frac{(1.6 \times 10^{-19})^2 \times (15.3 \times 10^{14})^2}{6\pi (8.85 \times 10^{-12}) (3.0 \times 10^8)^3} \times \frac{1}{9.6 \times 10^{-13}}$

$\alpha_p = \frac{599 \times 10^{-10}}{4504 \times 10^{12}} \frac{1}{9.6 \times 10^{-13}} = 1.39 \times 10^{-11}$

(c) $a_e = a_p \times \frac{m_p}{m_e} = a_p \times 1800 \Rightarrow \alpha_e = \alpha_p \times (1800)^2 = 4.5 \times 10^{-5}$