Final Exam: Solutions

Problem 1

I. According to Faraday's law, $\mathcal{E} = -\frac{d\Phi_B}{dt}$, as we pull the sheet out, currents will circulate which oppose the change in flux. (Also, one can argue based on Lenz' law.)

II. Identical => means mutual inductance $M = L$

in series: $\mathcal{E} - 2L\frac{dI}{dt} = 0 \Rightarrow L_{eq} = 2L$

in parallel: $\mathcal{E} - L\frac{dI_1}{dt} - L\frac{dI_2}{dt} = 0 \quad I = I_1 + I_2$

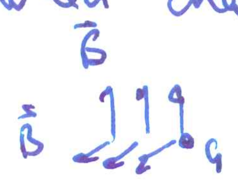
$\Rightarrow \mathcal{E} - L\frac{dI}{dt} = 0 \Rightarrow L_{eq} = L$

It can be zero if currents in serial connection generate opposing magnetic fields.

III. It can not. => magnetic field simply saturates and there will be no change in flux in the secondary coil

IV. two sources of force => $\vec{F}_E = q\vec{E} \quad \vec{F}_B = q\vec{v}\times\vec{B}$

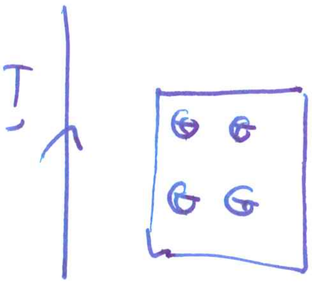
initially no magnetic force, only after $t=0$, when charge has velocity



- electric force direction ↑
- magnetic force direction initially ↦ but changes

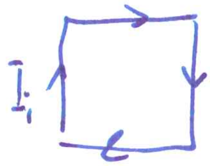
eventually forces balance: F_E ↑ F_B ↓ v_y constant

V attracted



⊙ B-field (direction)

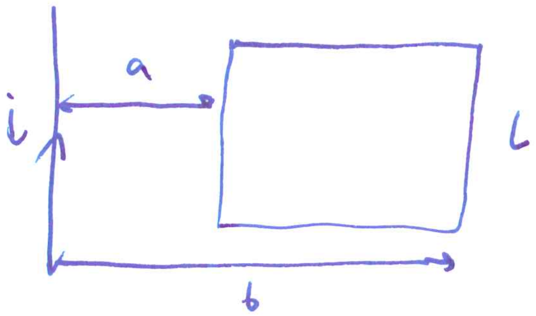
if I decreases, induced current will be (Lenz's law)



parallel current attracting, anti-parallel repulsive

parallel part is closer → attractive

Problem 2: Electromagnetic Induction



(a) what is B-field?

Ampère's law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

$$\Downarrow$$

$$B 2\pi r = \mu_0 i$$

$$B = \frac{\mu_0 i}{2\pi r}$$

(b) total flux:

$$\Phi_B = \int \vec{B}\cdot d\vec{A} = L \int_a^b \frac{\mu_0 i}{2\pi r} dr = \frac{\mu_0 i L}{2\pi} \ln b/a$$

(c) induced emf

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 L}{2\pi}\left[\ln b/a\right]\frac{di}{dt}$$

Problem 3: Inductance

(a) $L_1 \rightarrow \qquad L_1 I_1 = N_1 \Phi_B$

$$\Phi_B = BA$$

$$B 2\pi r = N_1 I_1 \mu_0$$

$$B = \frac{N_1 I_1 \mu_0}{2\pi r}$$

$$L_1 I_1 = \frac{N_1^2}{2\pi r} I_1 A \mu_0 \Rightarrow \quad L_1 = \frac{N_1^2 A}{2\pi r}\mu_0$$

for $L_2 \rightarrow$ same derivation leads to

$$L_2 = \frac{N_2^2 A}{2\pi r}\mu_0$$

(b) $M = ?$

$\cancel{M L_1 I_2 = N.}$

$$M I_1 = N_2 \Phi_B \qquad \Phi_B = B_1 A$$

$$B_1 = \frac{N_1 I_1}{2\pi r}\mu_0$$

$$M = \frac{N_1 N_2 A}{2\pi r}\mu_0$$

$$M^2 = \frac{N_1^2 N_2^2 A^2 \mu_0^2}{4\pi^2 r^2} = L_1 L_2$$

Problem 4: Electromagnetic radiation

wave equations: $$\frac{\partial^2 E(x,t)}{\partial t^2} = c^2 \frac{\partial^2 E(x,t)}{\partial x^2}$$

$$\frac{\partial^2 B(x,t)}{\partial t^2} = c^2 \frac{\partial^2 B(x,t)}{\partial x^2}$$

(a) $E(x,t) = E_m \sin(kx - \omega t)$

$$\frac{\partial E(x,t)}{\partial t} = -\omega E_m \cos(kx - \omega t)$$

$$\frac{\partial^2 E(x,t)}{\partial t^2} = -\omega^2 E_m \sin(kx - \omega t)$$

$$\frac{\partial^2 E(x,t)}{\partial x^2} = -k^2 E_m \sin(kx - \omega t)$$

$\Downarrow$

satisfies wave eqn. if $\underline{\omega^2 = k^2 c^2}$

same for $B(x,t)$

(b) $E(x,t) = E_m f(kx - \omega t)$

$$\frac{\partial E(x,t)}{\partial x} = k E_m f'(kx - \omega t)$$

$$\frac{\partial^2 E(x,t)}{\partial x^2} = k^2 E_m f''(kx - \omega t)$$

$$\frac{\partial^2 E(x,t)}{\partial t^2} = \omega^2 E_m f(kx - \omega t)$$

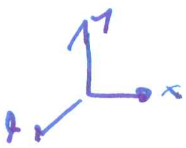
$\Downarrow$

satisfies wave eqn. if $\underline{\omega^2 = k^2 c^2}$

same for $B(x,t)$

(c) $E_z = E_m f(kx + \omega t) \longrightarrow$ velocity is in the negative x-direction

$\vec{E} \times \vec{B}$ must point in negative x-direction



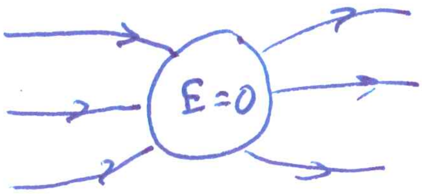
$\Downarrow$

magnetic field should point in +y direction

$B_x = 0 \quad B_y = B_m f(kx + \omega t) \quad B_z = 0$

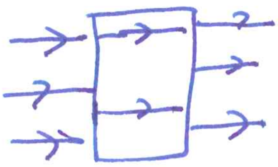
Problem 5: Electrostatics & Gauss' law

(1) (a)



field lines are always perpendicular to surface

(b)



less dense inside since

$$E_{inside} = \frac{E_{outside}}{K}$$

$K > 1$

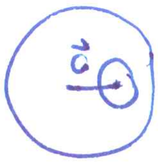
(2) (a) Gauss' law

$$\oint \vec{E} \cdot d\vec{A} = E_r 4\pi r^2 = \frac{Q_{encl}}{\varepsilon_0} = \frac{4\pi \rho r^3}{3}$$

$$\Rightarrow E_r = \frac{\rho r}{3\varepsilon_0}$$

in cartesian coordinates: $\vec{E} = \frac{\rho}{3\varepsilon_0} \vec{r}$

(6)



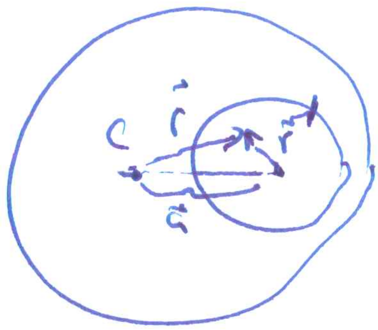
superposition principle

assume cavity corresponds to sphere with charge density $-\rho$

$\Rightarrow$

big sphere $\Rightarrow$ charge density ρ

small sphere $\Rightarrow$ charge density $-\rho$



$$\vec{E} = \frac{\rho \vec{r}}{3\varepsilon_0} - \frac{\rho \vec{r}'}{3\varepsilon_0} = \frac{\rho}{3\varepsilon_0}(\vec{r} - \vec{r}') = \frac{\rho}{3\varepsilon_0}(\vec{r} - \vec{r} + \vec{a}) = \frac{\rho}{3\varepsilon_0}\vec{a}$$

Problem 6: Electromotive force and circuits

(1)

(a) it it is "charged" when used regularly,

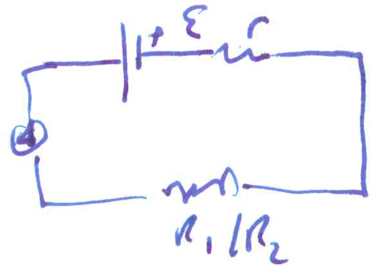
~~nor~~

$\varepsilon - Ir$ (r – internal resistance)

when charged

$\varepsilon + Ir$

(b) two resistors with known resistances R_1 & R_2

=> 

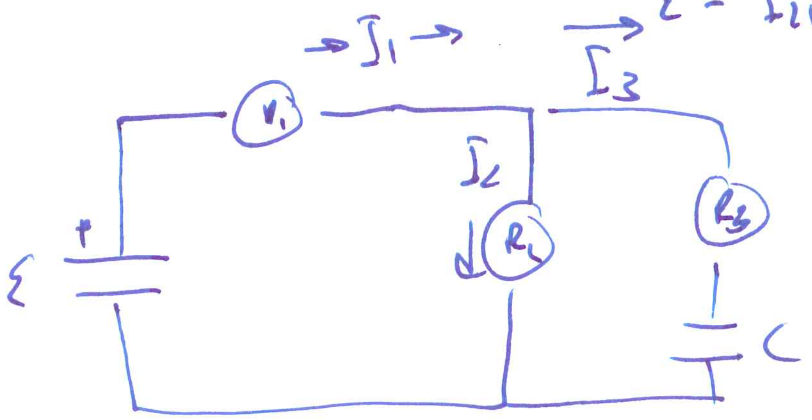
+ ammeter

=> two equations two unknowns (ε, r)

$$\varepsilon - I_1 r - I_1 R_1 = 0$$

$$\varepsilon - I_2 r - I_2 R_2 = 0$$

(2)



$$\varepsilon - I_1 R_1 - I_2 R_2 = 0$$

$$\varepsilon - I_1 R_1 - I_3 R_3 - \frac{Q_3}{C} = 0$$

$$I_1 = I_2 + I_3 \qquad I_3 = \frac{dQ_3}{dt}$$

$$\rightarrow \quad \varepsilon - I_1 R_1 - I_1 R_2 + I_3 R_3 = 0$$

$$\varepsilon - I_1 (R_1 + R_2) + I_3 R_3 = 0 \Rightarrow I_1 = \frac{\varepsilon + I_3 R_3}{R_1 + R_2}$$

$$\rightarrow \quad \underbrace{\mathcal{E} - \frac{\mathcal{E} R_1}{R_1+R_2}}_{} - \frac{I_3 R_3 R_1}{R_1+R_2} - I_3 R_3 - \frac{Q_3}{C} = 0$$

$$\mathcal{E}' - I_3 R' - \frac{Q_3}{C} = 0 \qquad R' = R_3\left(\frac{R_1}{R_1+R_2}+1\right)$$

$$I_3 = \frac{dQ_3}{dt}$$

$\Rightarrow$ solution: $$Q_3(t) = \mathcal{E}' C \left(1 - e^{-t/R'C}\right)$$

$$I_3(t) = \frac{\mathcal{E}'}{R'} e^{-t/R'C}$$

Find $I_1(t)$ from

$$I_1(t) = \frac{\mathcal{E} + I_3(t) R_3}{R_1+R_2} = \frac{\mathcal{E}}{R_1+R_2} + \frac{\mathcal{E}' R_3 e^{-t/R'C}}{R'(R_1+R_2)}$$

$$= \frac{\mathcal{E}}{R_1+R_2} + \mathcal{E}\left(1-\frac{R_1}{R_1+R_2}\right)\frac{\cancel{R_3}}{\cancel{R_3}\left(\frac{R_1}{R_1+R_2}+1\right)(R_1+R_2)} e^{-t/R'C}$$

$$= \frac{\mathcal{E}}{R_1+R_2} + \mathcal{E}\left(\frac{R_2}{R_1+R_2}\right)\frac{e^{-t/R'C}}{(2R_1+R_2)}$$

$$I_2(t) = I_1(t) - I_3(t)$$

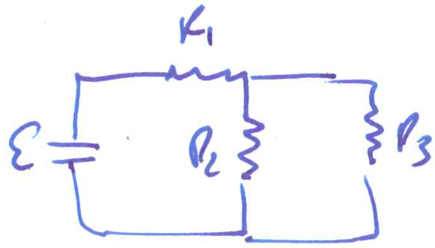
$$= \frac{\mathcal{E}}{R_1+R_2} + \frac{\mathcal{E} R_2 e^{-t/R'C}}{(R_1+R_2)(2R_1+R_2)} - \frac{\mathcal{E}\left(1-\frac{R_1}{R_1+R_2}\right) e^{-t/R'C}}{R_3\left(\frac{R_1}{R_1+R_2}+1\right)}$$

$$= \frac{\mathcal{E}}{R_1+R_2} + \frac{\mathcal{E} R_2 e^{-t/R'C}}{(R_1+R_2)(2R_1+R_2)} - \frac{\mathcal{E} R_2}{R_3(2R_1+R_2)} e^{-t/R'C}$$

$t \geq 0$ case:

$$Q_3(0) = 0$$

$$I_3(0) = \frac{\mathcal{E}'}{R'} \Rightarrow \text{as if}$$



$t \to \infty$ case

$$Q_3(0) = \mathcal{E}'/C$$

$$I_3(0) = 0$$

$\Rightarrow$ as if