Berry Phase, Polarization, Persistent Current ①

Berry phase:

Sources: Resta	ale2ts.ts.infn.it:6163/~resta/publ/notes-trois.ps.gz

- Pancharatnam (classical optics)
  precursor of Berry phase
- Hamiltonian of some system: $H(\vec{\xi})$

$\vec{\xi}$ – parameter (can be many dimensional)

$$H(\vec{\xi})|\Psi(\vec{\xi})\rangle = E(\vec{\xi})|\Psi(\vec{\xi})\rangle$$

example: electronic Hamiltonian depends parametrically on nuclear coordinates (nuclear coordinates · $\vec{\xi}$)

- for this example: assume non-degenerate ground state

$$\cancel{H(\vec{\xi}) = E} \quad H(\vec{\xi})|\Psi_0(\vec{\xi})\rangle = E_0(\vec{\xi})|\Psi_0(\vec{\xi})\rangle$$

- if we take two different points in parameter space $\vec{\xi}_1$ and $\vec{\xi}_2$, we can define quantity

$$e^{-i\Delta\varphi_{12}} = \frac{\langle\Psi(\vec{\xi}_1)|\Psi(\vec{\xi}_2)\rangle}{|\langle\Psi(\vec{\xi}_1)|\Psi(\vec{\xi}_2)\rangle|}$$

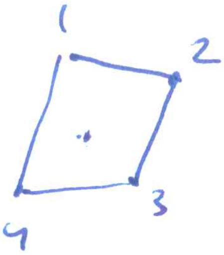
phase: $\Delta\varphi_{12} = -\operatorname{Im}\log\langle\Psi(\xi_1)|\Psi(\vec{\xi}_2)\rangle$

$\Delta\varphi_{12}$ · defined up to $2\pi m$, unless $\langle\Psi(\vec{\xi}_1)|\Psi(\vec{\xi}_2)\rangle = 0$ (orthogonal states)

- phase usually has no physical meaning: wavefunctions are undetermined up to a phase ⇒ phase is a "choice" gauge

- consider a finite number of points in $\vec{\xi}$-space

$$\gamma = \Delta\varphi_{12} + \Delta\varphi_{23} + \Delta\varphi_{34} + \Delta\varphi_{41} \longrightarrow$$



$$= -\operatorname{Im} \log \Big[\langle \Psi(\vec{\xi}_1) | \Psi(\vec{\xi}_2) \rangle \langle \Psi(\vec{\xi}_2) | \Psi(\vec{\xi}_3) \rangle \langle \Psi(\vec{\xi}_2) | \Psi(\vec{\xi}_4) \rangle \langle \Psi(\vec{\xi}_4) | \Psi(\vec{\xi}_1) \rangle \Big]$$

γ - gauge invariant! each $|\Psi(\vec{\xi}_i)\rangle$ has a $\langle \Psi(\vec{\xi}_i)|$ "pair" $\Rightarrow$ arbitrary phase is cancelled

$\Downarrow$ $\Downarrow$

γ - well-defined physical quantity $\rightarrow$ depends on path in $\vec{\xi}$-space

new aspect of γ: usually in quantum mechanics physically well-defined quantities correspond to Hermitian operators

- Berry argued for the physical significance of γ

- initial assumption: Hamiltonian is <u>parametric</u>

$\Rightarrow$ follows that isolated systems do not exhibit a Berry phase

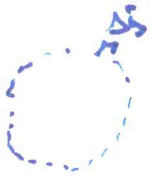
$\Downarrow$ $\Downarrow$

system + $\underbrace{\text{surroundings (rest of universe)}}_{\text{gives rise to Berry phase}}$

The continuum limit

$$e^{-i\Delta\varphi} = \frac{\langle\Psi(\vec\xi)|\Psi(\vec\xi+\Delta\vec\xi)\rangle}{|\langle\Psi(\vec\xi)|\Psi(\vec\xi+\Delta\vec\xi)\rangle|}$$

- consider case when $\Delta\vec\xi \to 0$, moreover construct circular path which is closed



$$\cancel{-i\Delta\varphi = \log\langle\Psi(\vec\xi)|\Psi(\vec\xi+\Delta\vec\xi)\rangle - \log|\langle\Psi(\vec\xi)|\Psi(\vec\xi+\Delta\vec\xi)\rangle|}$$

$$\cancel{= \log[\langle\Psi(\vec\xi)|\Psi(\vec\xi)\rangle + \langle\Psi(\vec\xi)|\nabla_\xi|\Psi(\vec\xi)\rangle\cdot\Delta\vec\xi] - \log|\langle\Psi(\vec\xi)|\Psi(\vec\xi)\rangle|}$$

$$\cancel{\mathrm{Im}}\ \Delta\varphi = -\mathrm{Im}\log\langle\Psi(\vec\xi)|\Psi(\vec\xi+\Delta\vec\xi)\rangle$$

$$= -\mathrm{Im}\log[\langle\Psi(\vec\xi)|\Psi(\vec\xi)\rangle + \langle\Psi(\vec\xi)|\nabla_\xi|\Psi(\vec\xi)\rangle\cdot\Delta\vec\xi] \quad \text{for small } \Delta\vec\xi$$

$$= -\mathrm{Im}\langle\Psi(\vec\xi)|\nabla_\xi\Psi(\vec\xi)\rangle\cdot\Delta\vec\xi$$

the circuit integral $\oint d\varphi$ around closed path given by $\Rightarrow \gamma = \oint d\varphi = -\mathrm{Im}\oint\langle\Psi(\vec\xi)|\nabla_\xi|\Psi(\vec\xi)\rangle\cdot d\vec\xi$

or equivalently: $\gamma = i\oint\langle\Psi(\vec\xi)|\nabla_\xi\Psi(\vec\xi)\rangle\cdot d\vec\xi$

- this limit can be carried out only if $|\Psi(\vec\xi)\rangle$ is differentiable, but the discrete Berry phase is always well-defined

Connection and Curvature

$$\gamma = \oint_C d\phi = i \oint_C \langle \Psi(\vec{\xi}) | \nabla_{\xi} \Psi(\vec{\xi}) \rangle \cdot d\vec{\xi}$$

relation to magnetism

vector potential: $\vec{A}(\vec{\xi})$

magnetic field: $\vec{B}(\vec{\xi}) = \vec{\nabla}_{\xi} \times \vec{A}(\vec{\xi})$

Stokes theorem: $\oint_C A(\vec{\xi}) \cdot d\vec{\xi} = \iint \vec{B}(\vec{\xi}) \cdot \vec{n}\, da$

line integral — surface integral

- Berry connection: $i \langle \Psi(\vec{\xi}) | \nabla_{\xi} \Psi(\vec{\xi}) \rangle = \vec{\chi}(\vec{\xi})$
- Berry curvature: $\cancel{Y_{\alpha\beta} = -\left(\partial_{\xi_\alpha} \chi_{\beta}(\vec{\xi}) - \partial_{\xi_\beta} \chi_{\alpha}(\vec{\xi}) \right)}$

$$Y_{\alpha\beta} = -\left(\frac{\partial \chi_\beta(\vec{\xi})}{\partial \xi_\alpha} - \frac{\partial \chi_\alpha(\vec{\xi})}{\partial \xi_\beta} \right)$$

consider one component of ~~curvature~~ connection

$$\chi_\alpha(\vec{\xi}) = i \langle \Psi(\vec{\xi}) | \frac{\partial}{\partial \xi_\alpha} | \Psi(\vec{\xi}) \rangle$$

$$\frac{\partial \chi_\alpha(\vec{\xi})}{\partial \xi_\beta} = i \langle \frac{\partial \Psi(\vec{\xi})}{\partial \xi_\beta} | \frac{\partial \Psi(\vec{\xi})}{\partial \xi_\alpha} \rangle + i \langle \Psi(\alpha) | \frac{\partial^2 \Psi(\vec{\xi})}{\partial \xi_\beta \partial \xi_\alpha} \rangle$$

$$Y_{\alpha\beta} = -i \left(\langle \frac{\partial \Psi(\vec{\xi})}{\partial \xi_\beta} | \frac{\partial \Psi(\vec{\xi})}{\partial \xi_\alpha} \rangle - \langle \frac{\partial \Psi(\vec{\xi})}{\partial \xi_\alpha} | \frac{\partial \Psi(\vec{\xi})}{\partial \xi_\beta} \rangle \right)$$

$$= 2\, \mathrm{Im} \langle \partial_{\xi_\alpha} \Psi(\vec{\xi}) | \partial_{\xi_\beta} \Psi(\vec{\xi}) \rangle$$

- real wave functions: $Y_{\alpha\beta} = 0$
  - nontrivial Berry phase occurs if domain $\vec{\xi}$ is not simply connected
- complex wavefunctions: nontrivial Berry phase can occur

<u>Perturbation expression for curvature</u>

$$\nabla_{\vec{\xi}} |\Psi_0(\vec{\xi})\rangle \cong \sum_n \frac{|\Psi_n(\vec{\xi})\rangle\langle\Psi_n(\vec{\xi})| \nabla_{\vec{\xi}} H(\vec{\xi}) |\Psi_0(\vec{\xi})\rangle}{E_n(\vec{\xi}) - E_0(\vec{\xi})}$$

(expression can be derived from first-order perturbation theory)

$$\Downarrow \qquad \Downarrow$$

$$Y_{\alpha\beta} = 2\,\mathrm{Im} \sum_{n\neq 0} \frac{\langle\Psi_0(\vec{\xi})| \frac{\partial H(\vec{\xi})}{\partial \xi_\alpha} |\Psi_n(\vec{\xi})\rangle\langle\Psi_n(\vec{\xi})| \frac{\partial H(\vec{\xi})}{\partial \xi_\beta} |\Psi_0\rangle}{(E_0(\vec{\xi}) - E_n(\vec{\xi}))^2}$$

curvature diverges at degeneracies

for real Ψ non-trivial Berry phase obtained if singularities present

complex wavefunctions can always give a non-trivial Berry phase

Physical Example of Berry Phase: Crystalline Polarization

Definition: $\vec{P} = -e \int d\vec{r}\, \vec{r}\, n(\vec{r})$ → valid for finite systems (molecules)

for crystals: definition is not valid due to absence of a valid position operator for periodic systems

- constructing one is easier if we switch to k-space: assume system is periodic in L

$$\langle \Psi | \hat{X} | \Psi \rangle = \langle \Psi | \sum_i x_i | \Psi \rangle$$

in k-space: $i \langle \Psi | \sum_i \frac{\partial}{\partial k_i} | \Psi \rangle$

$$= i \int dk_1 \dots dk_N \, \Psi^*(k_1 \dots k_N) \left(\sum_i \frac{\partial}{\partial k_i} \right) \Psi(k_1, \dots, k_N)$$

- write wave function as

$$\Psi(k_1 + K, \dots, k_N + K)$$

$$\sum_i \frac{\partial}{\partial k_i} \Psi(k_1 + K, \dots, k_N + K)$$

$$= \frac{\partial}{\partial K} \Psi(k_1 + K, \dots, k_N + K)$$

- in crystalline system: K - crystalline momentum has to be sampled (integrated over)

$$\rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} dK \dots$$

thus: polarization becomes $\frac{iL}{2\pi}\int_{-\pi}^{\pi} dk \langle \Psi | \frac{\partial}{\partial k} | \Psi \rangle$

- of interest is the discretized version:

$\frac{iL}{2\pi}\int_{-\pi}^{\pi} dk \langle \Psi | \frac{\partial}{\partial k} | \Psi \rangle$ is a Berry Phase since k can be viewed as a parameter

- we have shown in the beginning that a discretized Berry phase is related to a continuous one as

$$i \oint \langle \Psi | \partial_k | \Psi \rangle dk = \operatorname{Im} \log \prod_{i=1}^{N} \langle \Psi(q_i) | \Psi(q_{i+1}) \rangle$$

$$\Downarrow$$

$$\langle X \rangle = \frac{iL}{2\pi} \int dk \langle \Psi | \partial_k | \Psi \rangle = \frac{L}{2\pi} \operatorname{Im} \log \prod_{i=0}^{N-1} \langle \Psi(k_i) | \Psi(k_{i+1}) \rangle$$

where k_i defines a set of grid points

$\frac{2\pi}{NL}$ $\qquad$ L - length of system (periodicity)

- suppose there exists an operator which shifts the total momentum of the system by $\frac{2\pi}{NL}$

let's denote it as $\hat{U}\left(\frac{2\pi}{NL}\right)$

$$\hat{U}\left(\frac{2\pi}{NL}\right) | \Psi(k_1, \dots, k_n) \rangle = | \Psi(k_1 + \tfrac{2\pi}{NL}, \dots, k_n + \tfrac{2\pi}{NL}) \rangle$$

$$\Downarrow$$

$$\hat{U}\left(\frac{2\pi}{NL}\right) | \Psi(k_i) \rangle = | \Psi(k_{i+1}) \rangle$$

$$\hat{U}\left(\frac{2\pi}{NL}m\right)|\Psi(k_0)\rangle = |\Psi(k_m)\rangle$$

- if we are lucky this operator will also have the property

$$\cancel{\langle\Psi(k_m)|\hat{U}\left(-\frac{2\pi}{NL}m\right) = \langle\Psi(k_0)|}$$

$$\langle\Psi(k_0)|\hat{U}\left(-\frac{4\pi}{NL}m\right) = \langle\Psi(k_m)|$$

and most likely it will since it must be unitary (but we'll see that when we construct it)

- For any "i" I can write

$$\langle\Psi(k_i)|\Psi(k_{i+1})\rangle = \langle\Psi(k_i)|\hat{U}\left(\frac{2\pi}{NL}\right)|\Psi(k_i)\rangle$$

$$= \langle\Psi(k_0)|\hat{U}\left(-\frac{2\pi}{NL}i\right)\hat{U}\left(\frac{2\pi}{NL}\right)\hat{U}\left(\frac{2\pi}{NL}i\right)|\Psi(k_0)\rangle$$

$$= \langle\Psi(k_0)|\hat{U}\left(\frac{2\pi}{NL}\right)|\Psi(k_0)\rangle$$

$\Rightarrow$ polarization: $\langle x\rangle = \frac{1}{2\pi}\,\mathrm{Im}\log\left[\langle\Psi|\hat{U}\left(\frac{2\pi}{NL}\right)|\Psi\rangle\right]^N$

$$= \frac{NL}{2\pi}\,\mathrm{Im}\log\left[\langle\Psi|\hat{U}\left(\frac{2\pi}{NL}\right)|\Psi\rangle\right]$$

where $|\Psi\rangle$ is the wavefunction at any of the points k_i

(choose $k_i = 0$)

- remaining task construct total momentum shift

- second quantized formalism: fermions

$$[c_i^\dagger, c_j^\dagger] = [c_i, c_j] = 0$$

$$[c_i, c_j^\dagger] = \delta_{ij}$$

permutation operator: $P_{ij} = 1 - (c_i^\dagger - c_j^\dagger)(c_i - c_j)$

$$P_{ij} = P_{ij}^\dagger \qquad P_{ij} = P_{ji}$$

$$P_{ij} c_j = c_i P_{ij} \qquad P_{ij} c_i^\dagger = c_j^\dagger P_{ij} \qquad P_{ij} c_j^\dagger = c_i^\dagger P_{ij}$$

~~Product~~

Proof of $\rightarrow$ $[1 - (c_i^\dagger - c_j^\dagger)(c_i - c_j)]c_j$

$= c_j - c_i^\dagger c_i c_j + c_j^\dagger c_i c_j$

$= c_j - (1 - c_i c_i^\dagger) c_j - c_i c_j^\dagger c_j$

$= c_i c_i^\dagger c_j - c_i c_j^\dagger c_j$

$= c_i [c_i^\dagger c_j - c_j^\dagger c_j]$

$= c_i \underbrace{[(c_i^\dagger - c_j^\dagger) c_j]}_{\text{not quite } P_{ij}} c_i$

missing: $1 - (c_i^\dagger - c_j^\dagger) c_i$

$c_i [1 - (c_i^\dagger - c_j^\dagger) c_i] =$

$= c_i - c_i c_i^\dagger c_i + c_i c_j^\dagger c_i$

$= c_i (1 - c_i^\dagger c_i) + c_i c_j^\dagger c_i$

$= c_i c_i c_i^\dagger - c_i c_i c_j^\dagger = 0$ ✓

QED

$$P_{ij} P_{jk} = P_{ik} P_{ij} = P_{jk} P_{ik}$$

$$P_{ij} P_{ij} = 1$$

$$[P_{ij}, P_{kl}] = 0 \qquad \text{(commutator)}$$

these properties can be shown using the rules of second quantization (anti-commutator)

~~spin-dependent generalization also possible~~

- define:

$$\hat{U}_n = \hat{P}_{n-1\,n}, \dots P_{23} P_{12} \qquad n = 2, \dots, L$$

$\hat{U}_L$ satisfies:

$$\hat{U}_L c_j = \begin{cases} c_{j-1} \hat{U}_L & j = 2, \dots, L \\ c_L \hat{U}_L & j = 1 \end{cases}$$

$$\hat{U}_L^\dagger = P_{12} \dots P_{n-1\,n} \Rightarrow U_L^\dagger U_L = 1$$

U_L = unitary

- suppose now that the states c_j are momentum eigenstates: ~~let's denote them as $\tilde{c}_j$~~

c_j - momentum eigenstates $\to$ $k_j = \frac{2\pi}{NL} j$

$\tilde{c}_j$ - position eigenstates

if I have a state: $|\Psi\rangle = c_{j_1}^\dagger \dots c_{j_n}^\dagger |0\rangle$

$$\hat{U}_L |\Psi\rangle = c_{j_1-1}^\dagger \dots c_{j_n-1}^\dagger |0\rangle$$

the shift is to the negative direction with modulo$\left(\frac{2\pi}{L}\right)$

~~a~~ the operator constructed is indeed the momentum shift operator which gives the polarization

- there is another interesting way to express it (derived by very different means first by Resta, PRL (1998))

consider a one-particle system:

$$|\Psi\rangle = \tilde{c}_{x_1}|0\rangle$$

~~eq~~ $\tilde{c}_{x_1}$ - position basis

$$\tilde{c}_{x_1} = \frac{1}{\sqrt{N}} \sum_k e^{i\frac{2\pi}{NL}kx_1}$$

$$\hat{u}\,\tilde{c}_{x_1}|0\rangle = \hat{u}\frac{1}{\sqrt{N}} \sum_k e^{i\frac{2\pi}{NL}kx_1} c_k|0\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_k e^{i\frac{2\pi}{NL}kx_1} \hat{u}c_k|0\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_k e^{i\frac{2\pi}{NL}kx_1} c_{k-1}|0\rangle$$

sum over k runs over all points in the Brillouin zone, so we can shift the sum

$$= \frac{1}{\sqrt{N}} \sum_k e^{i\frac{2\pi}{NL}(k+1)x_1} c_k|0\rangle$$

$$= e^{i\frac{2\pi}{NL}x_1}\,\tilde{c}_{x_1}|0\rangle$$

we can then also write the momentum shift operator as

$$\hat{U}\, c^{+}_{x_1} \dots c^{+}_{x_N} |0\rangle = e^{i\frac{2\pi}{NL}(x_1 + x_2 + \dots x_N)} c^{+}_{x_1} \dots c^{+}_{x_N} |0\rangle$$

and the polarization as

$$\langle X \rangle = \frac{NL}{2\pi} \operatorname{Im} \log \langle \Psi | e^{i\frac{2\pi}{NL}\hat{X}} | \Psi \rangle$$

$$\hat{X} = \sum_i x_i$$

---

the same formalism, with appropriate modifications can be used to develop a Berry phase theory for the persistent current

definition of persistent current:

$$J(\Phi) = \frac{\partial E(\Phi)}{\partial \Phi}$$

Φ - perturbation (momentum shift, or vector potential)

$$E(\Phi) = \langle \Psi(\Phi) | H(\Phi) | \Psi(\Phi) \rangle$$

$|\Psi(\Phi)\rangle$ = ground state energy

$H(\Phi)$ = Hamiltonian

both under the perturbation

$$H(\bar{Q}) = \sum_i \frac{(p_i + \bar{Q})^2}{2m} + \hat{V}$$

$\hat{V}$ - some potential

current:
$$\frac{\partial \langle \Psi(\bar{Q}) | H(\bar{Q}) | \Psi(\bar{Q}) \rangle}{\partial \bar{Q}} = \langle \Psi(\bar{Q}) | \frac{\partial H(\bar{Q})}{\partial \bar{Q}} | \Psi(\bar{Q}) \rangle$$

$$\frac{\partial H(\bar{Q})}{\partial \bar{Q}} = \sum_i \frac{(p_i + \bar{Q})}{m} = \frac{N}{m}\bar{Q} + \frac{\sum_i p_i}{m}$$

$$J(\bar{Q}) = \cancel{\#} \frac{N}{m}\bar{Q} + \langle \Psi(\bar{Q}) | \frac{\sum_i p_i}{m} | \Psi(\bar{Q}) \rangle$$

$$= \frac{N}{m}\bar{Q} + \frac{i}{m} \langle \Psi(\bar{Q}) | \sum_i \partial_{x_i} | \Psi(\bar{Q}) \rangle$$

we can write wavefunction as

$$\sum_i \partial_{x_i} \Psi(x_1 + X, \dots x_N + X; \bar{Q})$$

$$= \partial_X \Psi(x_1 + X, \dots, x_N + X; \bar{Q})$$

and average over X

$$J(\bar{Q}) = \frac{N}{m}\bar{Q} + \cancel{\frac{i}{m}\frac{1}{L}\int_0^L dX\, \Psi^*(x_1 + X, \dots, x_N + X)\partial_X \Psi(x_1 + \dots}$$

$$+ \underbrace{\frac{i}{m}\frac{1}{L}\int_0^L dX \langle \Psi^*(X;\bar{Q}) | \partial_X | \Psi(X;\bar{Q}) \rangle}_{\text{Berry phase}}$$

$=$

$$\langle x_1, \dots, x_N | \Psi(X, \bar{Q}) \rangle = \Psi(x_1 + X, \dots, x_N + X; \bar{Q})$$

- One can apply the same analysis as before to obtain

$$\frac{1}{L}\int dx \, \langle \Psi(x,\varphi) | \partial_x | \Psi(x,\varphi) \rangle$$

$$\simeq \frac{1}{L} \operatorname{Im} \ln \prod_{i=0}^{N-1} \langle \Psi(x_i) | \Psi(x_{i+1}) \rangle$$

$$= \cancel{\frac{1}{L} \operatorname{Im} \ln} = \frac{1}{\Delta x} \operatorname{Im} \ln \langle \Psi(0) | U(\Delta x) | \Psi(0) \rangle$$

$U(\Delta x)$ is the total position shift rather than the total momentum shift

it also holds that

$$U(\Delta x) = e^{i \Delta x \underbrace{(k_1 + \dots k_N)}_{\text{total momentum}}} = e^{i \Delta x \hat{K}}$$

- current response for some variational wavefunctions for the Hubbard model

Gutzwiller wavefunction: $|\Psi_G\rangle = e^{-\gamma \sum_i n_{i\uparrow} n_{i\downarrow}} |FS\rangle$

Baeriswyl wavefunction: $|\Psi_B\rangle = e^{-\alpha \hat{T}} |\Psi_\infty\rangle$

mixed wavefunctions:

$$|\Psi_{BG}\rangle = e^{-\alpha \hat{T}} e^{-\gamma \sum_i n_{i\uparrow} n_{i\downarrow}} |FS\rangle$$

$$|\Psi_{GB}\rangle = e^{-\gamma \sum_i n_{i\uparrow} n_{i\downarrow}} e^{-\alpha \hat{T}} |\Psi_\infty\rangle$$

$\hat{T}$ - kinetic energy operator $\quad n_{i\uparrow} n_{i\downarrow}$ - double occupation the projector operators $e^{-\gamma \sum_i n_{i\uparrow} n_{i\downarrow}}$ and $e^{-\alpha \hat{T}}$ both commute with $\hat{U}(\Delta x)$

- to see this: $\hat{U}(\Delta x) e^{-\gamma \sum_i n_{i\uparrow} n_{i\downarrow}}$

  $\hat{U}(\Delta x)$ shifts the position of all particles $\Rightarrow$ the number of double occupations cannot change

  $$\hat{U}(\Delta x) e^{-\alpha \hat{T}} = e^{i\Delta x \hat{K}} e^{-\alpha \hat{T}}$$

  Both $\hat{K}$ and $\hat{T}$ are diagonal in momentum space

- we can say that the function on which the projector acts determines the current response for these wavefunctions

  example Gutzwiller

$$\text{phase: } \frac{1}{\Delta x} \operatorname{Im} \ln \frac{\langle \Psi_G | e^{i\Delta x \hat{K}} | \Psi_G \rangle}{\langle \Psi_G | \Psi_G \rangle}$$

$$= \frac{1}{\Delta x} \operatorname{Im} \ln \frac{\langle FS | e^{-2\gamma \sum_i n_{i\uparrow} n_{i\downarrow}} e^{i\Delta x \hat{K}} | FS \rangle}{\langle FS | e^{-2\gamma \sum_i n_{i\uparrow} n_{i\downarrow}} | FS \rangle}$$

$$= \frac{1}{\Delta x} \operatorname{Im} \ln e^{i\Delta x K_{FS}} = K_{FS}$$

$$\hat{K} | FS \rangle = K_{FS} | FS \rangle$$

- for the Baeriswyl wave functions the function $|\Psi_\infty\rangle$ is not an eigen function of $\hat{U}(\Delta x)$)

Sources:

1.) http://ale2ts.ts.infn.it/6163/vresta/publ/notes_trois.ps.gz

2.) Resta Phys. Rev. Lett. 82 370 (1999).

3.) M.V. Berry Proc. Roy. Soc. London A392 45 (1984).

4.) Essler, Frahm, Göhmann, Klümper, Korepin: The One-Dimensional Hubbard Model Cambridge (2005).

---

1. B.H. J Phys A 42 412003 (2009)

2.) B.H. J Phys. Soc. Japan 81 023701 (2012).

3. B.H. arxiv.org/1208.0671