The topology of ideal conduction

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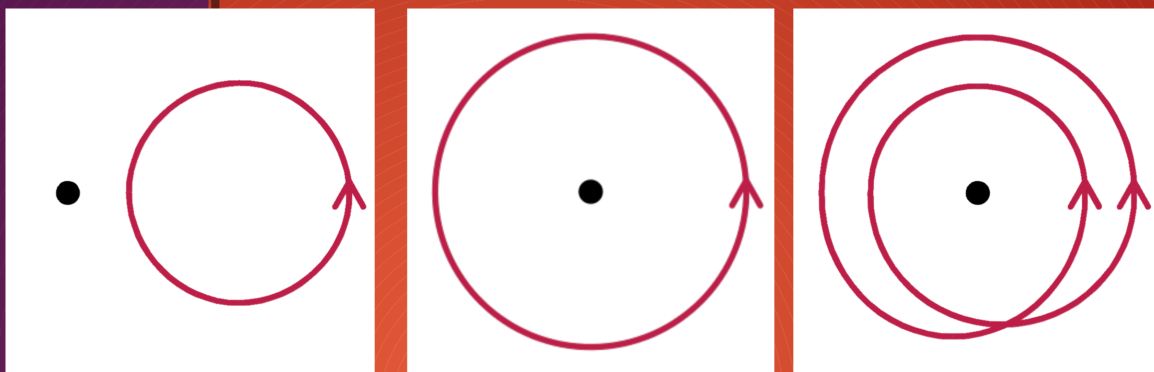
Outline

- Topology, quantum Hall effect, topological insulators
  - Topological invariants
  - Quantization and edge states
- Kohn`s tenet: Drude weight and localization
- Transport weights: Drude weight, superfluid weight
  - Topological invariants
  - Quantization and edge states
- Conclusion

Topology



Winding number

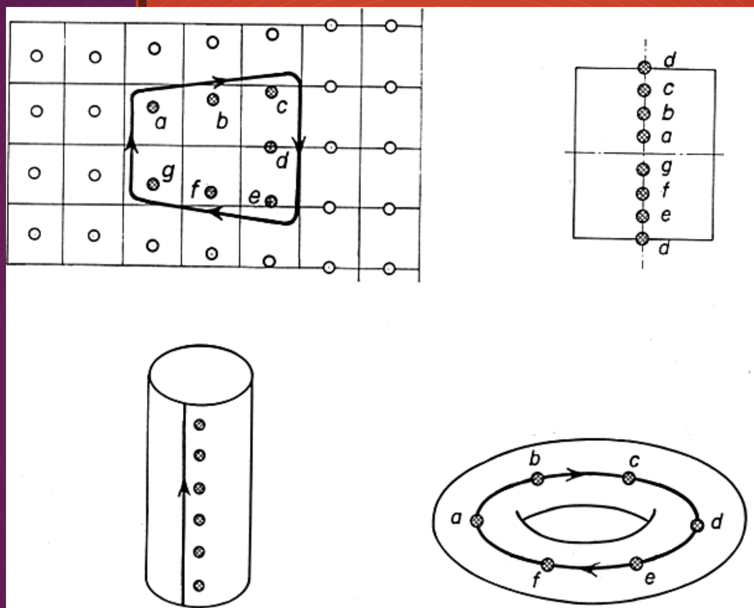
Genus g: "number of holes"

Related to integral over the curvature of the surface:

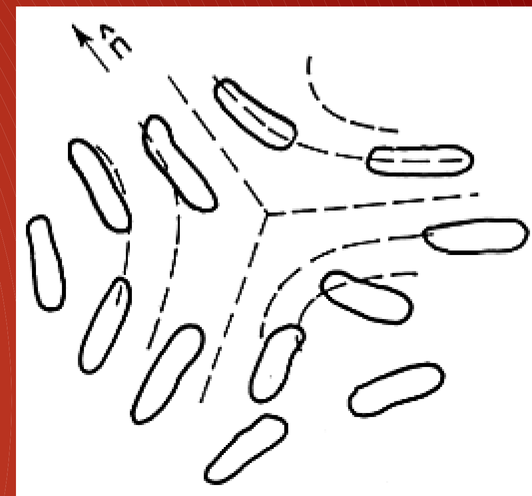
$$n(1-g) = (1/2\pi) \int_S K dS$$

Pictures from Sethna: wikipedia (Jim Belk, geek3, Oleg Alexandrov)

Topology and physics



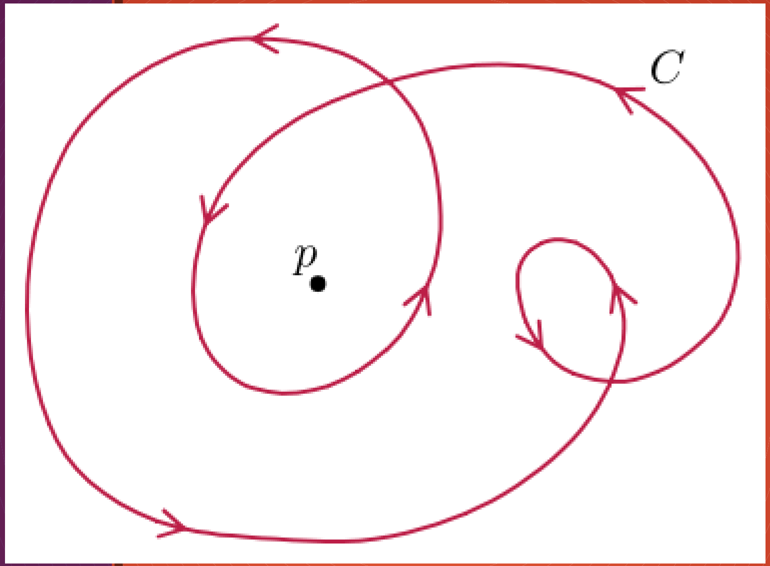
Dislocations in crystals



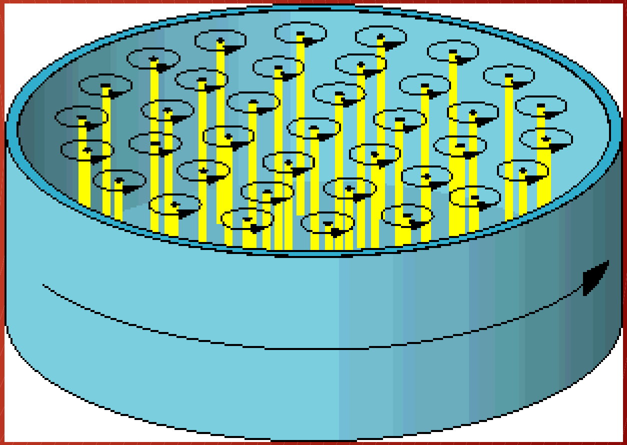
Liquid crystals

Pictures from Sethna: Entropy, Order Parameters and Complexity (Oxford, 2006)

Topology and physics



Winding number



Vortex quantization in rotating supefluids

Pictures from wikipedia; Erkki Thuneberg, ltl.tkk.fi

Topology in physics

- Crystal defects, dislocations, liquid crystals
- Winding number: quantized flow in superfluids, superfluid weight
- Edge states in quantum Hall effect
- Topological insulators, skyrmions
- **Ideal conduction: superfluidity, superconductivity**

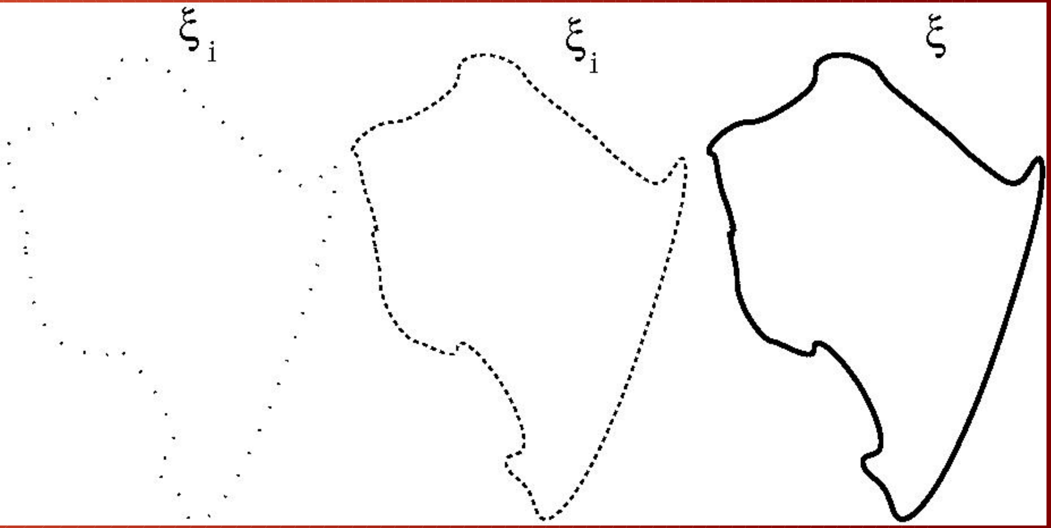
Berry phase

- Arises when a quantum system is taken around an adiabatic cycle:
- Discrete Berry phase:

$$\Phi = \mathrm{Im}\ln\prod_{i=1}^{N}\langle\Psi(\xi_i)|\Psi(\xi_{i+1})\rangle$$

- Continuous Berry phase:

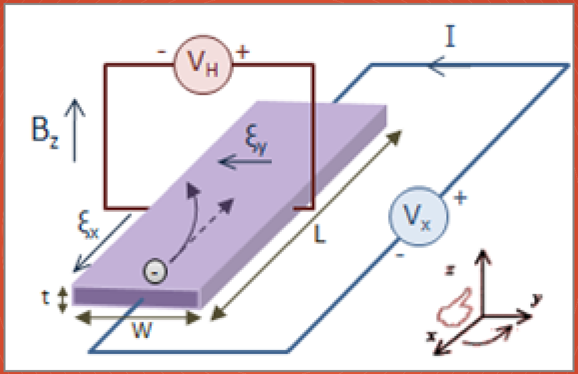
$$\Phi = i\oint d\xi\cdot\langle\Psi(\xi)|\nabla_\xi|\Psi(\xi)\rangle$$



- Stokes theorem: $\Phi = i\int\int d\sigma\cdot\nabla_\xi\times\langle\Psi(\xi)|\nabla_\xi|\Psi(\xi)\rangle$
  - Integral over the curvature: can be a **topological quantity**

Classical Hall effect

- Discovered by Edwin Hall in 1879

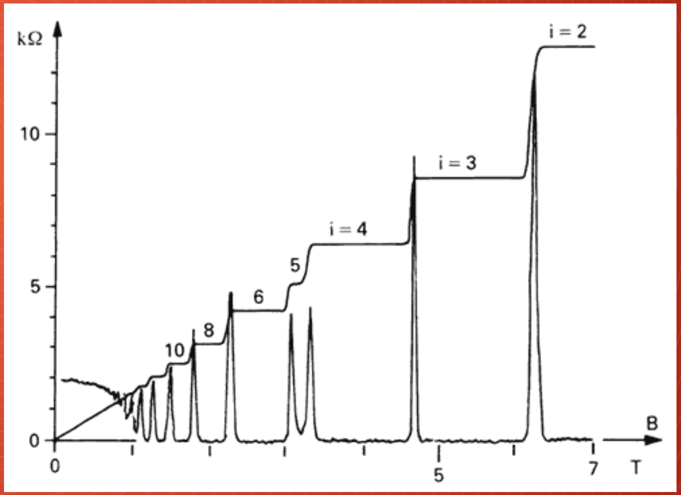


- Hall conductance: $\sigma_{yx} = \frac{nec}{H}$
- Based on the direction of polarization one can determine the sign of charge carriers in a material

Picture credit: wikimedia commons (Steven Byrnes)

Integer quantum Hall effect

- Discovered by von Klitzing, Dorda, and Pepper (*PRL*, 1980)

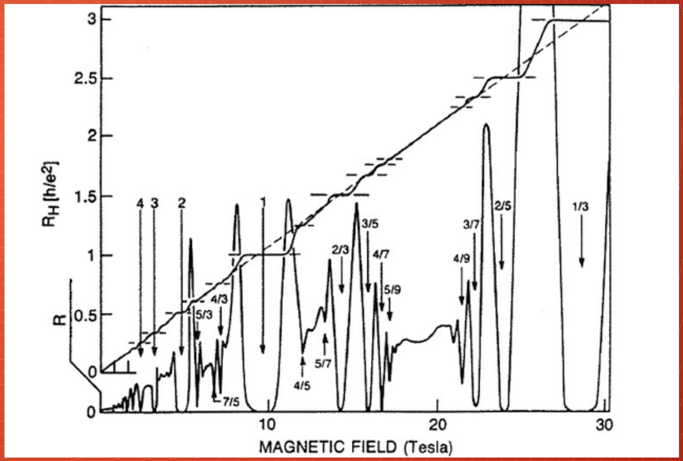


- Low-*T*, high-*B* → Hall conductance: $\sigma_{yx} = \frac{ie^2}{h}$
- Conductivity quantum

Picture credit: nobelprize.org

Fractional quantum Hall effect

- Discovered by Tsui, Stormer, and Gossard (*PRL*, 1982)



- Hall conductance: $\sigma_{yx} = \frac{p}{q} \frac{e^2}{h}$
- Pronounced at particular fractions (1/3, 2/5, 4/7, 5/11, 2/3, ...)

Picture credit: nobelprize.org

Laughlin argument

- Laughlin (*PRB,* 1981) considers a ribbon geometry

$$\hat{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + eE_0 y$$

Landau gauge: $\mathbf{A} = (H_0 y\hat{x}, 0, 0)$

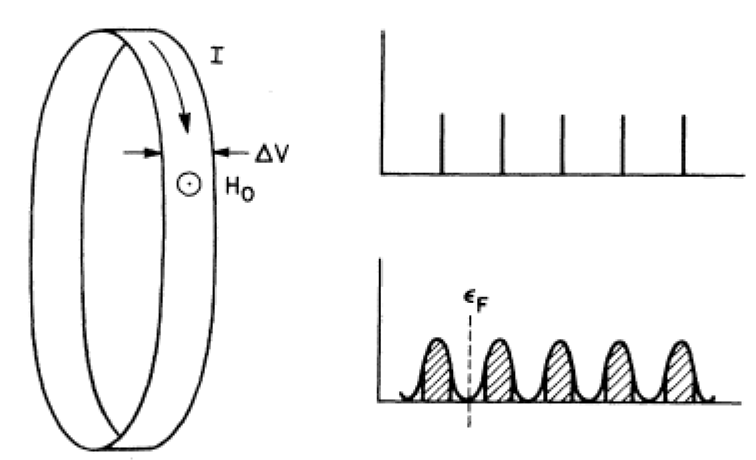


FIG. 1. Left: Diagram of metallic loop. Right: Density of states without (top) and with (bottom) disorder. Regions of delocalized states are shaded. The dashed line indicates the Fermi level.

- Gauge invariance results in: $I = \frac{ne^2V}{h}$

QHE and topology

- Thouless, Kohmoto, Nightingale, and den Nijs (*PRL*, 1982)
  - In the presence of a magnetic field Bloch condition is generalized
  - Hall conductance:

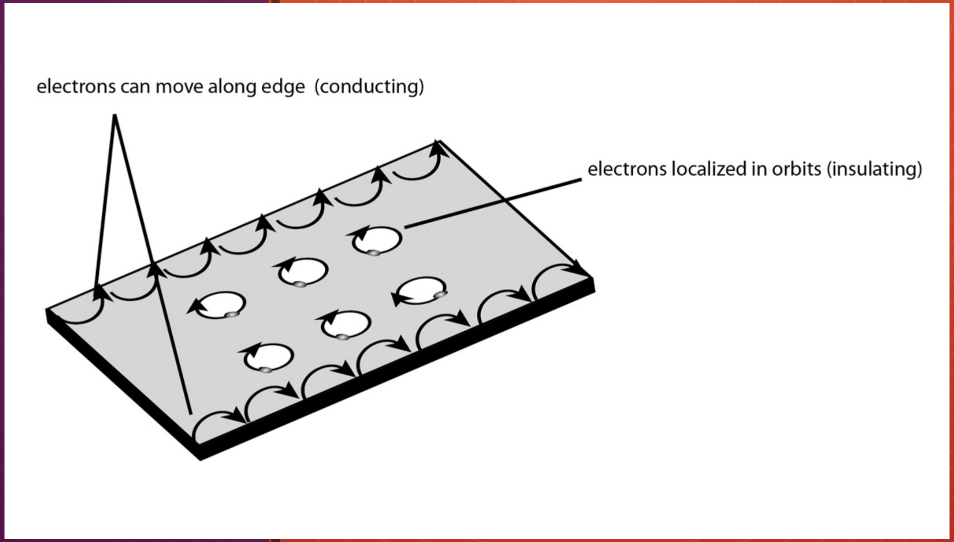
$$\sigma_{xy} = \frac{ie^2}{2\pi h} \sum_{i=1}^{n_{occ}} \int\int dk_x dk_y \left(\langle \partial_{k_x} u_i | \partial_{k_y} u_i \rangle - \langle \partial_{k_y} u_i | \partial_{k_x} u_i \rangle \right)$$

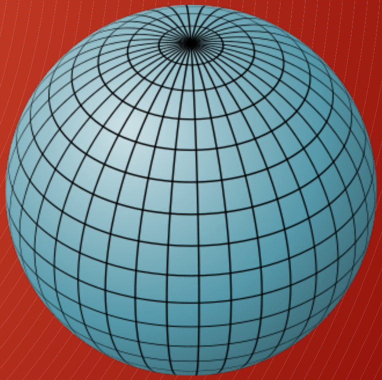
  - Integral over a curvature: Berry phase (quantized)
  - Integral over the two-dimensional Brillouin zone: a torus

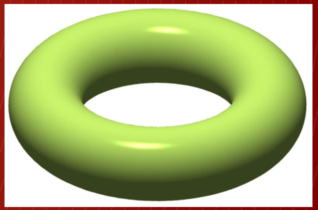
Topological: TKNN invariant

Edge states

- Bulk-boundary correspondence principle (Halperin, *PRB*, 1982): quantum Hall systems have current carrying edge states at their boundaries







Two objects can not be continuously deformed into each other

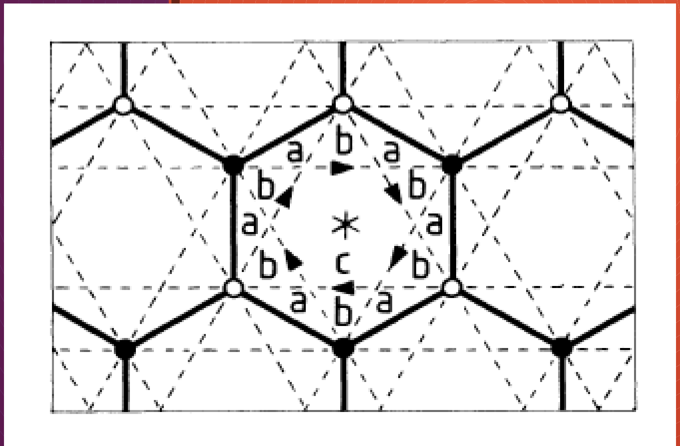
- At **interfaces** between systems with different topological invariants: **topological invariant undefined**

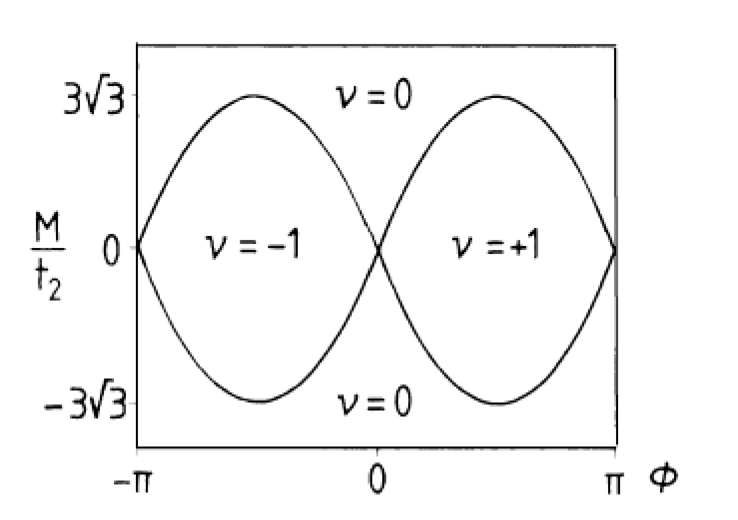
EDGE CURRENTS ARISE

Picture credit: jqi.umd.edu

Haldane model: QHE without B-field

- Haldane (*PRL*, 1988) suggested a model which exhibits chiral edge states in the **absence** of a magnetic field



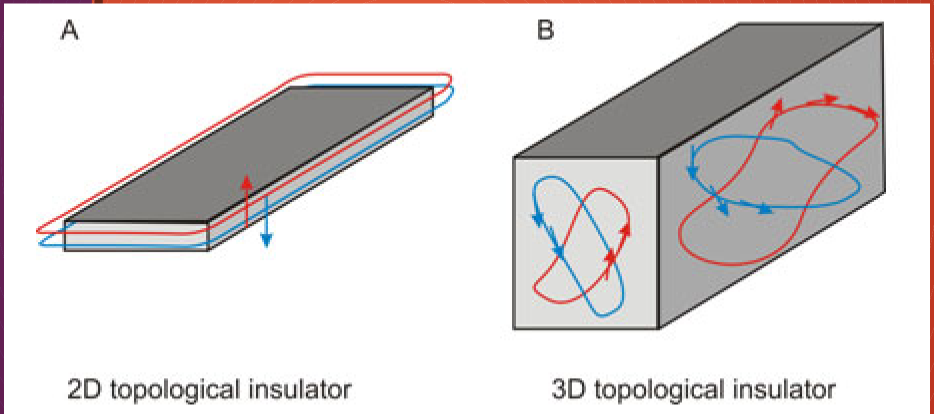


- Hall conductance:

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

Topological insulators

- Insulators in the bulk, but have conducting states on their edges



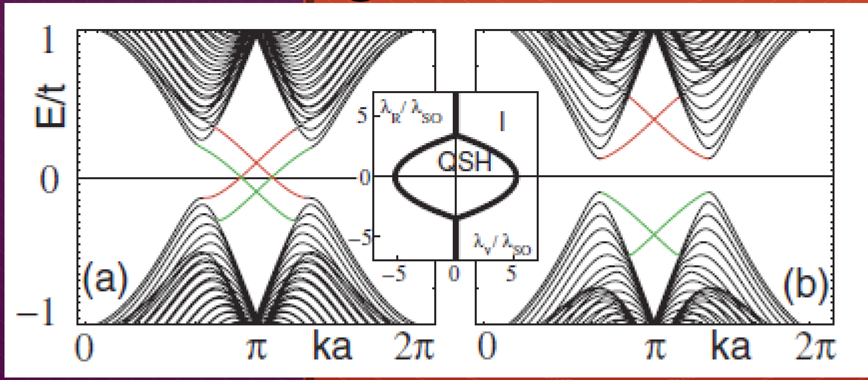
- Realizations:
  - 2D: HgTe, CdTe quantum wells (Bernevig et al., *Science*, 2006; Konig, *Science*, 2007)
  - 3D generalizations: “weak” and “strong” topological insulator (Fu, Kane, Mele, *PRL* 2007; Moore and Balents, *PRL* 2007; Roy *PRB* 2009)

2D Topological insulators: Kane-Mele model

- Kane and Mele (*PRL*, 2005) proposed a model in which chiral current carrying edge states are spin-resolved

$$H = -t \sum_{\langle i,j \rangle} c_i^\dagger c_j + i\lambda_{SO} \sum_{\langle\langle i,j \rangle\rangle} \nu_{ij} c_i^\dagger \sigma_z c_j + i\lambda_R \sum_{\langle i,j \rangle} c_i^\dagger (\mathbf{s} \times \hat{\mathbf{d}}_{ij})_z c_j + \lambda_\nu \sum_i \xi_i c_i^\dagger c_i$$

- Edge states and phase diagram:



Topological invariant:

$$I = \int d\mathbf{k} \cdot \nabla_{\mathbf{k}} \ln(P(\mathbf{k}) + i\delta)$$

Summary of topological phases

- Certain phases of matter are topological
  - Not characterized by symmetry breaking
  - Characterized by topological invariants (defined on the band structure) and quantization
  - Exhibit edge states
- Examples:
  - QHE (gapped bulk, edge currents)
  - Topological insulators (gapped bulk, edge currents)

Drude weight

- Derived by Kohn (*PR*, 1964) in a seminal paper on the theory of insulation
  - In this work insulation is connected with many-body localization
- Drude weight is the strength of the δ-peak of the conductivity at zero frequency

$$\sigma(\omega) = D\delta(0) + \sigma_{\mathrm{reg}}(\omega)$$

- Measure of ideal (non-diffusive) conductivity
- Related to other transport quantities: superfluid weight and Meissner weight

Why are some materials conductors and others insulators?

- Kohn's comprehensive answer:

Theory of the Insulating State*

WALTER KOHN
University of California, San Diego, La Jolla, California
(Received 30 August 1963)

In this paper a new and more comprehensive characterization of the insulating state of matter is developed. This characterization includes the conventional insulators with energy gap as well as systems discussed by Mott which, in band theory, would be metals. The essential property is this: Every low-lying wave function Φ of an insulating ring breaks up into a sum of functions, $\Phi = \sum_{-\infty}^{\infty} \Phi_M$, which are localized in disconnected regions of the many-particle configuration space and have essentially vanishing overlap. This property is the analog of localization for a single particle and leads directly to the electrical properties characteristic of insulators. An Appendix deals with a soluble model exhibiting a transition between an insulating and a conducting state.

- **Insulation** is a result of **many-body localization** (*PR* **133** A171 (1964))

Drude weight/dc conduction

- Introduce frequency dependent vector potential and take the

  zero frequency limit → constant shift of momenta:

$$H = \sum_i \frac{(p_i + \Phi)^2}{2m} + \hat{V}$$

$$J(\Phi) = \frac{\partial E_0(\Phi)}{\partial \Phi}$$

$$D = \frac{1}{V} \left[\frac{\partial^2 E_0(\Phi)}{\partial \Phi^2} \right]_{\Phi=0}$$

Ideal conduction vs. diffusive conduction

- In an ideal conductor charges accelerate with electric field (Newton's law is obeyed):

$$m\frac{dv}{dt} = qE$$

- In a non-ideal conductor collisions slow electrons (dissipation occurs)
  - Drude model (collision time τ) → Ohm`s law

$$m\frac{v}{\tau} = qE \quad \rightarrow \quad j = \sigma_0 E = \frac{nq^2\tau}{m}E$$

- Finite σ_0 → zero *D*, infinite σ_0 → finite *D*

Meissner weight

- Perfect conductor: $$\mathbf{E} = \frac{1}{n^{(s)}} \frac{\partial \mathbf{j}}{\partial t}$$
- Maxwell equation for curl of electric field leads to: $$\frac{\partial}{\partial t}(\nabla \times \mathbf{j} + n^{(s)}\mathbf{B}) = 0$$
- London equation: $\mathbf{j} = n^{(s)}\mathbf{A}$
- Associating the vector potential $\mathbf{A}$ with Φ we obtain $$n^{(s)} = \frac{1}{\mathrm{V}} \left[\frac{\partial^2 E(\Phi)}{\partial \Phi^2} \right]_{\Phi=0}$$

Non-classical rotational inertia

- Liquid helium, rotating bucket experiment: below the critical point rotational inertia of rotating He^4 is reduced (Andronikashvili, Hess & Fairbank experiments)
- Work to rotate above T_c : $\Delta W = \frac{I}{2}\Phi^2$
- Work to rotate below T_c : $\Delta W = \frac{I^{(n)}}{2}\Phi^2$
- Rotational inertia associated with the superfluid fraction: $I^{(s)} = \frac{\partial^2 E^{(s)}(\Phi)}{\partial \Phi^2}$ (Φ angular velocity)

Total position ill-defined

- Ill-defined nature of the position operator gives rise to **difficulties** in calculating **physical quantities: polarization**, quantifying **localization,** and even **conductivity**
- As emphasized by Resta (RMP, 1994), this issue **invalidates** the **Clausius-Mosotti** picture of dielectrics, where a unique boundary for the periodic unit cell is assumed. It also invalidates the polarization expressions given in a number of textbooks.
- **Modern theory of polarization**: **total position** corresponds to a **geometric phase**

Polarization theory

- We are after $\langle X \rangle = \langle \Psi_0 | \hat{X} | \Psi_0 \rangle$ where $\hat{X} = \sum_{i=1}^{N} \hat{x}_i$.
- Write the position operators in reciprocal space as $\hat{X} = i \sum_{j=1}^{N} \frac{\partial}{\partial k_j}$
- For a wavefuntion in reciprocal space it holds that

$$\hat{X} \Psi_0(k_1 + K, ..., k_N + K)\rangle = i \frac{\partial}{\partial K} \Psi_0(k_1 + K, .., k_N + K)$$

Polarization theory

- Averaging over K results in

$$\langle X \rangle = \frac{i}{2\pi} \int_{BZ} \mathrm{d}K \langle \Psi_0(K) | \frac{\partial}{\partial_K} | \Psi_0(K) \rangle$$

- To arrive at a practical scheme: discretize the variable K which results in

$$\langle X \rangle = \lim_{\Delta K \to 0} \frac{1}{\Delta K} \mathrm{Im} \ln \langle \Psi_0 | e^{i \Delta K \hat{X}} | \Psi_0 \rangle$$

Quantifying localization

- Expression for the spread

$$\sigma_X^2 = -\lim_{\Delta K \to 0} \frac{2}{\Delta K^2} \mathrm{Re} \ln \langle \Psi_0 | e^{i \Delta K \hat{X}} | \Psi_0 \rangle$$

  can be used to quantify localization
- Divergent spread: conductor
- Finite spread: insulator
- It has been applied in a large number of cases, band structure calculations, strongly correlated systems, Anderson localized systems, etc. and the **results are always in agreement with Kohn's hypothesis regarding localization**

Current as a geometric phase

- Similarly to the theory of polarization the current can also be expressed as a geometric phase
  - Hamiltonian $H = \sum_i \frac{(p_i + \Phi)^2}{2m} + V$
  - Current: $J(\Phi) = \frac{\partial E_0(\Phi)}{\partial \Phi}$

$$J(\Phi) = \frac{N\Phi}{m} - \frac{i}{ma} \int_0^a \mathrm{d}X \langle \Psi(X) | \frac{\partial}{\partial X} | \Psi(X) \rangle$$

BH, *J. Phys. Soc. Japan*, (2012)

Drude weight as a topological invariant

- Starting with the current

$$J(\Phi) = \frac{N\Phi}{m} - \frac{i}{ma}\int_0^a \mathrm{d}X \langle\Psi(X)|\frac{\partial}{\partial X}|\Psi(X)\rangle$$

and taking the derivative with respect to Φ
and averaging over the Brillouin zone results in

$$D = \frac{N}{m} - \frac{i}{2\pi m}\int_{BZ}\int_0^a \mathrm{d}K\mathrm{d}X \left(\left\langle \frac{\partial}{\partial K}\Psi \middle| \frac{\partial}{\partial X}\Psi \right\rangle - \left\langle \frac{\partial}{\partial X}\Psi \middle| \frac{\partial}{\partial K}\Psi \right\rangle\right)$$

BH, *Phys. Rev. B* **87** 235123 (2013).

Drude weight as a topological invariant

$$D = \frac{N}{m} - \frac{i}{2\pi m}\int_{BZ}\int_0^a \mathrm{d}K\mathrm{d}X\left(\left\langle \frac{\partial}{\partial K}\Psi \middle| \frac{\partial}{\partial X}\Psi\right\rangle - \left\langle \frac{\partial}{\partial X}\Psi \middle| \frac{\partial}{\partial K}\Psi\right\rangle\right)$$

Expression for the Hall conductance:

$$\sigma_{xy} = \frac{ie^2}{2\pi h}\sum_{i=1}^{n_{occ}}\int\int dk_x dk_y \left(\langle \partial_{k_x} u_i | \partial_{k_y} u_i\rangle - \langle \partial_{k_y} u_i | \partial_{k_x} u_i\rangle\right)$$

- Thouless, Kohmoto, Nightingale, and den Nijs, *Phys. Rev. Lett.* **49** 405 (1982).

- Drude weight consists of a TKNN-like topological invariant

Drude weight as a topological invariant

$$D = \frac{N}{m} - \frac{i}{2\pi m}\int_{BZ}\int_0^a \mathrm{d}K\mathrm{d}X\left(\left\langle \frac{\partial}{\partial K}\Psi \middle| \frac{\partial}{\partial X}\Psi\right\rangle - \left\langle \frac{\partial}{\partial X}\Psi \middle| \frac{\partial}{\partial K}\Psi\right\rangle\right)$$

- The TKNN term consists of the commutator of two "heuristic" many-body operators $i\frac{\partial}{\partial K}$ $i\frac{\partial}{\partial X}$
- First term can also be written as $\frac{i}{m}\sum_{j=1}^{N}\langle\Psi|[\partial_{k_j},\partial_{x_j}]|\Psi\rangle$
- Drude weight is the difference of commutator of single body momenta and positions and commutator of total momentum and position

Quantization of the Drude weight

- London equation: $\nabla \times \mathbf{j}_s = -\frac{n_s e^2}{mc}\mathbf{B}$
- Integrate over surface of a cavity:

$$\oint \mathbf{j}_s \cdot d\mathbf{l} = -\frac{n_s e^2}{mc}\Phi_B$$

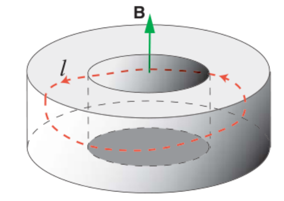


Figure 1.3: Magnetic flux through the hole in a superconductor is quantized.

- Replace current with expectation value of current:

$$\frac{ie\hbar}{m}\oint \langle\Psi|\nabla_{\mathbf{r}}|\Psi\rangle \cdot d\mathbf{l} = -\frac{n_s e^2}{mc}\Phi_B$$

  - For superconductors it is an experimental fact that the magnetic flux inside a cavity is quantized in units of $\Phi_B = jhc/e$
  - The integral of the current around a cavity is a Berry phase, also quantized

Picture credit: Kopnin lecture notes, U. of Buffalo

Edge states?

- Interface between insulator and ideal conductor
  - Simple tight-binding model (ideal conductor):

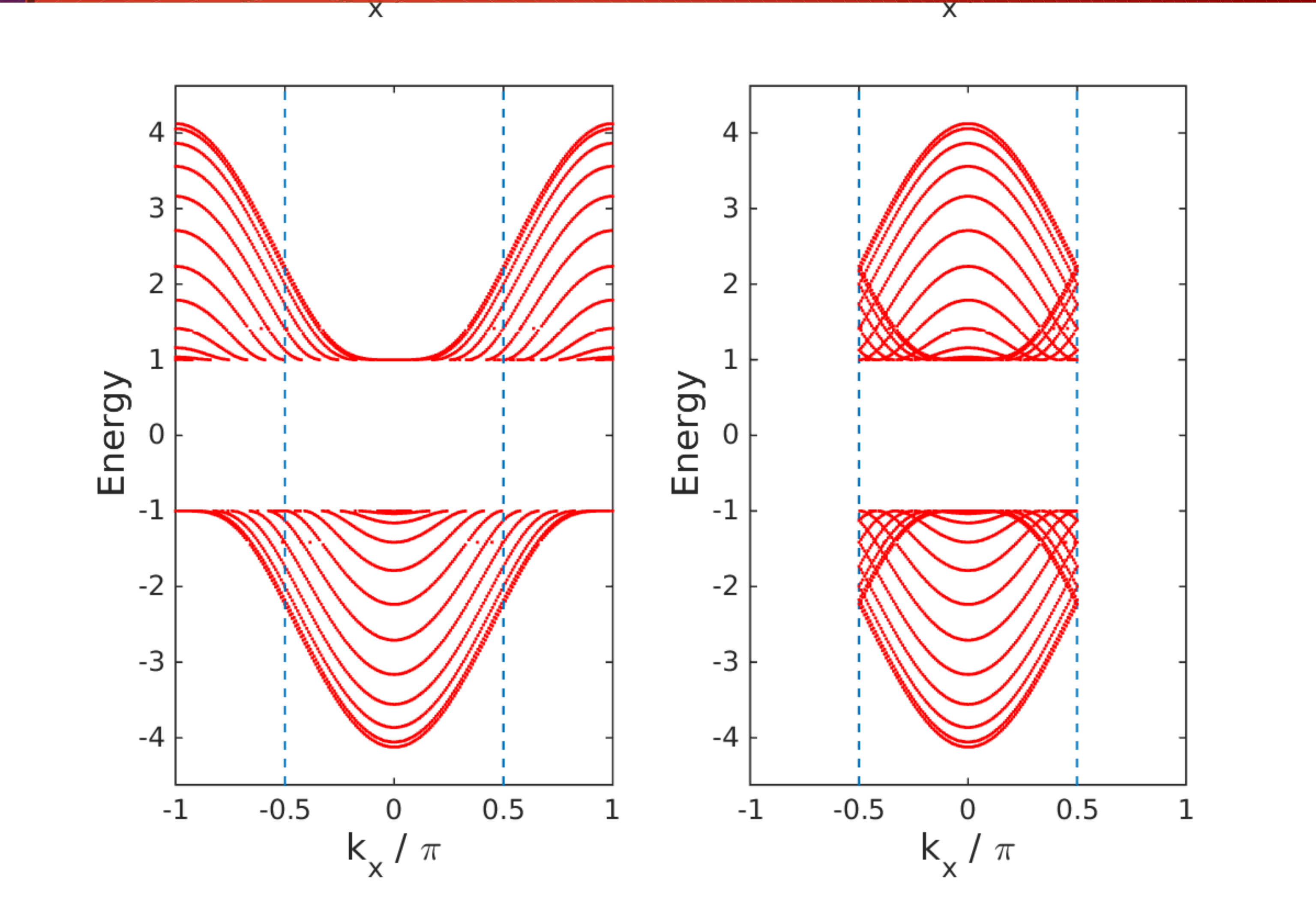
$$\hat{H}_{TB} = -t \sum_{\langle i,j \rangle} (c_i^\dagger c_j + c_j^\dagger c_i)$$

  - Crystalline insulator:

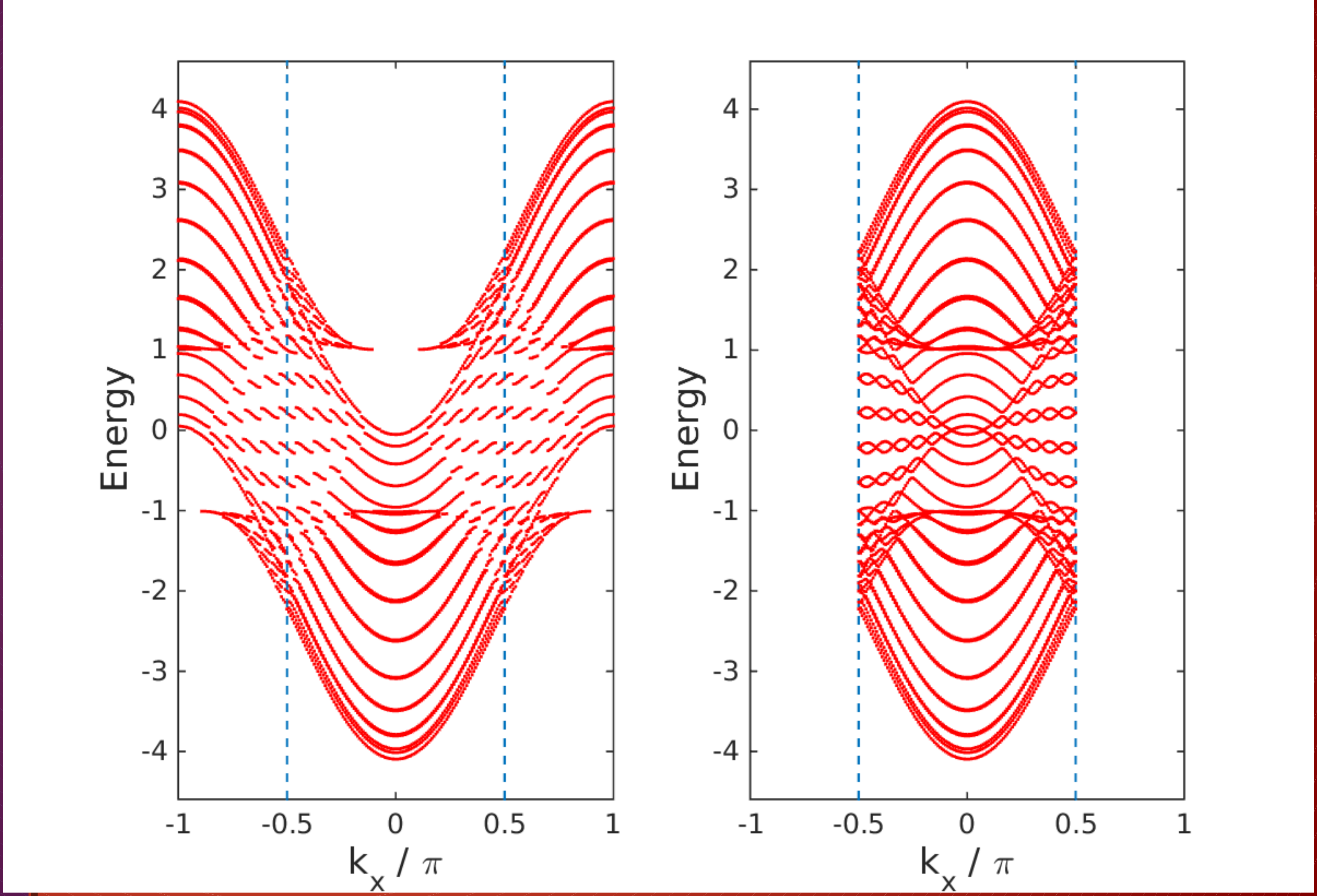
$$\hat{H}_{CI} = \hat{H}_{TB} + U \sum_i \xi_i \hat{n}_i$$

$$\xi_i = \pm 1$$

Band structure of crystalline insulator

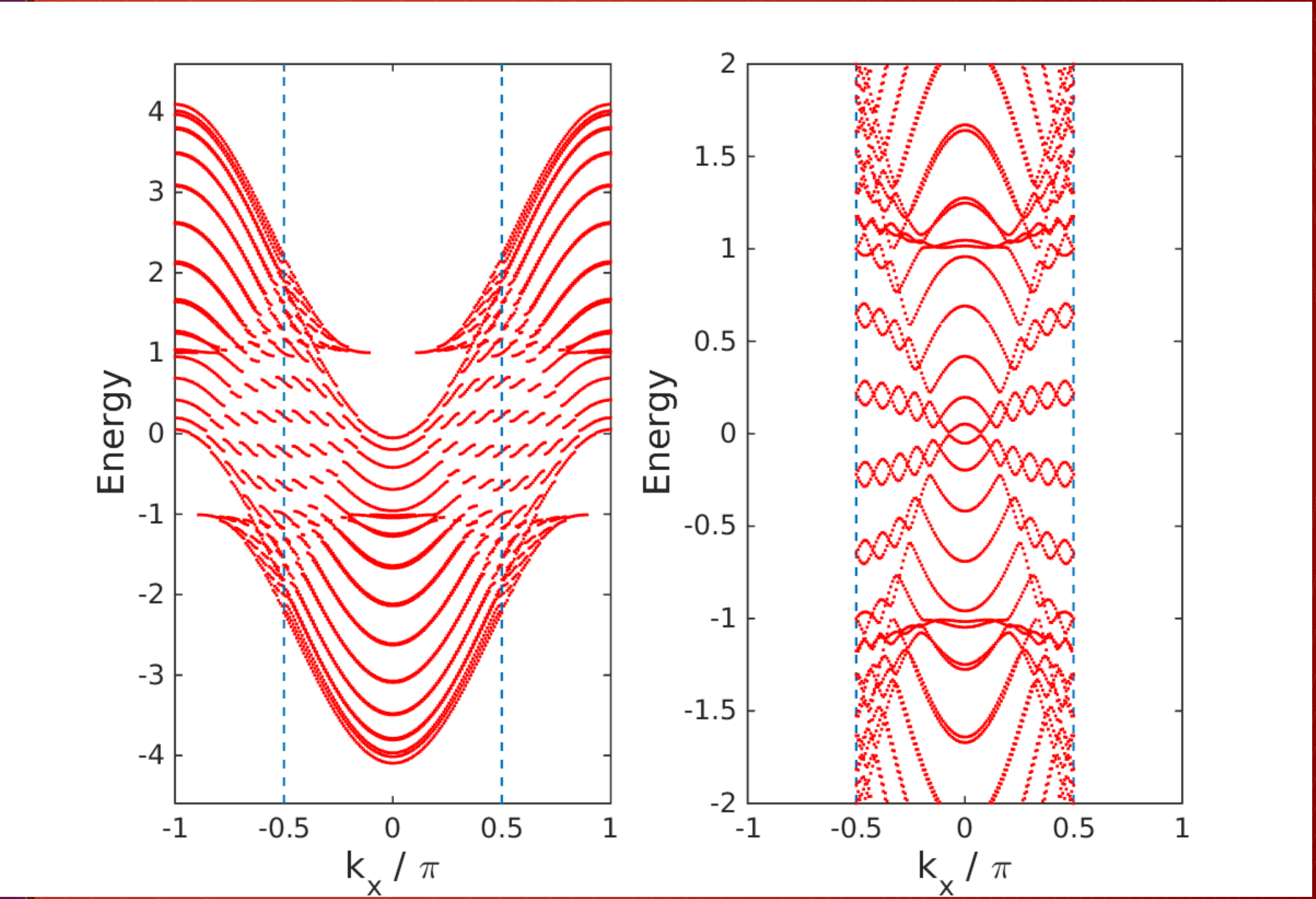


Edge states



Energy
4
3
2
1
0
-1
-2
-3
-4
-1
-0.5
0
0.5
1
k_x / π
Energy
4
3
2
1
0
-1
-2
-3
-4
-1
-0.5
0
0.5
1
k_x / π

Edge states (zoom)



Drude weight as a topological invariant

- Can also start with the discretized expression

$$J(\Phi) = \frac{N\Phi}{m} - \lim_{\Delta X \rightarrow 0} \frac{1}{m \Delta X} \mathrm{Im} \ln \langle \Psi(\Phi) | e^{i \Delta X \hat{K}} | \Psi(\Phi) \rangle$$

and taking the derivative with respect to Φ and averaging over the Brillouin zone results in

BH, *Phys. Rev. B* **87** 235123 (2013).

Drude weight as a topological invariant

- In this case the Drude weight becomes

$$D = \frac{N}{m} + \frac{\gamma}{m}$$

$$\gamma = -\lim_{\Delta X, \Delta K \rightarrow 0} \frac{1}{\Delta X \Delta K} \mathrm{Im} \left[\frac{\langle \Psi | e^{i\Delta K \hat{X}} e^{i \Delta X \hat{K}} | \Psi \rangle + \langle \Psi | e^{i\Delta X \hat{K}} e^{-i \Delta K \hat{X}} | \Psi \rangle}{\langle \Psi | e^{i \Delta X \hat{K}} | \Psi \rangle} \right]$$

- Taking the limits one can expand the exponentials and the result is that $D = 0$

BH, *Phys. Rev. B* **87** 235123 (2013).

Drude weight as a topological invariant

- In this case the Drude weight becomes

$$D = \frac{N}{m} + \frac{\gamma}{m}$$

$$\gamma = -\lim_{\Delta X, \Delta K \rightarrow 0} \frac{1}{\Delta X \Delta K} \mathrm{Im} \left[\frac{\langle \Psi | e^{i \Delta K \hat{X}} e^{i \Delta X \hat{K}} | \Psi \rangle + \langle \Psi | e^{i \Delta X \hat{K}} e^{-i \Delta K \hat{X}} | \Psi \rangle}{\langle \Psi | e^{i \Delta X \hat{K}} | \Psi \rangle} \right]$$

- Such an expansion is not always valid. If the wavefunction is an eigenstate of the total current (and has eigenvalue zero, eigenvalue of an unpeturbed system) then $D = \frac{N}{m}$

Connection to localization

- Recall that the spread is given by

$$\sigma_X^2 = -\lim_{\Delta K \to 0} \frac{2}{\Delta K^2} \mathrm{Re}\ln\langle\Psi_0|e^{i\Delta K\hat{X}}|\Psi_0\rangle$$

an expectation value of the total momentum shift.

- If the wavefunction is an eigenstate of the total current than the scalar product

$$\langle\Psi_0|e^{i\Delta K\hat{X}}|\Psi_0\rangle = 0$$

since the shifted wavefunction will necessarily be orthogonal to the unshifted one. Q.E.D.

Example: Anderson localization

- Hamiltonian: $$H = -t\sum_i c_i^\dagger c_{i+1} + \mathrm{H.\,c.} + U\sum_i \xi_i n_i$$
  - ξ_i Drawn from a Gaussian distribution
- Well known insulator for finite U.
- Exactly diagonalize for system with 1024 lattice sites and 512 particles
- Quantities calculated: DC conductivity, spread in position, spread in current ,kinetic energy,

Example: Anderson localization

- Spread in current:

$$\sigma_K^2 = -\lim_{\Delta X \to 0} \frac{2}{\Delta X^2} \mathrm{Re} \ln \langle \Psi_0 | e^{i \Delta X \hat{K}} | \Psi_0 \rangle$$

  - Zero for eigenstate of the current.

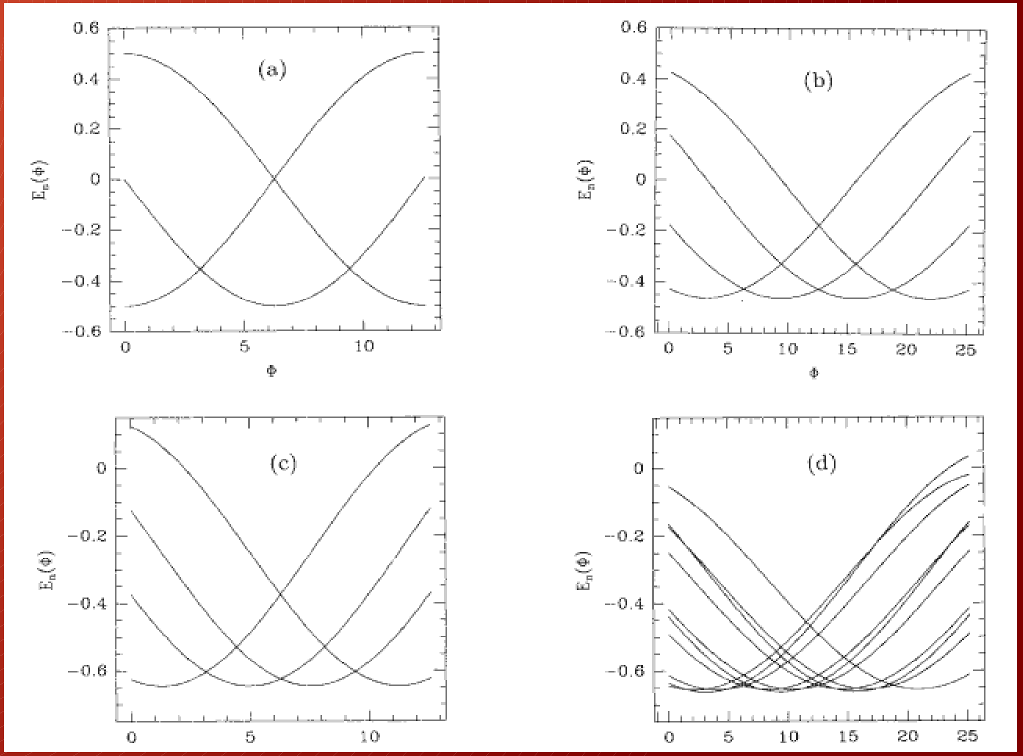
U	$D^{(c)}$	-½ Kinetic energy	Spread in current	Spread in position
0	327.95	327.95	0	----------
1	0.01154(4)	297(7)	5.8(2)	38(4)
2	0.0087(1)	233(3)	9.8(2)	17.8(9)
3	0.0066(2)	175(5)	12.8(2)	11.7(5)
4	0.0051(2)	136(5)	15.1(2)	8.4(4)
5	0.0041(2)	110(5)	16.7(3)	6.5(3)

Drude weight, Meissner weight, NCRI of superfluids

- Drude weight: $$D = \frac{1}{V}\left[\frac{\partial^2 E(\Phi)}{\partial \Phi^2}\right]_{\Phi=0}$$
- Meissner weight: $$n^{(s)} = \frac{1}{V}\left[\frac{\partial^2 E(\Phi)}{\partial \Phi^2}\right]_{\Phi=0}$$
- Reduction in rotational inertia of bosonic superfluid: $$I^{(s)} = \left[\frac{\partial^2 E^{(s)}(\Phi)}{\partial \Phi^2}\right]_{\Phi=0}$$
- **Question: how can these quantities be distinguished?**

Adiabatic vs. envelope derivative

- Scalapino, White and Zhang (PRL, 1992, PRB 1993)
    - For the Drude weight the second derivative with respect to the perturbing field is ambiguous
    - Energy levels can
    - cross!!!



Adiabatic derivative: Drude weight

„Envelope“ derivative: Meissner weight

Finite temperature generalization

- Zotos, Castella, and Prelovšek (PRL, 1992, PRB 1993): Drude weight

$$D(T) = \frac{\pi}{V} \sum_n \frac{\exp(-\beta E_n)}{Q} \left[\frac{\partial^2 E_n(\Phi)}{\partial \Phi^2} \right]_{\Phi=0}.$$

  - Weighted adiabatic derivatives

- Meissner weight based on this idea could possibly be of the form:

$$n^{(s)}(T) = \frac{\pi}{V} \left[\frac{\partial^2}{\partial \Phi^2} \sum_n \frac{\exp(-\beta E_n(\Phi))}{Q} E_n(\Phi) \right]_{\Phi=0}.$$

Variational theory

- A variational ground state energy is a weighted average over exact eigenstates:

$$E(\gamma) = \sum_n \langle \Psi(\gamma) | \tilde{\Psi}_n \rangle E_n \langle \tilde{\Psi}_n | \Psi(\gamma) \rangle = \sum_n P_n(\gamma) E_n .$$

which suggests that the Drude weight should be of the form (weighted adiabatic derivatives)

$$D_c = \frac{\pi}{V} \sum_n P_n(\gamma) \left[\frac{\partial^2 E_n(\Phi)}{\partial \Phi^2} \right]_{\Phi=0} .$$

and the Meissner weight of the form (one envelope derivative)

$$n^{(s)} = \frac{\pi}{V} \left[\frac{\partial^2}{\partial \Phi^2} \sum_n P_n(\gamma(\Phi)) E_n(\Phi) \right]_{\Phi=0} .$$

Variational theory

- In general the working assumption is that the variational ground state energy is a good approximation to the exact ground state, and $n^{(s)}$ is taken to be the Drude weight.
  - Example: for the Gutzwiller projected Fermi sea

$$\frac{\pi}{V}\left[\frac{\partial^2}{\partial\Phi^2}\sum_n P_n(\gamma(\Phi))E_n(\Phi)\right]_{\Phi=0} = \text{finite}.$$

Millis and Coppersmith, (PRB, 1991)

Expressions for the current

- Textbook expression for the current:

$$J(\Phi) = \frac{N}{m}\Phi + \langle\Psi(\Phi)| \left(\sum_i \hat{k}_i\right) |\Psi(\Phi)\rangle$$

- Alternatively,

$$J(\Phi) = \frac{N}{m}\Phi + \sum_i \langle\Psi(\Phi)|\hat{k}_i|\Psi(\Phi)\rangle$$

- **Are these two expressions the same?**

Expressions for the current

- Berry phase expression for the current:

$$J(\Phi) = \frac{N}{m}\Phi + \lim_{\Delta X \to 0} \frac{1}{m\Delta X} \operatorname{Im} \ln \langle \Psi(\Phi) | \exp\left(i\Delta X \sum_j \hat{k}_j \right) | \Psi(\Phi) \rangle$$

- Alternatively,

$$J(\Phi) = \frac{N}{m}\Phi + \lim_{\Delta X \to 0} \frac{N}{m\Delta X} \operatorname{Im} \ln \langle \Psi(\Phi) | \exp\left(i\Delta X \hat{k} \right) | \Psi(\Phi) \rangle$$

- **Are these two expressions the same?**

Expressions for the current

- Reduced density matrix of order m:

$$\rho_m(x_1, ...x_m; x'_1, ..., x'_m) = \int ... \int dx_{m+1}...dx_N$$

$$\Psi(x_1, ..., x_m, x_{m+1}, ..., x_N)\Psi(x'_1, ..., x'_m, x_{m+1}, ..., x_N)$$

Current expressions become:

$$J(\Phi) = \frac{N}{m}\Phi + \lim_{\Delta X \to 0} \frac{1}{m\Delta X} \mathrm{Im} \ln \mathrm{Tr} \left\{ \hat{\rho}_N \exp\left(i\Delta X \sum_j \hat{k}_j \right) \right\}$$

$$J(\Phi) = \frac{N}{m}\Phi + \lim_{\Delta X \to 0} \frac{N}{m\Delta X} \mathrm{Im} \ln \mathrm{Tr} \left\{ \hat{\rho}_1 \exp\left(i\Delta X \hat{k} \right) \right\}$$

- **Two expressions are not the same!!!**

Current vs. total current

- Expression for current:

$$J_N(\Phi) = \frac{\partial E(\Phi)}{\partial \Phi} = -\frac{Ne}{mc}\Phi + \frac{e}{mc}\langle \Psi(\Phi)| \left[\sum_{i=1}^{N} \hat{p}_i\right] |\Psi(\Phi)\rangle.$$

- Generator of one-body translations: one-body momentum

$$|\Psi(x+\delta x)\rangle = \exp(i\hat{p}\delta x)|\Psi(x)\rangle.$$

$$\hat{p} = -i\frac{\partial}{\partial x}.$$

- Generator of center of mass translations: total momentum

$$|\Psi(x_1 + x_{cm} + \delta x, ..., x_N + x_{cm} + \delta x)\rangle =$$

$$\exp(i\hat{P}_{cm}\delta x)|\Psi(x_1 + x_{cm}, ..., x_N + x_{cm})\rangle$$

$$\hat{P}_{cm} = -i\frac{\partial}{\partial x_{cm}} = \sum_{i=1}^{N} \hat{p}_i$$

Reduced density matrix/ODLRO

- Ordering in the reduced density matrix, known as off-diagonal long range order (ODLRO)

$$\lim_{|x_1 - x_1'| \to \infty} \rho_1(x_1; x_1') = \text{finite}$$

Penrose and Onsager (PRB, 1956)

  - ODLRO in $\hat{\rho}_1$: condensation of single particles (example: superfluidity in He^4)
  - ODLRO in $\hat{\rho}_2$: condensation of pairs of particles (example: Cooper pairs in superconductors)

Yang (RMP, 1959)

  - ODLRO in $\hat{\rho}_N$: ideal conductor

BH (PRB, 2013; arXiv:1310.4144)

Different kinds of currents

- General expression for the current:

$$J_p(\Phi) = \frac{N}{m}\Phi + \lim_{\Delta X \to 0} \frac{(N/p)}{m\Delta X} \mathrm{Im}\ln\mathrm{Tr}\left\{\hat{\rho}_p \exp\left(i\Delta X \sum_{j=1}^{p} \hat{k}_j\right)\right\}$$

- General expression for transport quantities:

$$D_p = \left[\frac{\partial J_p(\Phi)}{\partial \Phi}\right]_{\Phi=0}$$

- Eigenstates of *p*-momenta contribute to *p*-transport coefficients

Transport quantities

- D_1 : $I^{(s)}$ non-classical rotational inertia
- D_2 : $n^{(s)}$ Meissner weight
- D_N : $D^{(c)}$ Drude weight

Associated flux quantization:

- D_p: $\Phi_B = nhc/(pe)$ as $p \rightarrow \infty$ flux is no longer quantized
- Hubbard model phase transition: between a quantized flux vs. continuous flux state

Transport quantities

- Yang (RMP, 1962): if ODLRO found in $\hat{\rho}_p$ it will also be found in $\hat{\rho}_m$ for all $m > p.$

Ideal conduction, different types:

- If D_p is finite, so is D_m for all $m>p$.
- If the minimum p for which D_p is finite is a microscopic number ($p<<N$), then ideal conduction with flux quantization
- If $p\sim N$ ideal conduction without flux quantization
- If all D_p are zero, then material is insulating or non-ideal conductor

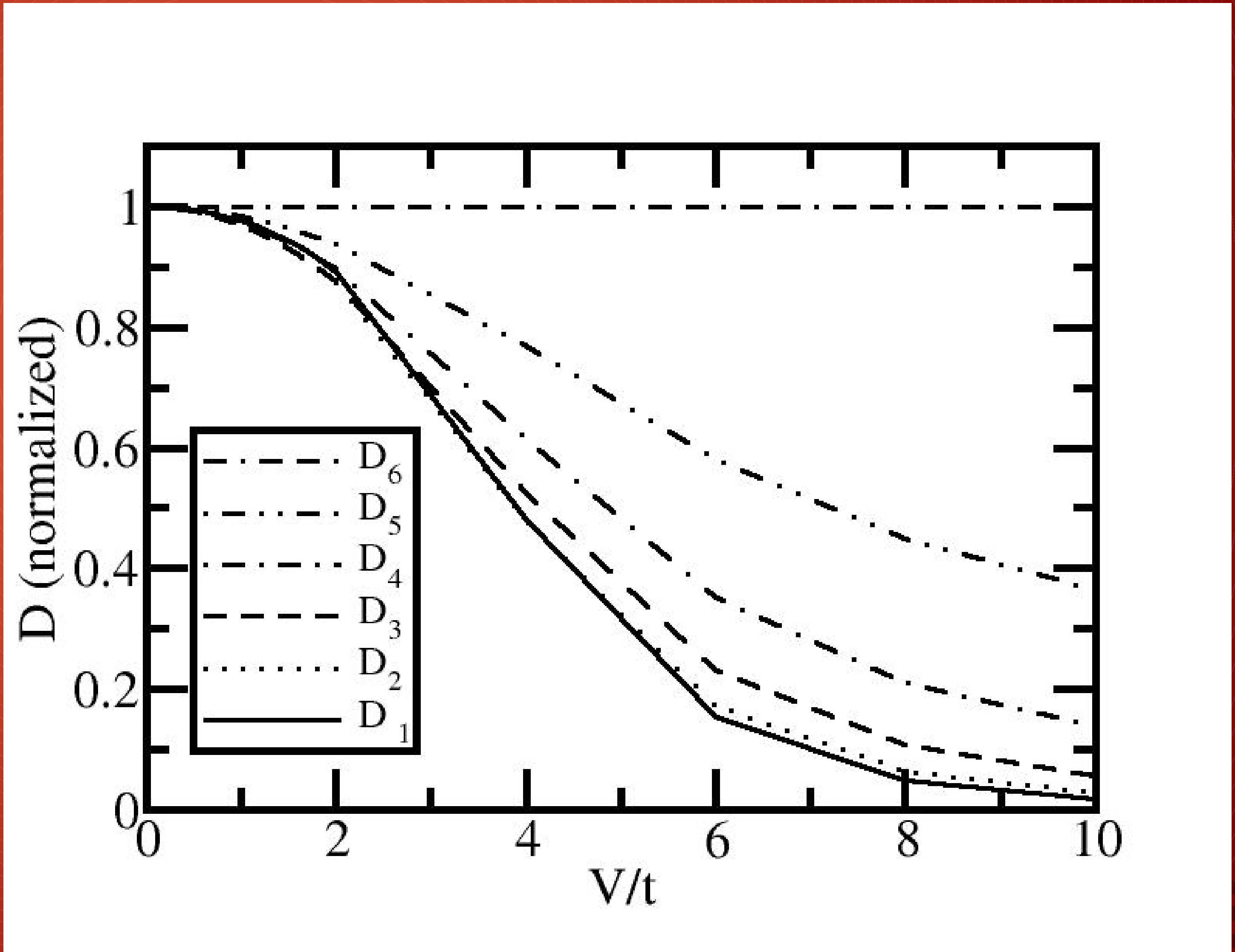
Surprising result for some strongly correlated models

- Hubbard model Hamiltonian:

$$H = -t\sum_i (c^\dagger_{i+1}c_i + c^\dagger_i c_{i+1}) + U\sum_i n_{i\uparrow}n_{i\downarrow}$$

- Hamiltonian commutes with total position shift operator
- As a result: D_N will be **finite,** and the Hubbard model is an **ideal conductor (Fröhlich superconductor)**
- Previous calculations on the Hubbard model (Lieb and Wu, Shastry and Sutherland) use D_1 or equivalent quantity (single-particle gap, chemical potential, etc.)

Surprising result for some strongly correlated models



Conclusions

- Drude weight corresponds to a topological invariant
- Evidence for “topological” behavior: fractional quantization, edge states
- Generalization of transport coefficients to account for “basic groups” of charge carriers
- Reinterpretation of the nature of conduction in many strongly correlated models

Thank you for your attention!