

MATH 101-007 Quiz 9

Why is the following integral an improper integral? Evaluate it.

$$I = \int_1^{\infty} \frac{x}{e^x} dx.$$

This integral is improper because of the upper bound ∞ . The interval of integration is infinite.

$$I = \lim_{c \rightarrow \infty} \int_1^c \frac{x}{e^x} dx = \lim_{c \rightarrow \infty} \int_1^c x e^{-x} dx$$

$u = x, \quad dv = e^{-x} dx$
 $du = dx, \quad v = -e^{-x}.$

$$\begin{aligned} I &= \lim_{c \rightarrow \infty} \left(-x e^{-x} \Big|_1^c + \int_1^c e^{-x} dx \right) \\ &= \lim_{c \rightarrow \infty} \left(-c e^{-c} + e^{-1} - e^{-x} \Big|_1^c \right) \\ &= \lim_{c \rightarrow \infty} \left(-\underbrace{\frac{c}{e^c}}_{\downarrow (*)} + \frac{1}{e} - \underbrace{\left(\frac{e^{-c}}{e} + e^{-1} \right)}_{\downarrow 0} \right) = \frac{1}{e} + \frac{1}{e} = \frac{2}{e}. \end{aligned}$$

$$\begin{aligned} (*) \quad & \lim_{c \rightarrow \infty} \frac{c}{e^c} \left[\frac{\infty}{\infty} \right] \text{ l'Hôpital's rule} \\ &= \lim_{c \rightarrow \infty} \frac{1}{e^c} = 0. \end{aligned}$$