Name Surname: November 25, 2010

MATH 101-007 Quiz 7 Solutions

a) Find $\lim_{n\to\infty} \dfrac{1^6+2^6+3^6+\cdots+n^6}{n^7}$.

Solution. Let $S_n = \dfrac{1^6+2^6+3^6+\cdots+n^6}{n^7}$. Then

$$\begin{aligned} S_n &= \frac{1}{n}\,\frac{1^6+2^6+3^6+\cdots+n^6}{n^6} = \frac{1}{n}\left[\left(\frac{1}{n}\right)^6+\left(\frac{2}{n}\right)^6+\left(\frac{3}{n}\right)^6+\cdots+\left(\frac{n}{n}\right)^6\right] \\ &= \frac{1}{n}\left(\frac{1}{n}\right)^6+\frac{1}{n}\left(\frac{2}{n}\right)^6+\frac{1}{n}\left(\frac{3}{n}\right)^6+\cdots+\frac{1}{n}\left(\frac{n}{n}\right)^6 \end{aligned}$$

We try to perceive S_n as a Riemann sum. Let

$$f(x)=x^6, \quad c_1=\frac{1}{n},\ c_2=\frac{2}{n},\ c_3=\frac{3}{n},\ldots,c_n=\frac{n}{n}.$$

Then as $n\to\infty$, we have $c_1\to 0$, and $c_n\to 1$. Thus $a=0$ and $b=1$. With n subintervals of equal length, $\Delta x=\frac{1}{n}$. Then $\|P\|=\frac{1}{n}$, and $\|P\|\to 0$ if and only if $n\to\infty$. So

$$S_n=\Delta x f(c_1)+\Delta x f(c_2)+\Delta x f(c_3)+\cdots+\Delta x f(c_n)=\sum_{k=1}^{n}\Delta x f(c_k).$$

So

$$\lim_{n\to\infty}\frac{1^6+2^6+3^6+\cdots+n^6}{n^7}=\lim_{n\to\infty}S_n=\lim_{\|P\|\to 0}\sum_{k=1}^{n}\Delta x f(c_k)=\int_0^1 x^6\,dx=\frac{x^7}{7}\bigg|_0^1=\frac{1}{7}.$$

b) Evaluate $I=\int x\sqrt{x-1}\,dx$.

Solution. Let $u=x-1$. Then $du=dx$, and $x=u+1$. So

$$I=\int (u+1)\sqrt{u}du=\int\left(u^{3/2}+u^{1/2}\right)du=\frac{u^{5/2}}{\frac{5}{2}}+\frac{u^{3/2}}{\frac{3}{2}}+C=\frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C.$$