Name Surname: October 14, 2010

MATH 101-007 Quiz 4

Question. Assume the equation $xy + y^3 = 3$ defines y as an implicit function of x. Find $\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)}$ and $\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,1)}$ by using implicit differentiation.

Show all your work.

Solution. I will write y' for $\frac{dy}{dx}$ and y'' for $\frac{d^2y}{dx^2}$.

$$\begin{aligned} xy + y^3 = 3 \quad &\Rightarrow \quad y + xy' + 3y^2\, y' = 0 \quad (*) \\ &\Rightarrow \quad y' + y' + xy'' + 6yy'y' + 3y^2y'' = 0 \quad (**) \end{aligned}$$

In (*), substitute $x = 2,\ y = 1$.

$$1 + 2y' + 3y' = 0 \Rightarrow y' = -\frac{1}{5}.$$

In (**), substitute $x = 2,\ y = 1,\ y' = -\frac{1}{5}$.

$$-\frac{1}{5} - \frac{1}{5} + 2y'' + 6\left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right) + 3y'' = 0 \Rightarrow 5y'' = \frac{2}{5} - \frac{6}{25} = \frac{4}{25} \Rightarrow y'' = \frac{4}{125}.$$

So

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = -\frac{1}{5}, \text{ and } \left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,1)} = \frac{4}{125}.$$

Second solution.

$$xy + y^3 = 3 \Rightarrow y + xy' + 3y^2\, y' = 0 \Rightarrow y'(x + 3y^2) = -y \Rightarrow y' = \frac{-y}{x + 3y^2}.$$

We take the derivative of y' with respect to x by using the quotient rule and other differentiation rules.

$$y'' = \frac{-y'(x + 3y^2) - (-y)(1 + 6yy')}{(x + 3y^2)^2}.$$

Next we substitute $y' = \frac{-y}{x + 3y^2}$ into y''.

$$\begin{aligned} y'' &= \frac{-\frac{-y}{x+3y^2}\,(x + 3y^2) + y\left(1 + 6y\,\frac{-y}{x+3y^2}\right)}{(x + 3y^2)^2} \\ &= \frac{y + y\,\frac{x+3y^2-6y^2}{x+3y^2}}{(x + 3y^2)^2} = \frac{y(x + 3y^2) + y(x - 3y^2)}{(x + 3y^2)^3} = \frac{2xy}{(x + 3y^2)^3}. \end{aligned}$$

So

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \left.\frac{-y}{x + 3y^2}\right|_{(x,y)=(2,1)} = -\frac{1}{5}, \text{ and } \left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,1)} = \left.\frac{2xy}{(x + 3y^2)^3}\right|_{(x,y)=(2,1)} = \frac{4}{125}.$$