Name Surname: December 30, 2010

MATH 101-007 Quiz 11 Solutions

Determine whether each of the following series converges or diverges. Show all your work.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{n+1}}{e^n}$.

Solution 1. We write this series as

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{n+1}}{e^n}=\sum_{n=1}^{\infty}(-\pi)\left(-\frac{\pi}{e}\right)^n.$$

This is a geometric series with $a=-\pi$ and $r=-\frac{\pi}{e}$. Since $r<-1$, that is r does not satisfy $-1<r<1$, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{n+1}}{e^n}$ is divergent.

Solution 2. We use Ratio Test.

$$\rho=\lim_{n\to\infty}\frac{\left|(-1)^{n+2}\frac{\pi^{n+2}}{e^{n+1}}\right|}{\left|(-1)^{n+1}\frac{\pi^{n+1}}{e^n}\right|}=\lim_{n\to\infty}\frac{\pi^{n+2}}{e^{n+1}}\frac{e^n}{\pi^{n+1}}=\frac{\pi}{e}.$$

Since $\rho=\frac{\pi}{e}>1$, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{n+1}}{e^n}$ is divergent by the ratio Test.

Solution 3. We have $a_n=(-1)^{n+1}\frac{\pi^{n+1}}{e^n}=(-1)^{n+1}\pi\,\left(\frac{\pi}{e}\right)^n$. Since $\frac{\pi}{e}>1$, $\lim_{n\to\infty}\left(\frac{\pi}{e}\right)^n=\infty$. Since $(-1)^{n+1}$ alternates the sign as $+$ and $-$, $\lim_{n\to\infty}a_n=\lim_{n\to\infty}(-1)^{n+1}\pi\,\left(\frac{\pi}{e}\right)^n$ doesn't exist. Thus $\lim_{n\to\infty}a_n\neq 0$, so by the n-th Term Test the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^{n+1}}{e^n}$ is divergent.

b. $\displaystyle\sum_{n=1}^{\infty}\frac{n+10}{\sqrt{n^5+1}}$.

Solution. We have $a_n=\frac{n+10}{\sqrt{n^5+1}}>0$. We use Limit Comparison Test. We choose $b_n=\frac{n}{\sqrt{n^5}}=\frac{1}{n^{3/2}}$. Then

$$L=\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n+10}{\sqrt{n^5+1}}\frac{\sqrt{n^5}}{n}=\frac{n+10}{n}\sqrt{\frac{n^5}{n^5+1}}=1.$$

Since $0<L<\infty$, the series $\sum a_n$ and $\sum b_n$ behave the same.

The series $\displaystyle\sum_{n=1}^{\infty}b_n=\sum\frac{1}{n^{3/2}}$ is convergent ($p=\frac{3}{2}>1$). So by the Limit Comparison Test the series $\displaystyle\sum_{n=1}^{\infty}\frac{n+10}{\sqrt{n^5+1}}$ is also convergent.