Date: 12 July 2010 Time: 08:40-10:30

MATH 114 MIDTERM 2 SOLUTIONS

Q 1. Let f be a function of two variables with continuous first and second order partial derivatives, and let g be a differentiable function of one variable. Let $z = f(x^2 g(y), x+y^2)$. Given that

$$g(3) = -1,\ g'(3) = -3,\ f_1(-4,11) = 2,\ f_{11}(-4,11) = -4,\ f_{12}(-4,11) = -2, f_{22}(-4,11) = 3,$$

find

$$\left.\frac{\partial^2 z}{\partial y \partial x}\right|_{(x,y)=(2,3)}.$$

Solution. Let $u = x^2 g(y)$, $v = x + y^2$. Then we have $z = f(u,v)$. By the Chain Rule

$$\begin{aligned}\frac{\partial z}{\partial x} &= f_1(u,v)\frac{\partial u}{\partial x} + f_2(u,v)\frac{\partial v}{\partial x} \\ &= f_1(u,v)2xg(y) + f_2(u,v).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y\, \partial x} &= \left(f_{11}(u,v)\frac{\partial u}{\partial y} + f_{12}(u,v)\frac{\partial v}{\partial y}\right) 2xg(y) + f_1(u,v)2xg'(y) + f_{21}(u,v)\frac{\partial u}{\partial y} + f_{22}(u,v)\frac{\partial v}{\partial y} \\ &= \left[f_{11}(u,v)x^2 g'(y) + f_{12}(u,v)2y\right] 2xg(y) + f_1(u,v)2xg'(y) + f_{21}(u,v)x^2 g'(y) + f_{22}(u,v)2y\end{aligned}$$

Next we substitute $x = 2, y = 3$. At this point we have $u = 4g(3) = -4$ and $v = 2+9 = 11$. Thus

$$\begin{aligned}\left.\frac{\partial^2 z}{\partial y \partial x}\right|_{(x,y)=(2,3)} &= [f_{11}(-4,11)4g'(3) + f_{12}(-4,11)6]\,4g(3) + f_1(-4,11)4g'(3) \\ &+ f_{21}(-4,11)4g'(3) + f_{22}(-4,11)6 \\ &= [(-4)\cdot 4\cdot(-3) + (-2)\cdot 6]\,4\cdot(-1) + 2\cdot 4\cdot(-3) + (-2)\cdot 4\cdot(-3) + 3\cdot 6 \\ &= (48-12)\cdot(-4) + (-24) + 24 + 18 = -144 + 18 = -126.\end{aligned}$$

Q 2. Find the values of the constants a, b and c such that the directional derivative of $f(x,y,z) = axy^2 + byz + cz^2x^3$ at the point $(1,2,-1)$ has a maximum value of 64 in a direction parallel to the z-axis.

Solution. We know that maximum directional derivative is obtained in the direction of $\nabla f(1,2,-1)$. It is given that this direction is parallel to the z-axis. Thus $\nabla f(1,2,-1) = t\mathbf{k}$ for some scalar t. Maximum directional derivative is

$$\nabla f(1,2,-1)\cdot\frac{\nabla f(1,2,-1)}{|\nabla f(1,2,-1)|} = \frac{\nabla f(1,2,-1)\cdot\nabla f(1,2,-1)}{|\nabla f(1,2,-1)|} = \frac{|\nabla f(1,2,-1)|^2}{|\nabla f(1,2,-1)|} = |\nabla f(1,2,-1)|.$$

Thus $|\nabla f(1,2,-1)| = 64$, that is $|t\mathbf{k}| = 64$, that is $|t| = 64$, which gives $t = 64$ or $t = -64$.

On the other hand

$$\begin{aligned}\nabla f(x,y,z) &= (ay^2 + 3cz^2x^2)\mathbf{i} + (2axy + bz)\mathbf{j} + (by + 2czx^3)\mathbf{k} \Rightarrow \\ \nabla f(1,2,-1) &= (4a+3c)\mathbf{i} + (4a-b)\mathbf{j} + (2b-2c)\mathbf{k}\end{aligned}$$

Since $\nabla f(1,2,-1) = t\mathbf{k}$, we have that

$$4a + 3c = 0,\ 4a - b = 0,\ 2b - 2c = t.$$

This system has solution

$$b = 4a,\ c = -\frac{4a}{3},\ t = \frac{32a}{3} \text{ or } a = \frac{3t}{32},\ b = \frac{3t}{8},\ c = -\frac{t}{8}.$$

So

$$\begin{aligned} t = 64 \ &\Rightarrow\ a = 6,\ b = 24,\ c = -8, \\ t = -64 \ &\Rightarrow\ a = -6,\ b = -24,\ c = 8. \end{aligned}$$

Q 3. Find and classify all the critical points of the following function:

$$f(x,y) = x^2y + y^3 - 3y^2.$$

Solution. First we find the critical points.

$$\begin{aligned} f_1(x,y) &= 2xy = 0 &(1) \\ f_2(x,y) &= x^2 + 3y^2 - 6y &(2) \end{aligned}$$

From (1), we get $x = 0$ or $y = 0$.
When we substitute $x = 0$ in (2), we get $3y^2 - 6y = 0$, which has roots $y = 0$, $y = 2$. Thus we have points $P_1(0,0)$ and $P_2(0,2)$.
When we substitute $y = 0$ in (2), we get $x = 0$, that is the point $(0,0)$ which is already found.

Thus critical points are: $P_1(0,0)$ and $P_2(0,2)$.

Next we apply the Second Derivative Test.

$$f_{11}(x,y) = 2y,\ f_{12}(x,y) = 2x,\ f_{22}(x,y) = 6y - 6.$$

$$H(0,0) = \begin{vmatrix} 0 & 0 \\ 0 & -6 \end{vmatrix} = 0.$$ Thus no information. But we'll check this point some other way,

$$H(0,2) = \begin{vmatrix} 4 & 0 \\ 0 & -6 \end{vmatrix} = -24 < 0,$$ so a saddle point at $P_2(0,2)$.

Back to $P_1(0,0)$. We take (x,y) near $(0,0)$. Then $x \approx 0$ and $y \approx 0$. So

$$\Delta = f(x,y) - f(0,0) = x^2y + y^3 - 3y^2.$$

When $x = 0$, $\Delta = y^3 - 3y^2 = y^2(y-2) < 0$. Thus on $x = 0$, we have $f(x,y) < f(0,0)$.
When $y = \frac{1}{3}x^2$, then $x^2 = 3y^2$, and so $\Delta = y^3 > 0$. Thus on $y = \frac{1}{3}x^2$, we have $f(x,y) > f(0,0)$. So there is a saddle point at $P_1(0,0)$.

Q 4. Let $f(x,y,z) = xy + z^2$. Choose a point $P_0(a,b,c)$ on the ellipsoid $\dfrac{x^2}{16} + \dfrac{y^2}{81} + \dfrac{z^2}{4} = 1$, and calculate $K = f(a,b,c)$. You will get K points from this question.

Solution. Naturally we would like to have K as large as possible. So we find the maximum value of $f(x,y,z) = xy + z^2$ subject to the constraint $\frac{x^2}{16} + \frac{y^2}{81} + \frac{z^2}{4} - 1 = 0$. Let

$g(x,y,z) = \frac{x^2}{16} + \frac{y^2}{81} + \frac{z^2}{4} - 1$. Then by the method of Lagrange multipliers we find the critical points of

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) = xy + z^2 + \lambda\left(\frac{x^2}{16} + \frac{y^2}{81} + \frac{z^2}{4} - 1\right).$$

$$L_1(x,y,z,\lambda) = 0 \quad\Rightarrow\quad y + \lambda\frac{2x}{16} = 0 \tag{3}$$
$$L_2(x,y,z,\lambda) = 0 \quad\Rightarrow\quad x + \lambda\frac{2y}{81} = 0 \tag{4}$$
$$L_3(x,y,z,\lambda) = 0 \quad\Rightarrow\quad 2z + \lambda\frac{2z}{4} = 0 \tag{5}$$
$$L_4(x,y,z,\lambda) = 0 \quad\Rightarrow\quad \frac{x^2}{16} + \frac{y^2}{81} + \frac{z^2}{4} - 1 = 0 \tag{6}$$

(5) gives $z = 0$ or $\lambda = -4$.

If $\lambda = -4$, then

$$\left.\begin{array}{llll} (3): & y - 4\,\frac{2x}{16} = 0 & \Rightarrow & y = \frac{x}{2} \\ (4): & x - 4\,\frac{2y}{81} = 0 & \Rightarrow & x = \frac{8y}{81} \end{array}\right\} \Rightarrow x = 0,\ y = 0.$$

Thus by (6), $z = \mp 2$, so we have $f(0,0,\mp 2) = 4$.

If $z = 0$, then

$$\left.\begin{array}{llllll} (3): & y = -\frac{2\lambda x}{16} & \Rightarrow & y^2 = -\frac{2\lambda xy}{16} & \Rightarrow & -2\lambda xy = 16y^2 \\ (4): & x = -\frac{2\lambda y}{81} & \Rightarrow & x^2 = -\frac{2\lambda xy}{81} & \Rightarrow & -2\lambda xy = 81x^2 \end{array}\right\} \Rightarrow 16y^2 = 81x^2.$$

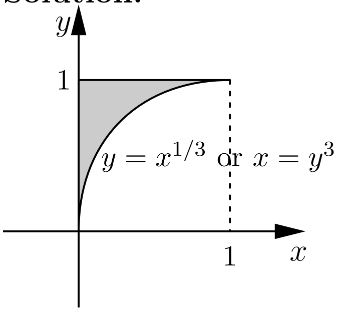
Substituting into (5) (together with $z = 0$), we get $x^2 = 8$, $y^2 = \frac{81}{2}$. So we get the points $x = \mp 2\sqrt{2}$, $y = \mp\frac{9}{\sqrt{2}}$, $z = 0$. So $f(x,y,z) = xy = \mp 18$.

Thus we choose $x = a = 2\sqrt{2}$, $y = b = \frac{9}{\sqrt{2}}$, $z = c = 0$, and get $K = f(a,b,c) = 18$.

Q 5. Evaluate the following iterated integral:

$$I = \int_0^1 dx \int_{x^{1/3}}^1 \frac{1}{\sqrt{1+y^4}}\, dy.$$

Solution.



We reverse the order of integration and the integral becomes

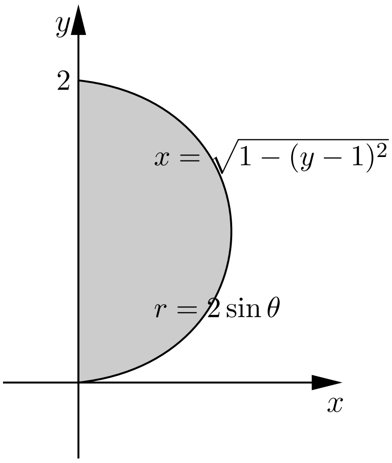
$$\begin{aligned} I &= \int_0^1 dy \int_0^{y^3} \frac{1}{\sqrt{1+y^4}}\, dx \\ &= \int_0^1 dy\, \frac{1}{\sqrt{1+y^4}}\, x\Big|_0^{y^3} = \int_0^1 \frac{y^3}{\sqrt{1+y^4}}\, dy \\ &= \frac{2}{4}\int_0^1 \frac{4y^3}{2\sqrt{1+y^4}}\, dy = \frac{1}{2}\sqrt{1+y^4}\Big|_0^1 = \frac{\sqrt{2}-1}{2}. \end{aligned}$$

Q 6. Evaluate the following iterated integral:

$$I = \int_0^2 \int_0^{\sqrt{1-(y-1)^2}} \sqrt{4 - x^2 - y^2}\, dx\, dy.$$

Solution. We use polar coordinates. The region of integration is described as

$$0 \le y \le 2,\ 0 \le x \le \sqrt{1-(y-1)^2}.$$



$x = \sqrt{1-(y-1)^2}$ is the right semicircle. Squaring both sides we get $x^2 + y^2 - 2y = 0$, and in polar coordinates it becomes $r = 2\sin\theta$. Thus the given integral becomes:

$$I = \int_0^{\pi/2} \int_0^{2\sin\theta} \sqrt{4-r^2}\, r\, dr\, d\theta.$$

Then

$$\begin{aligned}
I &= \int_0^{\pi/2} -\frac{1}{3}(4-r^2)^{3/2}\Big|_0^{2\sin\theta}\, d\theta = -\frac{1}{3}\int_0^{\pi/2} \left((4-4\sin^2\theta)^{3/2} - 4^{3/2}\right)\, d\theta \\
&= -\frac{1}{3}\int_0^{\pi/2} \left(8\cos^3\theta - 8\right)\, d\theta = \frac{8}{3}\int_0^{\pi/2} \left(1-\cos^3\theta\right)\, d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \left(1-(1-\sin^2\theta)\cos\theta\right)\, d\theta = \frac{8}{3}\left(\theta - \left(\sin\theta - \frac{\sin^3\theta}{3}\right)\right)\Big|_0^{\pi/2} \\
&= \frac{8}{3}\left(\frac{\pi}{2} - 1 + \frac{1}{3}\right) = \frac{4\pi}{3} - \frac{16}{9}.
\end{aligned}$$