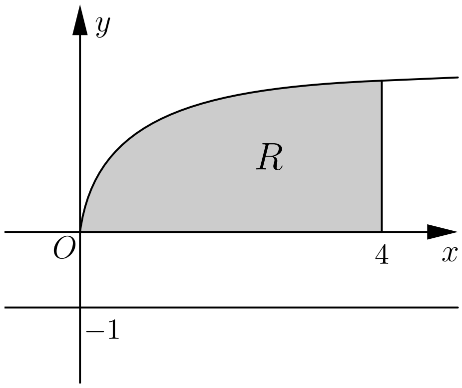
Date: 28 June 2010 Time: 08:40-10:30

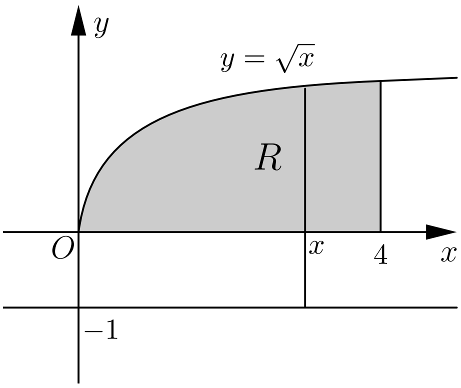
MATH 114 MIDTERM 1 Solutions

Q 1. Let R be the region in the first quadrant bounded by $y = \sqrt{x}$, the x-axis, and the line $x = 4$. Assume R is rotated about the line $y = -1$ and a solid is generated.

(a) Draw the region R.



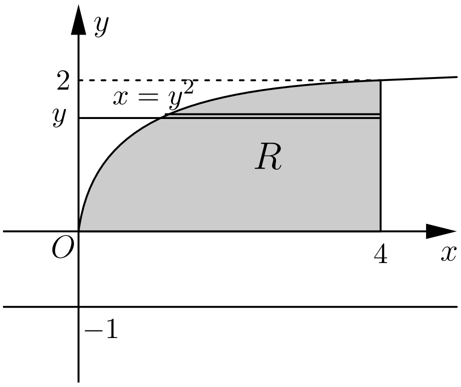
(b) Set up the integral for the volume of the solid by using the slicing method. Do not evaluate the integral.



Solution. The cross section at the position x is an annulus with
outer radius $R(x) = \sqrt{x} - (-1) = \sqrt{x} + 1$, and
inner radius $r(x) = 0 - (-1) = 1$.
So $A(x) = \pi(R(x))^2 - \pi(r(x))^2 = \pi(\sqrt{x} + 1)^2 - \pi\, 1^2$.

$$V = \int_0^4 \pi((\sqrt{x} + 1)^2 - 1)\, dx.$$

(c) Set up the integral for the volume of the solid by using the cylindrical shells method. Do not evaluate the integral.



Solution. The thin strip at the position y generates a cylindrical shell with
shell radius $r(y) = y - (-1) = y + 1$, and
shell height $h(y) = 4 - y^2$.
So $dV = 2\pi r(y)h(y)dy = 2\pi(y + 1)(4 - y^2)\, dy$.

$$V = \int_0^2 2\pi(y + 1)(4 - y^2)\, dy.$$

Q 2. Consider the following table.
In column I you're given some series $\sum a_n$. In column II, write the name of the test that you would use to check whether the series $\sum a_n$ is convergent.
If the test you named in column II, is Comparison Test or Limit Comparison Test, write the comparison series $\sum b_n$ in column III. Otherwise leave this column empty. In column IV, write your conclusion (as **convergent** or **divergent**) about the given series $\sum a_n$ in column I.
No other explanation is required.

Column I	Column II	Column III	Column IV
Series $\sum a_n$	Which test?	If the test in Column II is Comparison or Limit Comparison, write the series $\sum b_n$	Is the series $\sum a_n$ convergent or divergent?
$\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$	Ratio Test		Convergent
$\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n+5}{9n-7}$	n-th term test		Divergent
$\displaystyle\sum_{n=1}^{\infty} \frac{3+\sqrt{n}}{\sqrt{n^2+5n+2}}$	Limit Comparison Test	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$	Divergent
$\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ \ln n\ (\ln(\ln n))^3}$	Integral Test		Convergent

Q 3. Let

$$F(x) = \int_0^x \sin(t^2)\, dt.$$

Find a polynomial $P(x)$ that approximates $F(x)$ with $|\text{error}| < 10^{-5}$ for all x in the interval $0 \le x \le 1$.

Solution. We have

$$\sin(t^2) = t^2 - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} - \cdots + (-1)^n \frac{(t^2)^{2n+1}}{(2n+1)!} + \cdots, \quad -\infty < t < \infty.$$

Then term-by-term integration theorem for the power series implies that

$$\begin{aligned} F(x) &= \int_0^x \left(t^2 - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} - \cdots + (-1)^n \frac{(t^2)^{2n+1}}{(2n+1)!} + \cdots \right) dt \\ &= \frac{x^3}{3} - \frac{x^7}{3!\,7} + \frac{x^{11}}{5!\,11} - \cdots + (-1)^n \frac{x^{4n+3}}{(2n+1)!\,(4n+3)} + \cdots, \quad -\infty < x < \infty. \end{aligned}$$

The last series is an alternating series.
When $0 \le x \le 1$, the general term $a_n = (-1)^n \frac{x^{4n+3}}{(2n+1)!\,(4n+3)}$ has the following properties:

(i) $|a_n| \le \dfrac{1}{(2n+1)!\,(4n+3)}$ and so $\lim_{n\to\infty} a_n = 0$, and

(ii) $\dfrac{|a_{n+1}|}{|a_n|} = \dfrac{\frac{x^{4n+7}}{(2n+3)!\,(4n+7)}}{\frac{x^{4n+3}}{(2n+1)!\,(4n+3)}} = \dfrac{x^4}{(2n+2)(2n+3)}\,\dfrac{4n+3}{4n+7} \le \dfrac{1}{(2n+2)(2n+3)}\,\dfrac{4n+3}{4n+7} < 1.$

Thus $\{|a_n|\}$ is decreasing.

So we can use the Alternating Series Error Formula. If we stop at some term, then $|\text{error}| < |\text{first unused term}|$. That is if we take

$$F(x) \approx \frac{x^3}{3} - \frac{x^7}{3!\,7} + \frac{x^{11}}{5!\,11} - \cdots + (-1)^n \frac{x^{4n+3}}{(2n+1)!\,(4n+3)}$$

then for all x with $0 \le x \le 1$ we have

$$|\text{error}| < \left|(-1)^{n+1} \frac{x^{4n+7}}{(2n+3)!\,(4n+7)}\right| \le \frac{1}{(2n+3)!\,(4n+7)}.$$

Now we find a natural number n such that

$$\frac{1}{(2n+3)!\,(4n+7)} \le 10^{-5}.$$

Then $n = 3$ is good since $(2\cdot 3+3)!\,(4\cdot 3+7) = (9!)(19) = 6894720 > 10^5$. Thus we take

$$F(x) \approx \frac{x^3}{3} - \frac{x^7}{3!\,7} + \frac{x^{11}}{5!\,11} - \frac{x^{15}}{7!\,15} \quad \text{with } |\text{error}| < 10^{-5}.$$

So

$$P(x) = \frac{x^3}{3} - \frac{x^7}{3!\,7} + \frac{x^{11}}{5!\,11} - \frac{x^{15}}{7!\,15}.$$

Note that actually $|\text{error}| < \frac{1}{6894720} < 10^{-6}$.

Q 4. (a) Find the radius of convergence and the interval of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{\sqrt{n+1}\cdot 4^n}.$$

Solution. We apply the Ratio Test to the series $\sum \left|\frac{(x+1)^{2n}}{\sqrt{n+1}\cdot 4^n}\right|$. We have

$$\rho = \lim_{n\to\infty} \frac{\left|\frac{(x+1)^{2n+2}}{\sqrt{n+2}\cdot 4^{n+1}}\right|}{\left|\frac{(x+1)^{2n}}{\sqrt{n+1}\cdot 4^n}\right|} = \lim_{n\to\infty} \frac{(x+1)^2}{4}\,\frac{\sqrt{n+1}}{\sqrt{n+2}} = \frac{(x+1)^2}{4}.$$

The power series converges if $\frac{(x+1)^2}{4} < 1$, that is $(x+1)^2 < 4$, that is $-2 < x+1 < 2$, that is $-3 < x < 1$.

The power series diverges if $\frac{(x+1)^2}{4} > 1$, that is $x < -3$ or $x > 1$.
The end points are $x = -3$ and $x = 1$, which we check separately.

$$x = -3 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(-3+1)^{2n}}{\sqrt{n+1} \cdot 4^n} = \sum_{n=0}^{\infty} \frac{(-2)^{2n}}{\sqrt{n+1} \cdot 4^n} = \sum_{n=0}^{\infty} \frac{4^n}{\sqrt{n+1} \cdot 4^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \text{ is divergent,}$$
$$x = 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(1+1)^{2n}}{\sqrt{n+1} \cdot 4^n} = \sum_{n=0}^{\infty} \frac{2^{2n}}{\sqrt{n+1} \cdot 4^n} = \sum_{n=0}^{\infty} \frac{4^n}{\sqrt{n+1} \cdot 4^n} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \text{ is divergent.}$$

So interval of convergence is $(-3, 1)$ and radius of convergence is $R = 2$.

(b) Let $f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{\sqrt{n+1} \cdot 4^n}$. Find $f^{(126)}(-1)$ and $f^{(127)}(-1)$.

Solution. If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, then $a_n = \frac{f^{(n)}(c)}{n!}$, so $f^{(n)}(c) = n!\, a_n$.

In this problem $c = -1$, and $a_{2n} = \frac{1}{\sqrt{n+1} \cdot 4^n}$, $a_{2n+1} = 0$. So

$$\begin{aligned} a_{126} &= \frac{1}{\sqrt{63+1} \cdot 4^{63}} = \frac{1}{8 \cdot 4^{63}} \Rightarrow f^{(126)}(-1) = \frac{126!}{8 \cdot 4^{63}}, \\ a_{127} &= 0 \Rightarrow f^{(127)}(-1) = 0. \end{aligned}$$

Q 5.

(a) Prove that $\lim_{(x,y)\to(1,0)} \frac{(x-1)y^2}{(x-1)^2+y^2} = 0$.

Proof. Let $\varepsilon > 0$ be given. Assume we choose $\delta > 0$ such that $\delta = \varepsilon$. Let (x, y) be an arbitrary point such that $0 < \sqrt{(x-1)^2 + y^2} < \delta$. Then we have $(x, y) \neq (1, 0)$ and $|x - 1| < \delta$ and $|y| = |y - 0| < \delta$. So

$$\left| \frac{(x-1)y^2}{(x-1)^2+y^2} - 0 \right| = |x-1| \cdot \frac{y^2}{(x-1)^2+y^2} < |x-1| < \delta = \varepsilon.$$

(b) Prove that $\lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2}$ does not exist.

Solution. We consider the limit of the function along an arbitrary line $y = m(x-1)$ passing through the point $(1, 0)$.

$$\lim_{\substack{(x,y)\to(1,0) \\ y = m(x-1)}} \frac{(x-1)y}{(x-1)^2+y^2} = \lim_{x\to 1} \frac{(x-1)m(x-1)}{(x-1)^2+m^2(x-1)^2} = \frac{m}{1+m^2}.$$

Since the result depends on the slope m, for lines with different slopes, we have different limits. This means that $\lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2}$ does not exist.