Date: 26 July 2010 Time: 17:00-19:30

NAME SURNAME:..

STUDENT NO:..

MATH 114 FINAL EXAM SOLUTIONS

Q 1. (17 points) Given that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \ -1 < x < 1,$$

(a) find the Maclaurin series of $f(x) = \arctan x$ valid for $-1 < x < 1$. Which theorem(s) did you use?

Solution.

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots = \sum_{n=0}^{\infty} (-1)^n t^n, \ -1 < t < 1.$$

Replace t by $-t^2$. Then $-1 < -t^2 < 1$ if and only if $-1 < t < 1$.

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n t^{2n}, \ -1 < t < 1, \quad (*)$$

Take any x such that $-1 < x < 1$, and integrate both sides of (*) from 0 to x. Then

$$\int_0^x \frac{1}{1+t^2}\, dt = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots)\, dt.$$

Term-by-term integration theorem for power series gives

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots, \ -1 < x < 1.$$

(b) Show that the series that you found in part (a) converges also at the points $x = 1$ and $x = -1$ to $\arctan 1$ and $\arctan(-1)$? Which theorem(s) did you use?

Solution. For $x = 1$, the above series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, and by the Alternating Series Test it converges, since it is an alternating series, the sequence $\left\{\frac{1}{2n+1}\right\}$ decreases and $\lim_{n\to\infty} \frac{1}{2n+1} = 0$. Since the function $f(x) = \arctan x$ is continuous at $x = 1$, by **Abel's Theorem**, we have

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \lim_{x\to 1^-} \arctan x = \arctan 1.$$

For $x = -1$, the series becomes $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots$. By exactly the same argument as above we have that

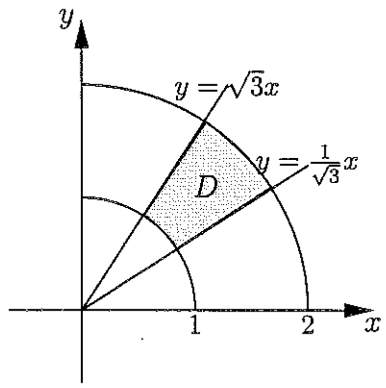
$$-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots = \arctan(-1).$$

Q 2. (17 points) Let D be the "rectangle-like" region in the first quadrant of the xy-plane, whose two sides are the lines $y = \sqrt{3}x$, $y = \frac{1}{\sqrt{3}}x$, and the other two sides are the circles $x^2 + y^2 = 1$, $x^2 + y^2 = 4$. Let C be the boundary of D, and assume C is oriented in the counterclockwise direction. Evaluate the following line integral

$$I = \oint_C (3x^2y^2 - y^3)\, dx + (2x^3y + x^3)\, dy$$

Solution. We use **Green's Theorem.**

$$\begin{aligned}
I &= \iint_D \left(\frac{\partial}{\partial x}(2x^3y + x^3) - \frac{\partial}{\partial y}(3x^2y^2 - y^3) \right) dA \\
&= \iint_D ((6x^2y + 3x^2) - (6x^2y - 3y^2))\, dA \\
&= \iint_D 3(x^2 + y^2)\, dA \quad \text{(use polar coordinates)} \\
&= 3 \int_{\pi/6}^{\pi/3} \int_1^2 r^2\, r\, dr d\theta = 3 \int_{\pi/6}^{\pi/3} \left. \frac{r^4}{4} \right|_1^2 d\theta \\
&= 3\, \frac{15}{4} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{15\pi}{8}.
\end{aligned}$$

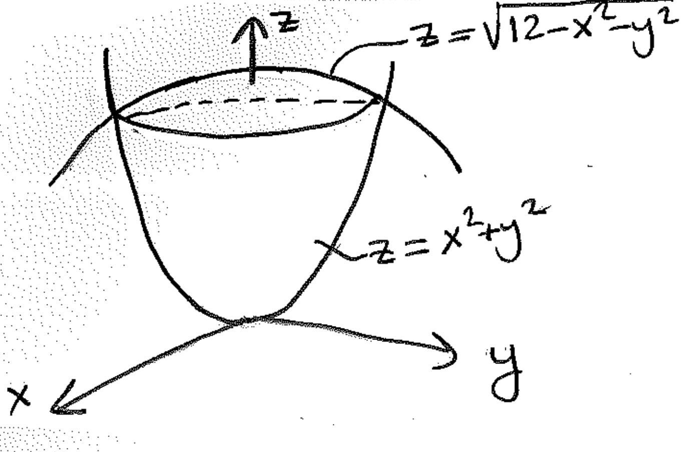


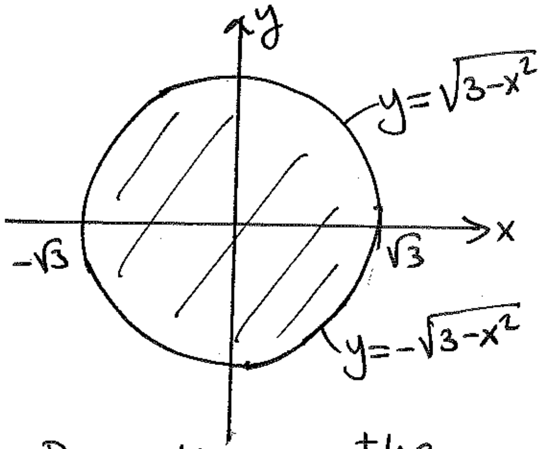
Q 3. (17 points) Let

$$I = \int_{-\sqrt{3}}^{\sqrt{3}} dx \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} dy \int_{x^2+y^2}^{\sqrt{12-x^2-y^2}} z\, dz.$$

Write I as equivalent iterated triple integrals in
(a) cylindrical coordinate,
(b) spherical coordinates.
Do not evaluate the integrals.

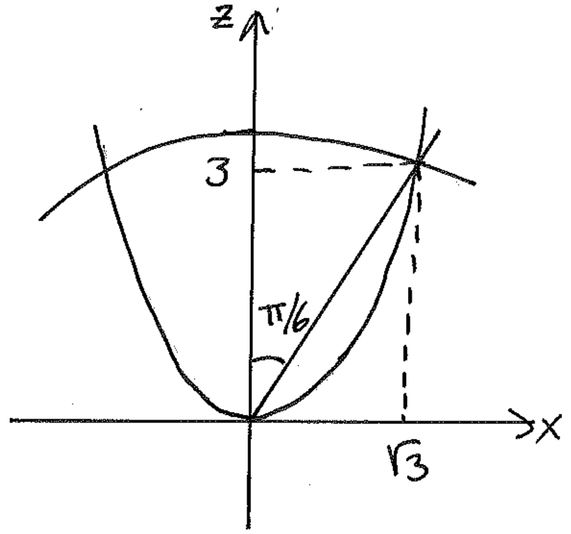
Solution. The solid of integration is bounded above by the upper part of the sphere $x^2 + y^2 + z^2 = 12$, and below by the circular paraboloid $z = x^2 + y^2$. Their intersection has equation $z + z^2 = 12$ which has positive root $z = 3$. That gives $x^2 + y^2 = 3$, this is the projection (shadow) of the solid on the xy-plane.





(a) $I = \displaystyle\int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2}^{\sqrt{12-r^2}} z\, r\, dz dr d\theta.$

(b) We divide the solid into two parts as shown in Thus the figure. We consider the parts inside the cone and outside the cone separately.
For the part inside the cone:



$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{\sqrt{12}} \rho\cos\phi\,\rho^2\sin\phi\,d\rho d\phi d\theta.$$

For the part outside the cone: in spherical coordinates the paraboloid $z = x^2 + y^2$ becomes

$$\begin{aligned}\rho\cos\phi &= \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = \rho^2\sin^2\phi \\ &\Rightarrow \rho\cos\phi = \rho^2\sin^2\phi \Rightarrow \rho = \frac{\cos\phi}{\sin^2\phi}\end{aligned}$$

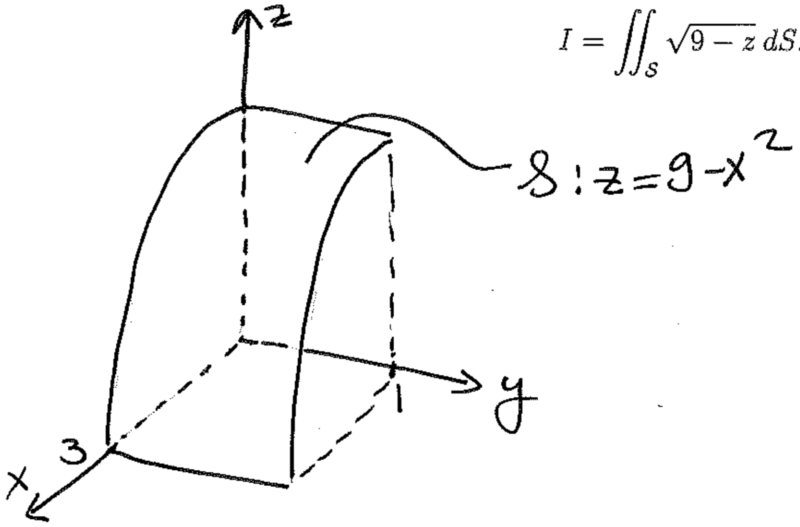
So we get

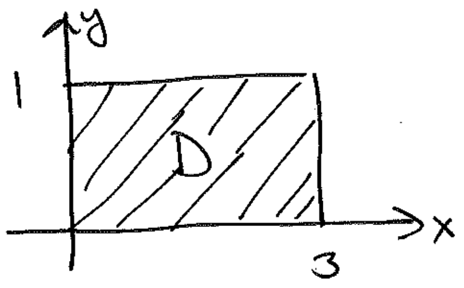
$$\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\frac{\cos\phi}{\sin^2\phi}} \rho\cos\phi\,\rho^2\sin\phi\,d\rho d\phi d\theta.$$

$$I = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{\sqrt{12}} \rho\cos\phi\,\rho^2\sin\phi\,d\rho d\phi d\theta + \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{\frac{\cos\phi}{\sin^2\phi}} \rho\cos\phi\,\rho^2\sin\phi\,d\rho d\phi d\theta.$$

Q 4. (17 points) Let $\mathcal{S}$ be the surface cut from the parabolic cylinder $z = 9 - x^2$ by the planes $z = 0$, $y = 0$, $y = 1$, $x = 0$. Evaluate

$$I = \iint_{\mathcal{S}} \sqrt{9 - z}\,dS.$$





Solution. We have

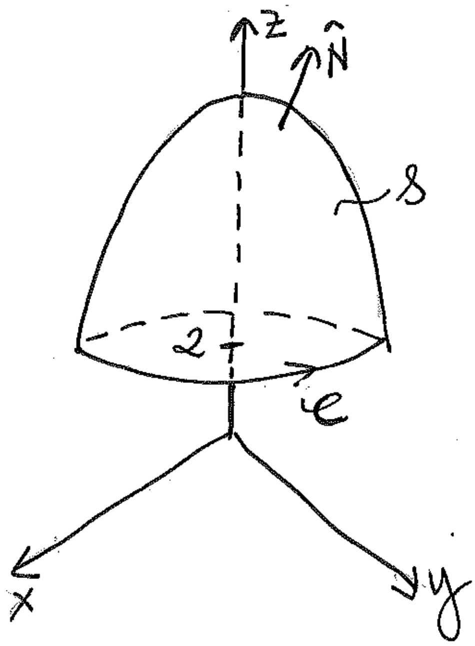
$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \sqrt{1 + 4x^2}\,dxdy$$

On $\mathcal{S}$, $\sqrt{9 - z} = \sqrt{9 - (9 - x^2)} = \sqrt{x^2} = |x|$. Then

$$\begin{aligned} I &= \iint_D |x|\sqrt{1 + 4x^2}\,dxdy = \int_0^1\int_0^3 x\sqrt{1 + 4x^2}\,dx\,dy \\ &= \int_0^1 \frac{1}{8}\frac{2}{3}(1 + 4x^2)^{3/2}\Big|_0^3 dy = \int_0^1 \frac{1}{12}(37\sqrt{37} - 1)\,dy = \frac{37\sqrt{37} - 1}{12}.\end{aligned}$$

Q 5. (17 points) Let S be the top of the paraboloid $z = 6 - x^2 - y^2$ cut by the plane $z = 2$, and assume S is oriented upward (that is away from the origin). Let

$$\mathbf{F} = -yz\mathbf{i} + xz\mathbf{j} + e^y \sin(x^3)\mathbf{k}.$$



By using a theorem, evaluate

$$I = \iint_S \text{curl}\mathbf{F} \bullet \hat{\text{N}}\, dS.$$

Which theorem did you use?

Solution. We use **Stokes' Theorem**. Let C be Then the boundary of S. Then C is oriented counter-clockwise when viewed from high above the xy-plane. By Stokes'Theorem we have

$$I = \iint_S \text{curl}\mathbf{F} \bullet \hat{\text{N}}\, dS = \oint_C \mathbf{F} \bullet d\mathbf{r}.$$

C is the circle $x^2 + y^2 = 4$ in the plane $z = 2$. So it has parametrization:

$$C : \mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\,\mathbf{k},\ 0 \le t \le 2\pi.$$

$$\begin{aligned} I &= \oint_C \mathbf{F} \bullet d\mathbf{r} \\ &= -yz\,dx + xz\,dy + e^y \sin(x^3)\,dz \\ &= \int_0^{2\pi} (-2\sin t\, 2\,(-2\sin t) + 2\cos t\; 2\; 2\cos t + 0)\,dt \\ &= \int_0^{2\pi} (8\sin^2 t + 8\cos^2 t)\,dt = 16\pi. \end{aligned}$$

Q 6. (17 points) A function $f : \mathbb{R}^3 \to \mathbb{R}$ has continuous second order partial derivatives and is such that

- $f(x, y, z) \neq 0$ for all (x, y, z),
- $|\nabla f|^2 = 2f$,
- $\text{div}(f\,\nabla f) = 5f$.

Evaluate $I = \displaystyle\iint_S \nabla f \bullet \hat{\text{N}}\, dS$, where S is the boundary of the solid cylinder $x^2 + y^2 \le 4$, $0 \le z \le 5$, and $\hat{\text{N}}$ is the outer unit normal vector to S.

Solution. by Divergence Theorem, we have that $I = \displaystyle\iiint_D \text{div}(\nabla f)\, dV$, where D is the solid cylinder $x^2 + y^2 \le 4,\ 0 \le z \le 5$.

Now we calculate $\text{div}(\nabla f)$ as follows.

$$\text{div} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},\quad \text{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

We are given

$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = 2f$ and $\operatorname{div}(f\,\nabla f) = 5f$. That is

$$\begin{aligned}
5f &= \frac{\partial}{\partial x}\left(f\,\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\,\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\,\frac{\partial f}{\partial z}\right) \\
&= \left(\frac{\partial f}{\partial x}\right)^2 + f\,\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial f}{\partial y}\right)^2 + f\,\frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial z}\right)^2 + f\,\frac{\partial^2 f}{\partial z^2} \\
&= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 + f\,\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)
\end{aligned}$$

Thus

$$5f = 2f + f\,(\operatorname{div}(\nabla f)) \Rightarrow f\,(\operatorname{div}(\nabla f)) = 3f.$$

Since $f \neq 0$, we have $\operatorname{div}(\nabla f) = 3$. So

$$I = \iiint_D \operatorname{div}(\nabla f)\,dV = \iiint_D 3\,dV = 3\,\text{Volume}(D) = 3\,(\pi 2^2)\,5 = 60\pi.$$