Date: 4 December 2010 Time: 15:15-17:15

NAME SURNAME:....................................................

STUDENT NO:....................................................

# MATH 112 SECOND MIDTERM EXAM SOLUTIONS

**Q 1. (10+10 points)** Evaluate the following improper integrals:

**(a)** $I = \displaystyle\int_0^\infty x\, e^{-x^2}\, dx.$

**Solution.** This integral is improper since the region of integration is unbounded. Otherwise the integrand is continuous on $[0, \infty)$.

$$\begin{aligned} I &= \lim_{b\to\infty} \int_0^b x\, e^{-x^2}\, dx \quad (u = x^2 \Rightarrow du = 2x\, dx.) \\ &= \lim_{b\to\infty} \int_0^{b^2} e^{-u} \frac{1}{2}\, du = \lim_{b\to\infty} -\frac{1}{2}\, e^{-u}\Big|_0^{b^2} = \lim_{b\to\infty} \left(-\frac{1}{2}\, e^{-b^2} + \frac{1}{2}\right) \\ &= \lim_{b\to\infty} \left(-\frac{1}{2e^{b^2}} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

**(b)** $\displaystyle\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\, dx.$

**Solution.** This integral is improper, since the integrand $f(x) = \frac{x+1}{\sqrt{x^2+2x}}$ becomes infinite at $x = 0$. Otherwise $f$ is continuous on $(0, 1]$. We have

$$\begin{aligned} I &= \lim_{c\to 0^+} \int_c^1 \frac{x+1}{\sqrt{x^2+2x}}\, dx \quad (u = x^2 + 2x \Rightarrow du = (2x+2)dx) \\ &= \lim_{c\to 0^+} \int_{c^2+2c}^3 \frac{\frac{1}{2}\, du}{\sqrt{u}} = \lim_{c\to 0^+} \int_{c^2+2c}^3 \frac{du}{2\sqrt{u}} = \lim_{c\to 0^+} \sqrt{u}\,\Big|_{c^2+2c}^3 \\ &= \lim_{c\to 0^+} (\sqrt{3} - \sqrt{c^2+2c}) = \sqrt{3} - 0 = \sqrt{3}. \end{aligned}$$

**Q 2. (10+10 points)** Determine whether the following improper integrals are convergent or divergent by using a test. Show all your work and write the name of the test that you use.

**(a)** $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^3}}\, dx.$

**Solution.** We use **Direct Comparison Test**. We have $f(x) = \dfrac{1}{\sqrt{1+x^3}}$. We let

$g(x) = \dfrac{1}{\sqrt{x^3}}$. Then $0 \le f(x) \le g(x)$ for all $x \ge 0$. Also

$$\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \int_1^\infty \frac{1}{x^{3/2}}\,dx \text{ is convergent } (p\text{-integral with } p = 3/2 > 1.)$$

Thus by **DCT**, $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^3}}\,dx$ is **convergent**.

**(b)** $\displaystyle\int_0^1 \frac{\sin x}{x^2}\,dx.$

**Solution.** We use **Limit Comparison Test**. We have $f(x) = \dfrac{\sin x}{x^2} > 0$ for $0 < x \le 1$, since $\sin xx > 0$ for $0 < x < \pi$. Moreover for $x \approx 0$, we have $\sin x \approx x$. Thus we let $g(x) = \frac{x}{x^2} = \frac{1}{x}$. Then

$$L = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

So $0 < L < 1$. Also $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x}\,dx$ is divergent ($p$-integral with $p = 1$). So by **LCT**, $\displaystyle\int_0^1 \frac{\sin x}{x^2}\,dx$ is also **divergent**.

**Q 3. (10+10 points)** Determine whether the following series are convergent or divergent. If convergent, find the sum.

**(a)** $\displaystyle\sum_{n=1}^{\infty} (\arctan n - \arctan(n+1))\,.$

**Solution.** This is a telescoping series. $a_n = \arctan n - \arctan(n+1)$. We have

$$\begin{aligned} S_n &= (\arctan 1 - \arctan 2) + (\arctan 2 - \arctan 3) + \cdots + (\arctan n - \arctan(n+1)) \\ &= \arctan 1 - \arctan(n+1). \end{aligned}$$

So

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\arctan 1 - \arctan(n+1)) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}.$$

So the series $\sum_{n=1}^{\infty} (\arctan n - \arctan(n+1))$ is **convergent** and has sum $s = -\frac{\pi}{4}$, that is $\sum_{n=1}^{\infty} (\arctan n - \arctan(n+1)) = -\frac{\pi}{4}$.

**(b)** $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{3^{n+1}}.$

**Solution.** We have

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{3^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right) (-1)^n \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right) \frac{(-1)^n\, 2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right) \left(\frac{-2}{3}\right)^n.$$

This is a geometric series $\sum_{n=0}^{\infty} a\, r^n$, with $a = -\frac{1}{3}$, and $r = -\frac{2}{3}$. Since $-1 < r < 1$, the series is **convergent**, and has sum

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{3^{n+1}} = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right) \left(\frac{-2}{3}\right)^n = \frac{a}{1-r} = \frac{-\frac{1}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{-\frac{1}{3}}{\frac{5}{3}} = -\frac{1}{5}.$$

**Q 4. (10+10 points)** Determine whether the following series are convergent or divergent by using a test. Show all your work and write the name of the test that you use.

**(a)** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}}$.

**Solution.** We have $a_n = \dfrac{1}{n^{1+(1/n)}}$. Then $a_n > 0$. Let $b_n = \dfrac{1}{n} > 0$. We use **Limit Comparison Test**. Then

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{1+(1/n)}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n\,n^{(1/n)}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1.$$

Thus $0 < L < \infty$. The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (the harmonic series is divergent). So by **LCT**, the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}}$ is also divergent.

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n+1)}{5^n\, n!}$.

**Solution.** We have $a_n = \frac{1\cdot 4\cdot 7\cdots(3n+1)}{5^n\, n!} > 0$. We use the **Ratio Test**.

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1\cdot 4\cdot 7\cdots(3n+1)\cdot(3n+4)}{5^{n+1}\,(n+1)!} \cdot \frac{5^n\, n!}{1\cdot 4\cdot 7\cdots(3n+1)} \\ &= \lim_{n\to\infty} \frac{3n+4}{5\,(n+1)} = \frac{3}{5}.\end{aligned}$$

Since $\rho < 1$, by the **Ratio Test**, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n+1)}{5^n\, n!}$ is **convergent**.

**Q 5. (10+10 points)** Determine whether the following series are convergent or divergent by using a test. Show all your work and write the name of the test that you use.

**(a)** $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$.

**Solution.** We have $a_n = \left(1+\frac{1}{n}\right)^n > 0$. If we first try the Root Test, we get $\rho = 1$, which means the Root Test gives no information. If we use the $n$**-the Term Test**, we have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 = e \neq 0.$$

Thus by the $n$**-th Term Test**, the series $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ is **divergent**.

**(b)** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\,\ln n\,\left(\ln(\ln n)\right)^2}$.

**Solution.** We have $a_n = \dfrac{1}{n\,\ln n\,\left(\ln(\ln n)\right)^2} > 0$. We use the **Integral Test**. Let

$$f(x) = \frac{1}{x\,\ln x\,\left(\ln(\ln x)\right)^2},\quad x \geq 3.$$

Then $f(x)$ is positive and continuous for $x \geq 3$. As $x$ increases through the values $\geq 3$, $x$, $\ln x$, $\ln(\ln x)$ all increase, so the denominator increases, therefore $f(x) = \frac{1}{\text{denominator}}$ decreases. So we can use the integral test.

$$\begin{aligned} \int_3^\infty f(x)\,dx &= \int_3^\infty \frac{1}{x\,\ln x\,\left(\ln(\ln x)\right)^2}\,dx \qquad \left\{ \begin{array}{ccc} u = \ln(\ln x) & \Rightarrow & du = \frac{1}{\ln x}\,\frac{1}{x}\,dx \\ x = 3 & \Rightarrow & u = \ln(\ln 3) \\ x \to \infty & \Rightarrow & u \to \infty \end{array} \right\} \\ &= \int_{\ln(\ln 3)}^\infty \frac{du}{u^2}. \end{aligned}$$

The last integral is convergent since $p = 2 > 1$ and $\ln(\ln 3) > 0$. Thus $\int_3^\infty f(x)\,dx$ is convergent, so by the **Integral Test**, the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\,\ln n\,\left(\ln(\ln n)\right)^2}$ is also **convergent**.