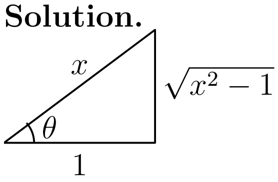
Date: 7 January 2011 Time: 15:30-17:30

MATH 112 FINAL EXAM SOLUTIONS

Q 1. (10+10 points)

(a) Evaluate the integral: $I = \displaystyle\int \frac{\sqrt{x^2-1}}{x^3}\, dx,\ x > 1.$

Solution.

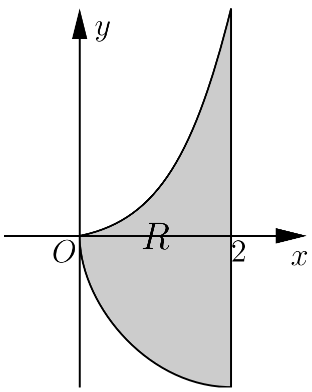


We substitute $x = \sec\theta$. Since $x > 1$, we have $0 \le \theta < \pi/2$. Then $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ and $dx = \sec\theta\tan\theta\, d\theta$. So

$$\begin{aligned} I &= \int \frac{\tan\theta}{\sec^3\theta} \sec\theta\tan\theta d\theta = \int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta = \int \sin^2\theta\, d\theta \\ &= \int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{2}\left(\theta - \sin\theta\cos\theta\right) + C \\ &= \frac{1}{2}\left(\text{arcsec}x - \frac{\sqrt{x^2-1}}{x}\,\frac{1}{x}\right) + C = \frac{1}{2}\left(\text{arcsec}x - \frac{\sqrt{x^2-1}}{x^2}\right) + C \end{aligned}$$

(b) Let R be the region in the plane bounded above by the parabola $y = x^2$, below by the lower half of the circle $(x-2)^2 + y^2 = 4$, and on the right by the line $x = 2$. Assume R is revolved about the y-axis, and a solid is generated. Draw the region R, and set up the integral for the volume of this solid by using the cylindrical shell method. **Do not evaluate.**

Solution.



We use the method of cylindrical shells, and get

$$\begin{aligned} V &= 2\pi \int_0^2 x\left(x^2 - (-\sqrt{4-(x-2)^2})\right)\, dx \\ &= 2\pi \int_0^2 x\left(x^2 + \sqrt{4-(x-2)^2}\right)\, dx \end{aligned}$$

Q 2. (10+10 points) Determine whether the following series are convergent or divergent by using a test. Show all your work and write the name of the test that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \sin(1/n^2)\, \sin(n^2).$

Solution. We have $a_n = \sin(1/n^2)\, \sin(n^2)$. We use the following two inequalities: i) $|\sin x| \le |x|$ (useful when $|x|$ is small), and ii) $|\sin x| \le 1$ (useful when $|x|$ is large). So

i) $|\sin(1/n^2)| \leq |1/n^2|$, and ii) $|\sin(n^2)| \leq 1$. Then

$$|a_n| = |\sin(1/n^2)|\,|\sin(n^2)| \leq \frac{1}{n^2};\ \text{choose } b_n = \frac{1}{n^2}.$$

since $\sum b_n = \sum \frac{1}{n^2}$ is convergent (p-series with $p = 2 > 1$), by DCT, the series $\sum |a_n|$ is convergent, so by ACT the series $\sum a_n = \sum \sin(1/n^2)\ \sin(n^2)$ is convergent.

(b) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n}{n^2+3n+10}$.

Solution. This is an alternating series $\sum(-1)^{n+1}u_n$, where $u_n = \frac{n}{n^2+3n+10}$. We check the conditions of AST: i) Is $\{u_n\}$ decreasing?, ii) is $\lim_{n\to\infty} u_n = 0$?
i) Let $f(x) = \frac{x}{x^2+3x+10}$, $x > 0$. Then $u_n = f(n)$. We have

$$\begin{aligned} f'(x) = \frac{-x^2+10}{(x^2+3x+10)^2} \quad &\Rightarrow\quad f'(x) < 0 \text{ for } x > \sqrt{10} \\ &\Rightarrow\quad f(x) \text{ is decreasing for } x\sqrt{10} \\ &\Rightarrow\quad u_n \text{ is decreasing for } n \geq 4. \end{aligned}$$

ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+3n+10} = 0$.

Thus by AST the series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n}{n^2+3n+10}$ is convergent.

Q 3. (24 points) Show all your work.
a) i. Find the interval of convergence of the following power series.
a) ii. For which x in this interval, is the series absolutely convergent?
a) iii. For which x in this interval is this series conditionally convergent?

$$\sum_{n=0}^{\infty}\frac{(x+3)^{2n}}{\sqrt{n+1}\cdot 4^n}.$$

Solution. We apply the Ratio Test (or the Root Test) to the series $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(x+3)^{2n}}{\sqrt{n+1}\cdot 4^n}\right|$.

$$\rho = \lim_{n\to\infty}\left|\frac{(x+3)^{2n+2}}{\sqrt{n+2}\cdot 4^{n+1}}\right|\left|\frac{\sqrt{n+1}\cdot 4^n}{(x+3)^{2n}}\right| = \lim_{n\to\infty}\frac{|x+3|^2}{4}\frac{\sqrt{n+1}}{\sqrt{n+2}} = \frac{|x+3|^2}{4}.$$

We have

$$\begin{aligned} \text{series converges absolutely} \quad &\text{if} \quad \rho < 1 \Leftrightarrow |x+3|^2 < 4 \Leftrightarrow |x+3| < 2 \Leftrightarrow -5 < x < -1 \\ \text{series diverges} \quad &\text{if} \quad \rho > 1 \Leftrightarrow |x+3|^2 > 4 \Leftrightarrow x < -5 \text{ or } x > -1 \\ \text{test is inconclusive} \quad &\text{if} \quad \rho = 1 \Leftrightarrow x = -5 \text{ or } x = -1. \end{aligned}$$

When $x = -5$ or $x = -1$, we have $x + 3 = \mp 2$, so the series becomes

$$\sum_{n=0}^{\infty}\frac{(\mp 2)^{2n}}{\sqrt{n+1}\cdot 4^n} = \sum_{n=0}^{\infty}\frac{4^n}{\sqrt{n+1}\cdot 4^n} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}},$$

and this series is divergent (p-series with $p = 1/2 \leq 1$).

Thus:
i. Interval of convergence is $(-5, -1)$.
ii. The series converges absolutely for all $x \in (-5, -1)$.
iii. There is no x in the interval for which the series converges conditionally.

b) Let $f(x) = \sum_{n=0}^{\infty} \frac{(x+3)^{2n}}{\sqrt{n+1} \cdot 4^n}$. Find $f^{(148)}(-3)$ and $f^{(149)}(-3)$.

Solution. Remember if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, then $c_n = \frac{f^{(n)}(a)}{n!}$, thus $f^{(n)}(a) = n!\, c_n$. In this question $a = -3$, and $c_n = 0$ if n is odd. More precisely,

$$\begin{aligned} c_{2n} &= \frac{1}{\sqrt{n+1} \cdot 4^n}, \\ c_{2n+1} &= 0. \end{aligned}$$

So

$$\begin{aligned} f^{(148)}(-3) &= 148!\, c_{148} = 148!\, c_{(2)(74)} = 148!\, \frac{1}{\sqrt{74+1} \cdot 4^{74}} = \frac{148!}{\sqrt{75}\, 4^{74}}, \\ f^{(149)}(-3) &= 149!\, c_{149} = 0. \end{aligned}$$

Q 4. (16 points) Find a polynomial $P(x)$ of the least degree which approximates the function

$$F(x) = \int_0^x e^{-t^2}\, dt$$

with $|\text{error}| < 1/1000$ for all $x \in [0,1]$.
Solution. We have

$$e^s = 1+s+\frac{s^2}{2!}+\frac{s^3}{3!}+\cdots+\frac{s^n}{n!}+\cdots, \text{ all } s \Rightarrow e^{-t^2} = 1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}+\cdots+(-1)^n\frac{t^{2n}}{n!}+\cdots, \text{ all } t.$$

Thus

$$\begin{aligned} F(x) &= \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + (-1)^n \frac{t^{2n}}{n!} + \cdots\right) dt \\ &= t - \frac{t^3}{3} + \frac{t^5}{2!\,5} - \cdots + (-1)^n \frac{t^{2n+1}}{n!\,(2n+1)} + \cdots \Big|_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{2!\,5} - \cdots + (-1)^n \frac{x^{2n+1}}{n!\,(2n+1)} + \cdots \end{aligned}$$

When $0 \le x$, the last power series is an alternating series with $u_n = \frac{x^{2n+1}}{n!\,(2n+1)}$. For $0 \le x \le 1$ we have

$$\frac{u_{n+1}}{u_n} = \frac{x^{2n+3}}{(n+1)!\,(2n+3)} \frac{n!\,(2n+1)}{x^{2n+1}} = \frac{x^2}{n+1} \frac{2n+1}{2n+3} \le 1 \Rightarrow u_{n+1} \le u_n.$$

Also,

$$0 \le x \le 1 \Rightarrow 0 \le u_n \le \frac{1}{n!\,(2n+1)} \Rightarrow \lim_{n\to\infty} u_n = 0.$$

Thus we can use the AST error formula, that is if we stop at the n-th term $(-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$ then the absolute error will be less than the absolute value of the first unused term. That is $|\text{error}| < \frac{x^{2n+3}}{(n+1)!\,(2n+3)}$. Thus

$$|\text{error}| < \frac{x^{2n+3}}{(n+1)!\,(2n+3)} \le \frac{1}{(n+1)!\,(2n+3)}.$$

Now we choose the smallest positive integer n such that $\frac{1}{(n+1)!\,(2n+3)} \leq \frac{1}{1000}$, that is $1000 \leq (n+1)!\,(2n+3)$. We see that for $n = 4$, $(n+1)!\,(2n+3) = 1320$ and for $n = 3$, $(n+1)!\,(2n+3) = 254$. Thus $n = 4$ is good. So we take

$$P(x) = x - \frac{x^3}{3} + \frac{x^5}{2!\,5} - \frac{x^7}{3!\,7} + \frac{x^9}{4!\,9},$$

that is the last term is $(-1)^n \frac{x^{2n+1}}{n!\,(2n+1)}$ with $n = 4$.

Q 5. (20 points) Given the point $P_0(0, -3, 1)$, and the line $L_1 : x = 5t - 3, y = -2t + 2, z = 4t - 4, -\infty < t < \infty$,

(a) Find the equation of the plane M which passes through the point P_0 and is perpendicular to the line L_1.

Solution. We have $\mathbf{n}_M = \mathbf{v}_{L_1} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. So

$$M:\ 5(x-0) - 2(y+3) + 4(z-1) = 0 \text{ or } M:\ 5x - 2y + 4z = 10.$$

(b) Find the coordinates of the intersection point Q_0 of the line L_1 and the plane M in part (a).

Solution. Let Q_0 have coordinates (x, y, z). Then

$$\begin{aligned} Q_0 \text{ is on } L_1 &\Rightarrow\ x = 5t - 3, y = -2t + 2, z = 4t - 4, \\ Q_0 \text{ is on } M &\Rightarrow\ 5x - 2y + 4z = 10, \\ \text{so} &\quad\ 5(5t-3) - 2(-2t+2) + 4(4t-4) = 10 \Rightarrow 45t = 45 \Rightarrow t = 1. \end{aligned}$$

So $x = 2, y = 0, z = 0$, that is, the intersection point is $Q_0(2, 0, 0)$.

(c) Find the parametric equations of the line L that passes through the point P_0 and intersects the line L_1 orthogonally.
Hint. Use part (b).

Solution. The direction vector of the line L is $\mathbf{v}_L = \vec{P_0Q_0} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Taking the point on the line L as $P_0(0, -3, 1)$, we get

$$L:\ x = 2t,\ y = 3t - 3,\ z = -t + 1,\ -\infty < t < \infty.$$