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NAME:.....

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MATH 624 Lie groups Final Exam

Please return the cover page together with the solutions and write your name in BLOCK LETTERS.

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Problem 1. Evaluate dimensions of the classical Lie groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, and exceptional groups G_2 and F_4.
Hint: Root system of the latter group can be found in Problem 2, MT 2.

Classical isomorphisms. You should be aware from lecture notes that $\boxed{\mathrm{Spin}(3) \simeq \mathrm{SU}(2) \simeq \mathrm{Sp}(1).}$ Below is a complete list of the remaining isomorphisms between classical simply connected compact Lie groups.

Problem 2. Consider bilateral action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on quaternions

$$q_1 \times q_2 : \mathbb{H} \to \mathbb{H}, \quad x \mapsto q_1 x q_2^*, \quad q_1, q_2 \in \mathrm{Sp}(1).$$

Prove that

(a) The action respects the metric and induces homomorphism $\varphi : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(\mathbb{H}) = \mathrm{SO}(4)$

(b) which is actually the universal covering, and hence $\boxed{\mathrm{Spin}(4) \simeq \mathrm{Sp}(1) \times \mathrm{Sp}(1)}$

Problem 3. Let $W \simeq \mathbb{E}^5$ be Euclidean space of traceless quaternionic Hermitian 2×2 matrices $M = \begin{bmatrix} x & q \\ q^* & -x \end{bmatrix}$, $x \in \mathbb{R}$, $q \in \mathbb{H}$ with metric $|M|^2 = -\det M = x^2 + |q|^2$. Prove that

(a) The space W is invariant with respect to adjoint action $g : M \mapsto gMg^{-1}, \quad g \in \mathrm{Sp}(2), \quad M \in W$,

(b) the action respects the metric, and induces a homomorphism $\varphi : \mathrm{Sp}(2) \to \mathrm{SO}(W) = \mathrm{SO}(5)$,

(c) which is in fact the universal covering, that is $\boxed{\mathrm{Spin}(5) \simeq \mathrm{Sp}(2)}$

Problem 4. Consider complex Hilbert space $\mathcal{H} \simeq \mathbb{C}^4$ of dimension 4 with inner product $\langle x, y\rangle$ together with its exterior square $\wedge^2\mathcal{H} = \mathcal{H} \wedge \mathcal{H} \simeq \mathbb{C}^6$ where $g \in \mathrm{SU}(\mathcal{H})$ acts by $g : x \wedge y \mapsto gx \wedge gy$. Prove that

(a) Equation $\langle x \wedge y, u \wedge v\rangle = \langle x, u\rangle\langle y, v\rangle - \langle x, v\rangle\langle y, u\rangle$ defines an inner product on $\wedge^2\mathcal{H}$

(b) invariant with respect to the natural action $\mathrm{SU}(\mathcal{H}) : \wedge^2\mathcal{H}$.

(c) In addition, the space $\wedge^2\mathcal{H}$ carries non-degenerate symmetric bilinear form

$$(x \wedge y, u \wedge v) = x \wedge y \wedge u \wedge v \in \wedge^4\mathbb{C}^4 \simeq \mathbb{C}$$

(d) also respected by the above action $\mathrm{SU}(\mathcal{H}) : \wedge^2\mathcal{H}$.

(e) This gives rise to a homomorphism $\varphi : \mathrm{SU}(\mathcal{H}) \to \mathrm{SO}(\wedge^2\mathcal{H}) \cap \mathrm{SU}(\wedge^2\mathcal{H}) \simeq \mathrm{SO}(6, \mathbb{C}) \cap \mathrm{SU}(6) \simeq \mathrm{SO}(6)$

(f) which is actually the universal covering, i.e. $\boxed{\mathrm{Spin}(6) \simeq \mathrm{SU}(4)}$

There are also *noncompact* versions of the above isomorphisms. Below is a simple example.

Problem 5. Prove that group $\mathrm{SO}^+(1,3)$ of Lorentz transformations $g : \mathbb{M}^4 \to \mathbb{M}^4$ in Minkowski space which respect metric $t^2 - x^2 - y^2 - z^2$, orientation $\det g = +1$, and the future cone $C^+ = \{t^2 - x^2 - y^2 - z^2 > 0, t > 0\}$ is isogeneous to $\mathrm{SL}(2, \mathbb{C})$. More precisely, there exists a two fold covering $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}^+(1,3)$, so that $\boxed{\mathrm{Spin}^+(1,3) \simeq \mathrm{SL}(2, \mathbb{C})}$

Hint: Realize $\mathbb{M}^4$ as space of Hermitian 2×2 matrices with quadratic form $(A, A) = \det A$ on which $g \in \mathrm{SL}(2, \mathbb{C})$ acts by $A \mapsto gAg^*$.