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Shift operators on harmonic Hilbert function spaces on real balls and von Neumann inequality

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On harmonic function spaces, we define shift operators using zonal harmonics and partial derivatives, and develop their basic properties. These operators turn out to be multiplications by the coordinate variables followed by projections on harmonic subspaces. This duality gives rise to a new identity for zonal harmonics. We introduce large families of reproducing kernel Hilbert spaces of harmonic functions on the unit ball of $\mathbb{R}^n$ and investigate the action of the shift operators on them. We prove a dilation result for a commuting row contraction which is also what we call harmonic type. As a consequence, we show that the norm of one of our spaces $\breve{\mathcal{G}}$ is maximal among those spaces with contractive norms on harmonic polynomials. We then obtain a von Neumann inequality for harmonic polynomials of a commuting harmonic-type row contraction. This yields the maximality of the operator norm of a harmonic polynomial of the shift on $\breve{\mathcal{G}}$ making this space a natural harmonic counterpart of the Drury-Arveson space.

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1. Introduction

After the pioneering work of Drury [10] and Arveson [6] on extending the von Neumann inequality to commuting operator tuples, there have been several other generalizations to Hilbert space operators in other settings. We can cite [20], [3], [16], [11], [13], and [8] to name a few. There is also the earlier [19] on noncommuting operator tuples. But in all von Neumann inequalities that we know of, the polynomials acting on the operators are holomorphic functions of their variables. It is the aim of this work to obtain a von Neumann inequality in which the polynomials are harmonic in the usual sense in $\mathbb{R}^n$.

Multivariable versions of von Neumann inequality often depend on shift operators on a specific Hilbert function space. This immediately brings out the first major obstacle in dealing with harmonicity. Even the definition of a shift operator on a space of harmonic functions has not been made before, because harmonicity is not preserved under multiplication, and a multiplication by a coordinate variable must be followed by some form of a projection on harmonic functions. We make a definition and check it by using another approach.

Another obstacle is to decide which space among harmonic function spaces plays a role like that of the Drury-Arveson space among holomorphic spaces. We find out that considering a family of reproducing kernel Hilbert spaces $\mathcal{G}_q$ of harmonic functions on the unit ball of $\mathbb{R}^n$ indexed by $q \in \mathbb{R}$ is more feasible since it exposes the compositions of the spaces better. Then it is easier to pick one of these spaces as the harmonic counterpart of the Drury-Arveson space using its extremal properties in the family.

One more obstacle is that the more complicated structure of harmonic functions persists at the operator level and we are obliged to restrict our attention in von Neumann inequality to a class of contractions that we call harmonic type.

We now present our major results; for them it helps to have some familiarity with the classical knowledge on harmonic polynomials summarized in Section 3. For $m = 0, 1, 2, \ldots$, let $\mathcal{P}_m$ and $\mathcal{H}_m$ denote the homogeneous polynomials of degree m and spherical harmonics of degree m on $\mathbb{R}^n$, respectively. Let $H_m : \mathcal{P}_m \to \mathcal{H}_m$ be the standard projection. The zonal harmonics $Z_m(x, y)$ are the reproducing kernels of the $\mathcal{H}_m$ with respect to the L^2 inner product on the unit sphere. For $j = 1, \ldots, n$, we define the shift operators $S_j : \mathcal{H}_m \to \mathcal{H}_{m+1}$ acting on x by

$$S_j Z_m(x, y) := \frac{1}{n + 2m} \frac{\partial}{\partial y_j} Z_{m+1}(x, y).$$

Our first main result shows that S_j is closely related to the operator of multiplication M_{x_j} by the jth coordinate variable.

Theorem 1.1. $S_j = H_{m+1} M_{x_j}$ *for all* $j = 1, \ldots, n$ *and* $m = 0, 1, 2, \ldots$.

In the course of proving this theorem, we obtain the following identities for the Gegenbauer polynomials C_m^λ and the Chebyshev polynomials T_m which seem new. There, K

is the Kelvin transform which transforms a harmonic function on the unit ball to one on its exterior.

Theorem 1.2. *For $\xi, \eta \in \mathbb{S}$, we have*

$$K\big[(\eta\cdot\partial)^m K[1]\big](\xi) = (-1)^m\, m!\, C_m^{n/2-1}(\xi\cdot\eta) \qquad (n \geq 3,\ m = 0, 1, 2, \ldots),$$
$$K\big[(\eta\cdot\partial)^m K[\log|\cdot|]\big](\xi) = (-1)^m (m-1)!\, T_m(\xi\cdot\eta) \qquad (n = 2,\ m = 1, 2, \ldots).$$

We define the space that we claim to be the harmonic version of the Drury-Arveson space as the reproducing kernel Hilbert space $\breve{\mathcal{G}}$ on the unit ball of $\mathbb{R}^n$ with reproducing kernel

$$\breve{G}(x,y) := \sum_{m=0}^{\infty} \frac{1}{A_m}\, Z_m(x,y),$$

where A_m is the coefficient of $(x\cdot y)^m$ in the expansion of $Z_m(x,y)$. We call commuting operators $(T_1,\ldots,T_n)$ a row contraction if they are a contraction as a tuple. A contractive norm is one in which the tuple of shift operators is a row contraction. Another main result of ours shows that the norm of $\breve{\mathcal{G}}$ is as large as possible.

Theorem 1.3. *If $\|\cdot\|$ is a contractive Hilbert norm on harmonic polynomials that respects the orthogonality of L^2, then $\|\cdot\| \leq \|\cdot\|_{\breve{\mathcal{G}}}\, \|1\|$.*

We call an operator tuple $(T_1,\ldots,T_n)$ *harmonic type* if $T_1T_1 + \cdots + T_nT_n = 0$. The harmonic shift $\breve{S} = (\breve{S}_1,\ldots,\breve{S}_n)$ on $\breve{\mathcal{G}}$ is the prime example of a harmonic-type operator. Our final main result is a von Neumann inequality.

Theorem 1.4. *Let $(T_1,\ldots,T_n)$ be a harmonic-type row contraction on a Hilbert space. If u is a harmonic polynomial, then $\|u(T_1.\ldots,T_n)\| \leq \|u(\breve{S}_1,\ldots,\breve{S}_n)\|$.*

All terminology is explained in detail in an appropriate section in the paper. After introducing in Section 2 the basic notation, we make a review of harmonic polynomials in Section 3 and give formulas for the Z_m, H_m, and K. In Section 4, we define the shift operators on harmonic spaces, explain the meaning of the coefficient $1/(n+2m)$, and then prove Theorem 1.1. In Section 5, we introduce a new family of reproducing kernel Hilbert spaces of harmonic functions and isolate one of them as $\breve{\mathcal{G}}$ by making it clear why we need the coefficients $1/A_m$ in $\breve{G}$. In Section 6, we find the basic properties of shift operators and their adjoints acting on the Hilbert spaces just introduced. In Section 7, we investigate the row contractions on harmonic Hilbert spaces, explain the term harmonic type, and prove an essential dilation result for harmonic-type and self-adjoint operator tuples. In Section 8, we prove Theorems 1.3 and 1.4.

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2. Notation

Let $\mathbb{B}$ and $\mathbb{S}$ be the open unit ball and its boundary the unit sphere in $\mathbb{R}^n$ with respect to the usual inner product $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ and the norm $|x| = \sqrt{x \cdot x}$, where always $n \geq 2$. We write $x = r\xi$, $y = \rho\eta$ with $r = |x|$, $\rho = |y|$, and $\xi, \eta \in \mathbb{S}$, and use these throughout without further comment. When $n = 2$, the ball is just the unit disc $\mathbb{D}$ in the complex plane bounded by the unit circle $\mathbb{T}$, and $x, y \in \mathbb{D}$ are complex numbers of modulus less than 1.

In a few places, we also use the complex space $\mathbb{C}^N$ and its Hermitian inner product $\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_N \overline{w}_N$. We continue to use $|\cdot|$, $\mathbb{B}$, and $\mathbb{S}$ in $\mathbb{C}^N$ too.

We let σ and ν be the surface and volume measures on $\mathbb{S}$ and $\mathbb{B}$ normalized as $\sigma(\mathbb{S}) = 1$ and $\nu(\mathbb{B}) = 1$. We abbreviate the all-important Lebesgue class $L^2(\sigma)$ to simply L^2. An overline $\overline{(\cdot)}$ denotes closure for sets and complex conjugation for elements; for polynomials in x, the conjugation affects only the coefficients naturally. The greatest integer less than or equal to a real number is shown by $\lfloor \cdot \rfloor$. The right side of $:=$ defines its left side.

Harmonic functions by definition are those sufficiently smooth functions annihilated by the usual *Laplacian* $\Delta := \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. We let $h(\mathbb{B})$ denote the space of complex-valued harmonic functions on $\mathbb{B}$ with the topology of uniform convergence on compact subsets.

In the multi-index notation, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $0^0 = 1$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Letting $\partial_j := \partial/\partial x_j$ and $\partial := (\partial_1, \ldots, \partial_n)$, we also have $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial^\alpha x^\alpha = \alpha!$ and $p(\partial) = \sum_\alpha a_\alpha \partial^\alpha$ for a polynomial $p(x) = \sum_\alpha a_\alpha x^\alpha$. So for $p(x) = |x|^2$,

$$|x|^2(\partial) = \Delta. \tag{1}$$

The *Pochhammer symbol* $(a)_b$ is defined by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. This is a shifted rising factorial since $(a)_k = a(a+1)\cdots(a+k-1)$ for positive integer k. In particular, $(1)_k = k!$ and $(a)_0 = 1$. Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b}, \qquad \frac{(c)_a}{(c)_b} \sim c^{a-b} \qquad (\operatorname{Re} c \to \infty), \tag{2}$$

where $A \sim B$ means that $|A/B|$ is bounded above and below by two positive constants, that is, $A = \mathcal{O}(B)$ and $B = \mathcal{O}(A)$, for all A, B of interest. So for example, $1-|x| \sim 1-|x|^2$ for all $x \in \mathbb{B}$. Such constants that are independent of the parameters and the functions in the equation are all denoted by the generic unadorned upper case C. We also use $A \lesssim B$ to mean $A = \mathcal{O}(B)$.

We denote an inner product on a function space H by $[\,\cdot,\cdot\,]_H$ and the associated norm by $\|\cdot\|_H$.

Definition 2.1. A function $k(x,y)$ is called the *reproducing kernel* of a Hilbert space H of functions defined on $\mathbb{B}$ if $k(x,\cdot) \in H$ for each $x \in \mathbb{B}$ and

$$u(x) = [\,u(\cdot),\, k(x,\cdot)\,]_H \qquad (u \in H,\ x \in \mathbb{B}).$$

There is a one-to-one correspondence between reproducing kernel Hilbert spaces and positive definite kernels. We use words like positive and increasing to mean nonnegative and nondecreasing.

The algebra of all bounded linear operators on a complex Hilbert space H is denoted $\mathcal{B}(H)$. An operator T on H is called *positive* and we write $T \geq 0$ if $[Tv, v]_H \geq 0$ for all $v \in H$. For $a, b \in H$, $a \otimes b$ denotes the rank-1 operator defined by $(a \otimes b)(v) = \big[v, b\big]_H a$ for $v \in H$.

3. Harmonic polynomials

We review the essentials of zonal harmonics and the Kelvin transform for completeness, because we refer to these facts many times in the paper. These results are mostly well-known and can be consulted in [7, Chapters 4 & 5].

For $m = 0, 1, 2, \ldots$, let $\mathcal{P}_m$ denote the complex vector space of all polynomials homogeneous (with respect to real scalars) of degree m on $\mathbb{R}^n$. It is immediate that

$$(x \cdot \partial) p_m = m\, p_m \qquad (p_m \in \mathcal{P}_m). \tag{3}$$

Let $\mathcal{H}_m$ be the subspace of $\mathcal{P}_m$ consisting of all harmonic homogeneous polynomials of degree m. By homogeneity, a $p_m \in \mathcal{P}_m$ is determined by its restriction to $\mathbb{S}$, and we freely identify p_m with its restriction. The restrictions of those $u_m \in \mathcal{H}_m$ to $\mathbb{S}$ are called *spherical harmonics*. We also let $\mathcal{P}$ and $\mathcal{H}$ denote all polynomials and all harmonic polynomials on $\mathbb{R}^n$.

We regard $\mathcal{P}_m$ and $\mathcal{H}_m$ as subspaces of L^2 in issues requiring a norm or inner product. For example, if $m \neq l$, then $\mathcal{H}_m$ is orthogonal to $\mathcal{H}_l$ with respect to $[\,\cdot,\cdot\,]_{L^2}$.

Proposition 3.1. *[7, Exercise 5.12] If $u_m, v_m \in \mathcal{H}_m$, then*

$$\begin{aligned} u_m(\partial)(v_m) &= n(n+2)\cdots(n+2m-2)\int_{\mathbb{S}} u_m\, v_m\, d\sigma \\ &= 2^m (n/2)_m \left[\, u_m,\, \overline{v}_m \,\right]_{L^2}. \end{aligned}$$

Proof. Let $u_m(x) = \sum_{|\alpha|=m} b_\alpha x^\alpha$ and $v_m(x) = \sum_{|\alpha|=m} d_\alpha x^\alpha$. By [7, Theorem 5.14], the right sides are $\sum_{|\alpha|=m} \alpha!\, b_\alpha d_\alpha$. It is easy to check that this is equal to $u_m(\partial)(v_m)$. □

If $p_m \in \mathcal{P}_m$, then there are unique $u_l \in \mathcal{H}_l$ such that

$$p_m(x) = u_m(x) + |x|^2 u_{m-2}(x) + \cdots + |x|^{2k} u_{m-2\lambda}(x) \qquad (x \in \mathbb{R}^n),$$

where $\lambda = \lfloor m/2 \rfloor$. This decomposition is simply $p_m = u_m + u_{m-2} + \cdots + u_{m-2\lambda}$ when restricted to $\mathbb{S}$. Let

$$H_m : \mathcal{P}_m \to \mathcal{H}_m, \qquad p_m \mapsto u_m$$

be the map that *projects* p_m to u_m. The explicit formula for H_m requires some constants c_m, $m = 1, 2, \ldots$, defined by

$$c_m := \begin{cases} (-1)^m 2^m (n/2-1)_m, & \text{if } n \geq 3, \\ (-1)^{m-1} 2^{m-1} (m-1)!, & \text{if } n = 2. \end{cases}$$

It also makes use of the *Kelvin transform* K defined on a function f by

$$K[f](x) := |x|^{2-n} f(x^*) \qquad (x \neq 0), \qquad \text{where} \qquad x^* := \frac{x}{|x|^2},$$

which reduces to $K[f](x) := f(x^*)$ when $n = 2$. Note that $K[f] = f$ on $\mathbb{S}$ for any $n \geq 2$. The Kelvin transform is linear and invertible with $K^{-1} = K$. A function is harmonic if and only if its Kelvin transform is harmonic. So $u(x) = |x|^{2-n}$ is harmonic especially for $n \geq 3$ wherever it is defined since it is the Kelvin transform of the constant 1. For $n = 2$, it is replaced in formulas by the harmonic function $u(x) = \log |x|$.

Theorem 3.2. *[7, Theorem 5.18] Let $m \geq 1$ and $p_m \in \mathcal{P}_m$.*

(a) $$H_m(p_m)(x) = \frac{1}{c_m} \begin{cases} K\big[p_m(\partial)|x|^{2-n}\big], & \textit{if } n \geq 3, \\ K\big[p_m(\partial) \log |x|\big], & \textit{if } n = 2. \end{cases}$$

(b) *When p_m is restricted to $\mathbb{S}$, then H_m (without any need for K) is an orthogonal projection with respect to $[\,\cdot,\cdot\,]_{L^2}$*

The spaces $\mathcal{H}_m$ are finite dimensional, hence closed subspaces of L^2, and

$$\delta_m := \dim \mathcal{H}_m = (n + 2m - 2)\frac{(n-1)_{m-1}}{m!} \qquad (m \geq 1), \tag{4}$$

which gives $\delta_m = 2$ for $m \geq 1$ when $n = 2$. When $m = 0$, $\mathcal{H}_0 = \mathbb{C}$ and $\delta_0 = 1$. The next few are $\delta_1 = n$, $\delta_2 = (n-1)(n+2)/2$, and $\delta_3 = (n-1)n(n+4)/6$. Thus evaluation functionals at points $\eta \in \mathbb{S}$ are bounded on $\mathcal{H}_m$, and $\mathcal{H}_m$ is a reproducing kernel Hilbert space. Its reproducing kernel $Z_m(\xi, \eta)$ with respect to $[\,\cdot,\cdot\,]_{L^2}$ is called the *zonal harmonic* of degree m; so Z_m is a positive definite function. Each Z_m is real valued and symmetric in its variables, hence it is a harmonic homogeneous polynomial in each if its two variables. The homogeneity of the Z_m gives $Z_m(x,y) := r^m \rho^m Z_m(\xi, \eta)$; so

$$Z_m(0, y) = Z_m(x, 0) = 0 \qquad (m \geq 1). \tag{5}$$

Their reproducing property written explicitly is

$$u_m(x) = \int_{\mathbb{S}} u_m(\eta)\, Z_m(x,\eta)\, d\sigma(\eta) = \big[\, u_m(\cdot),\, Z_m(x,\cdot)\,\big]_{L^2} \quad (x \in \mathbb{B},\, u_m \in \mathcal{H}_m). \tag{6}$$

The Poisson kernel is

$$P(x,\eta) := \frac{1-|x|^2}{|x-\eta|^n} = \sum_{m=0}^{\infty} Z_m(x,\eta) \qquad (x \in \mathbb{B},\ \eta \in \mathbb{S});$$

the series converges uniformly for x in a compact subset of $\mathbb{B}$.

There is an explicit formula for the Z_m which is of major interest to us; it is

$$\begin{aligned} Z_m(x,y) &= (n{+}2m{-}2) \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{n(n{+}2)\cdots(n{+}2m{-}2l{-}4)}{(-1)^l\, 2^l\, l!\, (m-2l)!} |x|^{2l} (x{\cdot}y)^{m-2l} |y|^{2l} \\ &= A_{m0}(x\cdot y)^m + A_{m1}|x|^2(x{\cdot}y)^{m-2}|y|^2 + A_{m2}|x|^4(x{\cdot}y)^{m-4}|y|^4 + \cdots, \end{aligned} \tag{7}$$

where $A_m := A_{m0}$ is the leading coefficient obtained for $l = 0$. Then

$$A_m := \frac{n(n+2)\cdots(n+2m-2)}{m!} = \frac{2^m (n/2)_m}{m!}, \tag{8}$$

where the numerator is the coefficient in the equation in Proposition 3.1. The first few are $A_0 = 1$, $A_1 = n$, $A_2 = n(n+2)/2$, and $A_3 = n(n+2)(n+4)/6$. Note that $A_m = 2^m$ for all $m = 0, 1, 2, \ldots$ when $n = 2$.

Interesting and useful relations include

$$|Z_m(\xi,\eta)| \leq Z_m(\xi,\xi) = \sum_{l=0}^{\lfloor m/2 \rfloor} A_{ml} = \delta_m \qquad (\xi,\eta \in \mathbb{S}). \tag{9}$$

If $\{Y_{m1},\dots,Y_{m\delta_m}\}$ is an orthonormal basis for $\mathcal{H}_m \subset L^2$, then

$$Z_m(\xi,\eta) = \sum_{k=1}^{\delta_m} Y_{mk}(\xi)\,\overline{Y_{mk}(\eta)}. \tag{10}$$

In particular, $Z_0 \equiv 1$ and we take $Y_{01} \equiv 1$; also $Z_1(x,y) = n(x \cdot y)$ and we can choose $Y_{1k}(x) = \sqrt{n}\,x_k$ for $k = 1,\dots,\delta_1 = n$. It is always possible to choose the Y_{mk} with real coefficients.

Theorem 3.3. *Every $u \in h(\mathbb{B})$ has the homogeneous expansion $u = \sum_{m=0}^{\infty} u_m$ in which $u_m \in \mathcal{H}_m$ and which converges absolutely and uniformly on compact subsets of $\mathbb{B}$. Letting $u_{mk} = \left[\,u_m,\, Y_{mk}\,\right]_{L^2} \in \mathbb{C}$, we also have*

$$u(x) = \sum_{m=0}^{\infty}\sum_{k=1}^{\delta_m} u_{mk}\,Y_{mk}(x) \qquad (x \in \mathbb{B})$$

with the same type of convergence.

When $n = 2$, we can use complex analysis and Fourier analysis to connect the above theory to better known objects. Then for all $m \geq 1$, an orthonormal basis for $\mathcal{H}_m$ is $\{\,Y_{m1}(x) = x^m,\, Y_{m2}(x) = \overline{x}^m : x \in \mathbb{C}\,\}$. The expansion of a $u \in h(\mathbb{B})$ with the Y_{mk} in Theorem 3.3 with suitable boundary behavior is its Fourier series on the unit circle. Letting $\xi = e^{i\phi}$ and $\eta = e^{i\psi}$, we can write

$$Z_m(x,y) = x^m\overline{y}^m + \overline{x}^m y^m = 2r^m\rho^m\cos(\phi-\psi) \qquad (m \geq 1).$$

It is amazing that this simple form is equal to the sum in (7) for $n = 2$, which is still complicated.

A comparison with the complex case done in [15, Section 3] when n is even is very instructive. We think of $\mathbb{R}^n$ as $\mathbb{C}^N$ by equating $n = 2N$. Spherical harmonics correspond to the space of holomorphic polynomials homogeneous of degree m which simply have the form $\sum_{|\alpha|=m} b_\alpha z^\alpha$. The dimension of this space is $\dfrac{(N)_m}{m!}$, which equals 1 for every m when $N = 1$. The counterparts of zonal harmonics are the sesquiholomorphic kernels

$$M_m(z,w) = \frac{(N)_m}{m!}\,\langle z,w\rangle^m.$$

Thus the complex version of the sum in (7) has only the leading term M_m, and its coefficient is exactly the dimension of the space of which M_m is the reproducing kernel. Note that $A_m \neq \delta_m$ in the harmonic case even when $n = 2$. However, writing (7) with $x = y = \xi$ and using (9), we see that $\sum_{l=0}^{\lfloor m/2 \rfloor} A_{ml} = \delta_m$. By (2), we have

$$\delta_m \sim m^{n-2} \qquad \text{and} \qquad A_m \sim 2^m m^{n/2-1} \qquad (m \to \infty). \tag{11}$$

Yet the reproducing kernel of the holomorphic Drury-Arveson space is $\sum_{m=0}^{\infty} \langle z, w \rangle^m$ and not $\sum_{m=0}^{\infty} M_m(z, w)$.

Therefore we must find the harmonic counterparts of the $\langle z, w \rangle^m$ and we are led to the

$$X_m(x, y) := \frac{1}{A_m} Z_m(x, y) = (x \cdot y)^m - \cdots , \tag{12}$$

which we call the *xonal harmonics*. The first few are $X_0 = 1$, $X_1(x, y) = x \cdot y$,

$$X_2(x, y) = (x \cdot y)^2 - \frac{|x|^2 |y|^2}{n}, \qquad X_3(x, y) = (x \cdot y)^3 - \frac{3}{n+2} |x|^2 (x \cdot y) |y|^2.$$

By homogeneity, (9), (4), and (8), we see that for all $x, y \in \mathbb{B}$,

$$|X_m(x, y)| = r^m \rho^m |X_m(\xi, \eta)| = (r\rho)^m \frac{|Z_m(\xi, \eta)|}{A_m} \leq (r\rho)^m \frac{\delta_m}{A_m} \leq (r\rho)^m < 1 \tag{13}$$

in complete analogy with $|\langle z, w \rangle^m| \leq |z|^m |w|^m < 1$ for z, w in the unit ball of $\mathbb{C}^N$.

4. Shift operators

We define the shift operators on harmonic functions first in an unusual way, but later show that they are equivalent essentially to multiplications by the coordinate variables.

The motivation for our definition lies in the observation

$$\frac{1}{m+1} \frac{\partial}{\partial \overline{w}_j} \big(\langle z, w \rangle^{m+1} \big) = z_j \, \langle z, w \rangle^m$$

for $z, w \in \mathbb{C}^N$ and the realization that $X_m(x, y)$ replaces $\langle z, w \rangle^m$.

Definition 4.1. For $1 \leq j \leq n$, we define the jth *shift operator* $S_j : \mathcal{H}_m \to \mathcal{H}_{m+1}$ acting on the variable x by first letting

$$S_j X_m(x, y) := \frac{1}{m+1} \frac{\partial}{\partial y_j} X_{m+1}(x, y)$$

and then extending to all of $\mathcal{H}_m$ by linearity and the density of the $X_m(\cdot, y)$ in $\mathcal{H}_m$.

Note that all this make sense; a partial derivative of a harmonic function is again harmonic, and finite linear combinations of the reproducing kernels Z_m and hence of the X_m are dense in $\mathcal{H}_m$ in $\|\cdot\|_{L^2}$. Also note that the shift acts on the first variable x, but the partial derivative is with respect to the second variable y. So occasionally we also use notation like S^x or ∂_y to indicate the variables on which they act. In terms of the more familiar zonal harmonics,

$$S_j^x Z_m(x,y) = \frac{A_m}{A_{m+1}} \frac{1}{m+1} \frac{\partial}{\partial y_j} Z_{m+1}(x,y) = \frac{1}{n+2m} \frac{\partial}{\partial y_j} Z_{m+1}(x,y).$$

Let's denote by $S_j^* : \mathcal{H}_m \to \mathcal{H}_{m-1}$ the adjoint of S_j with respect to $[\cdot,\cdot]_{L^2}$ in which the reproducing kernel of $\mathcal{H}_m$ is $Z_m = A_m X_m$. This is of course the jth *backward shift* operator. First we set $S_j^*(X_0) = S_j^*(1) = 0$. Next for $m \geq 1$, using (6), symmetry of X_m in its variables, and its real-valuedness, we obtain

$$\begin{aligned}
(S_j^x)^* X_m(x,y) &= \big[(S_j^t)^* X_m(t,y),\, A_{m-1} X_{m-1}(x,t)\big]_{L^2} \\
&= A_{m-1} \big[X_m(t,y),\, S_j^t X_{m-1}(x,t)\big]_{L^2} \\
&= A_{m-1} \left[X_m(t,y),\, \frac{1}{m} \frac{\partial}{\partial x_j} X_m(x,t)\right]_{L^2} \\
&= \frac{1}{m} \frac{A_{m-1}}{A_m} \left[A_m X_m(t,y),\, \frac{\partial}{\partial x_j} X_m(x,t)\right]_{L^2} \\
&= \frac{1}{m} \frac{A_{m-1}}{A_m} \frac{\partial}{\partial x_j} X_m(x,y),
\end{aligned}$$

where y acts just like a parameter. This last formula for S_j^* is independent of the particular form of the function on which it acts, so works equally well for $u_m \in \mathcal{H}_m$ in place of X_m by linearity and density again. Thus

$$S_j^* u_m = \frac{1}{m} \frac{A_{m-1}}{A_m} \partial_j u_m = \frac{1}{n+2m-2} \partial_j u_m \qquad (u_m \in \mathcal{H}_m,\ m \geq 1). \tag{14}$$

It is clear that the S_j^* commute with each other. So the shift $S = (S_1, \ldots, S_n)$ is a commuting tuple.

Shift operators on holomorphic function spaces are operators of multiplication by the coordinate variables. Here we make a distinction between the two, because the latter does not in general carry harmonic functions to harmonic functions unlike the former. If f, g are functions on the same domain, we let

$$M_g f = gf$$

be the *operator of multiplication* by g.

One of our results concerns a limited version of the obvious fact that if S is the tuple of shift operators on a space of holomorphic functions on a domain in $\mathbb{C}^N$ and p is a holomorphic polynomial in N complex variables, then $p(S) = M_p$. It is a first result on the connection between shifts and multiplication operators.

Proposition 4.2. *If u is a harmonic polynomial, then $u(S)(1) = u$. In other words, 1 is a cyclic vector for $h(\mathbb{B})$.*

Proof. It suffices to consider $u = u_m \in \mathcal{H}_m$. By (6), repeated use of (14), and Proposition 3.1,

$$\begin{aligned} u_m(S^x)(1)(x) &= \big[u_m(S^\eta)(Z_0)(\eta, y),\, Z_m(x, \eta)\big]_{L^2} \\ &= \big[Z_0(\eta, y),\, \overline{u_m}((S^\eta)^*)(Z_m)(x, \eta)\big]_{L^2} \\ &= \left[Z_0(\eta, y),\, \frac{1}{n(n+2)\cdots(n+2m-2)}\, \overline{u_m}(\partial_\eta)(Z_m)(x, \eta)\right]_{L^2} \\ &= \frac{1}{2^m (n/2)_m} \big[u_m(\partial_\eta)(Z_m)(x, \eta),\, Z_0(y, \eta)\big]_{L^2} \\ &= \frac{1}{2^m (n/2)_m}\, u_m(\partial_y)(Z_m)(x, y) \\ &= \int_{\mathbb{S}} u_m(\cdot)\, Z_m(x, \cdot)\, d\sigma(\cdot) = u_m(x). \end{aligned}$$

Above, S^η and ∂_η must be interpreted as $S^y\big|_{y=\eta}$ and $\partial_y\big|_{y=\eta}$, respectively. Note that we use the harmonicity of u_m only in passing to the last line. $\square$

If the function acted on is more complicated than 1, then we offer the following partial result.

Proposition 4.3. *If $p_m \in \mathcal{P}_m$, then*

$$p_m(S^x)(X_\ell)(x, y) = \frac{1}{(1+\ell)_m}\, p_m(\partial_y)(X_{m+\ell})(x, y).$$

Proof. Following the same idea and notation in the proof of Proposition 4.2,

$$\begin{aligned} p_m(S^x)(Z_\ell)(x, y) &= \big[p_m(S^\eta)(Z_\ell)(\eta, y),\, Z_{m+\ell}(x, \eta)\big]_{L^2} \\ &= \big[Z_\ell(\eta, y),\, \overline{p_m}((S^\eta)^*)(Z_{m+\ell})(x, \eta)\big]_{L^2} \\ &= \left[Z_\ell(\eta, y),\, \frac{\overline{p_m}(\partial_\eta)(Z_{m+\ell})(x, \eta)}{(n+2\ell)(n+2\ell+2)\cdots(n+2\ell+2m-2)}\right]_{L^2} \\ &= \frac{1}{2^m (n/2+\ell)_m} \big[p_m(\partial_\eta)(Z_{m+\ell}(x, \eta),\, Z_\ell(y, \eta)\big]_{L^2} \end{aligned}$$

$$= \frac{1}{2^m (n/2+\ell)_m} p_m(\partial_y)(Z_{m+\ell})(x,y).$$

Lastly, we pass to the xonal harmonics $X_\ell = Z_\ell / A_\ell$ and simplify the resulting coefficient. □

Our goal now is to express the shift operators $S_j : \mathcal{H}_m \to \mathcal{H}_{m+1}$ in terms of the operators of *multiplication by the coordinate variables* $M_{x_j} : \mathcal{H}_m \to \mathcal{P}_{m+1}$, where for both $j = 1, \dots, n$. It is Theorem 1.1 and we restate it.

Theorem 4.4. *For all $j = 1, \dots, n$ and $m = 0, 1, 2 \dots$, if $S_j : \mathcal{H}_m \to \mathcal{H}_{m+1}$, then $S_j = H_{m+1} M_{x_j}$.*

So the harmonic shifts are really Toeplitz operators. We facilitate the long proof of this theorem with some computational lemmas in which $j = 1, \dots, n$, y is a parameter, and all partial derivatives and Kelvin transforms are with respect to x.

Lemma 4.5. *For $a, b \in \mathbb{R}$, easy computations give*

$$\partial_j |x|^a = a|x|^{a-2} x_j \qquad \text{and} \qquad \partial_j (x \cdot y)^b = b(x \cdot y)^{b-1} y_j,$$
$$(y \cdot \partial)|x|^a = a|x|^{a-2}(x \cdot y) \qquad \text{and} \qquad (y \cdot \partial)(x \cdot y)^b = b(x \cdot y)^{b-1}|y|^2.$$

Lemma 4.6. *If a polynomial p has $|x|^2$ as a factor, then $p(\partial)|x|^{2-n} = 0$ and also $p(\partial) \log |x| = 0$ when $n = 2$. Consequently*

$$X_m(\partial, y)|x|^{2-n} = (y \cdot \partial)^m |x|^{2-n} \qquad \text{and}$$
$$X_m(\partial, y) \log |x| = (y \cdot \partial)^m \log |x| \qquad (n = 2).$$

Proof. The first statement follows immediately from (1) and the harmonicity of $|x|^{2-n}$ and of $\log |x|$ when $n = 2$. The second statement follows from the explicit forms of zonal harmonics in (7), because all the terms in $X_m(x, y)$ except the first have a factor of $|x|^2$. □

This lemma is very useful, because it lets us treat X_m as the single term $(x \cdot y)^m$ in the presence of K or H_m like its holomorphic counterpart $\langle z, w \rangle^m$.

Lemma 4.7. *For $m \geq 1$,*

$$(y \cdot \partial)^m |x|^{2-n} = c_m \frac{X_m(x,y)}{|x|^{n+2m-2}} \qquad (n \geq 3),$$
$$(y \cdot \partial)^m \log |x| = c_m \frac{X_m(x,y)}{|x|^{2m}} \qquad (n = 2);$$

and hence

$$K\big[(y\cdot\partial)^m|x|^{2-n}\big] = c_m X_m(x,y) \qquad (n\ge 3),$$
$$K\big[(y\cdot\partial)^m\log|x|\big] = c_m X_m(x,y) \qquad (n=2).$$

The identities for $m\ge 3$ are true also for $m=0$ if we set $c_0=1$.

From Lemma 4.7, Theorem 1.2 follows easily which we restate.

Theorem 4.8. *For $\xi,\eta\in\mathbb{S}$, we have*

$$K\big[(\eta\cdot\partial)^m K[1]\big](\xi) = (-1)^m\, m!\, C_m^{n/2-1}(\xi\cdot\eta) \qquad (n\ge 3,\ m=0,1,2,\dots),$$
$$K\big[(\eta\cdot\partial)^m K[\log|\cdot|\,]\big](\xi) = (-1)^m (m-1)!\, T_m(\xi\cdot\eta) \qquad (n=2,\ m=1,2,\dots),$$

where C_m^λ is the Gegenbauer (ultraspherical) polynomial of degree m and index λ, and T_m is the Chebyshev polynomial of the first kind of degree m.

Proof. Let's first note that $K[1]=|x|^{2-n}$ for $n=3$ and $K[\log|x|\,]=-\log|x|$ for $n=2$. The first identity is a consequence of the well-known relation between the zonal harmonics and Gegenbauer polynomials; see [12, (14.8)] for example. The second identity holds because of the close connection between Gegenbauer polynomials with parameter 0 and Chebyshev polynomials; see [18, 18.1.1] for example. □

The identities in Lemma 4.7 and Corollary 4.8 seem new; we are unable to locate them in standard references such as [18]. They also give further indication that the xonal harmonics X_m are important in their own right since the coefficients in the identities in Lemma 4.7 with the Z_m are not simple known ones.

Proof of Lemma 4.7. The second set of identities follows immediately from the first set, and for these we proceed by induction on m. We give the proof only for $n\ge 3$; the proof for $n=2$ is obtained by replacing $|x|^{2-n}$ by $\log|x|$ and setting $n=2$ in appropriate places. For $m=1$, by Lemma 4.5,

$$(y\cdot\partial)|x|^{2-n} = (2-n)|x|^{-n}(x\cdot y) = c_1\frac{X_1(x,y)}{|x|^n}.$$

Next we assume the first identity in Lemma 4.7 holds for m, and show that it also holds for $m+1$. By applying the induction hypothesis, differentiating with the quotient rule, and using Lemma 4.5, we obtain

$$\begin{aligned}(y\cdot\partial)^{m+1}|x|^{2-n} &= (y\cdot\partial)\left(c_m\frac{X_m(x,y)}{|x|^{n+2m-2}}\right)\\ &= c_m\frac{|x|^{n+2m-2}(y\cdot\partial)X_m - X_m(n+2m-2)|x|^{n+2m-4}(x\cdot y)}{|x|^{2n+4m-4}}\end{aligned}$$

$$= c_m \frac{|x|^2 (y\cdot\partial) X_m - (n+2m-2)(x\cdot y) X_m}{|x|^{n+2m}}.$$

Writing out the coefficients, we must show

$$(x\cdot y) X_m - \frac{|x|^2}{n+2m-2} (y\cdot\partial) X_m = X_{m+1}.$$

By (7) and Lemma 4.5, the left side equals

$$\begin{aligned}
&\frac{n+2m-2}{A_m} \sum_{l=0}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-4)}{(-1)^l\, 2^l\, l!\,(m-2l)!} |x|^{2l} (x\cdot y)^{m-2l+1} |y|^{2l} \\
&\quad - \frac{1}{A_m} \sum_{l=0}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-4)}{(-1)^l\, 2^l\, l!\,(m-2l)!}\, 2l\, |x|^{2l} (x\cdot y)^{m-2l+1} |y|^{2l} \\
&\quad - \frac{1}{A_m} \sum_{l=0}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-4)}{(-1)^l\, 2^l\, l!\,(m-2l)!} (m-2l) |x|^{2l+2} (x\cdot y)^{m-2l-1} |y|^{2l+2} \\
&= \frac{1}{A_m} \sum_{l=0}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-2)}{(-1)^l\, 2^l\, l!\,(m-2l)!} |x|^{2l} (x\cdot y)^{m-2l+1} |y|^{2l} \\
&\quad + \frac{1}{A_m} \sum_{l=1}^{\lfloor m/2\rfloor+1} \frac{n(n+2)\cdots(n+2m-2l-4)}{(-1)^l\, 2^{l-1}\, (l-1)!\,(m-2l+1)!} |x|^{2l} (x\cdot y)^{m-2l+1} |y|^{2l} \\
&= \frac{m+1}{A_m} \sum_{l=1}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-2)}{(-1)^l\, 2^l\, l!\,(m+1-2l)!} |x|^{2l} (x\cdot y)^{m+1-2l} |y|^{2l} + \text{extra term} \\
&\quad + (x\cdot y)^{m+1},
\end{aligned}$$

where the extra term is due to $l = \lfloor m/2\rfloor + 1$ in the second sum on the previous line. The right side equals

$$\frac{n+2m}{A_{m+1}} \sum_{l=1}^{\lfloor (m+1)/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-2)}{(-1)^l\, 2^l\, l!\,(m+1-2l)!} |x|^{2l} (x\cdot y)^{m+1-2l} |y|^{2l} + (x\cdot y)^{m+1}.$$

Since $(m+1)/A_m = (n+2m)/A_{m+1}$, the two sides are equal except for the extra term and that the sum on the right side ends perhaps at a larger value. When m is even, the l giving rise to the extra term equals $m/2+1$, and then $m-2l+1 = -1 < 0$ in that term, so there is really no extra term. Also $\lfloor (m+1)/2\rfloor = m/2 = \lfloor m/2\rfloor$, and the sums on both sides end at the same value. When m is odd, the l giving rise to the extra term equals $(m+1)/2$. Also $\lfloor (m+1)/2\rfloor = (m+1)/2$. So the extra term and the last term in the sum on the right are the same. $\square$

Proof of Theorem 4.4. Again we write the proof only for $n \geq 3$. It suffices to do the proof only for $u_m(\cdot) = X_m(\cdot, y)$ as in Definition 4.1. In view of Lemma 4.6 and by Theorem 3.2, all we need to show is

$$\frac{1}{m+1}\,\frac{\partial}{\partial y_j} X_{m+1}(x,y) = \frac{1}{c_{m+1}}\, K\left[\frac{\partial}{\partial x_j}(y\cdot\partial)^m |x|^{2-n}\right]$$

for all $m = 0, 1, 2, \ldots$, where K and ∂ are with respect to x. Applying Lemma 4.7 on the right and combining the constants, this equation takes the form

$$\frac{\partial}{\partial y_j} X_{m+1}(x,y) = -\frac{m+1}{n+2m-2}\, K\left[\frac{\partial}{\partial x_j}\left(\frac{X_m(x,y)}{|x|^{n+2m-2}}\right)\right] \tag{15}$$

By (7) and Lemma 4.5, the left side equals

$$\begin{aligned}
&\frac{n+2m}{A_{m+1}}\Bigg(\sum_{l=0}^{\lfloor (m+1)/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-2)}{(-1)^l\, 2^l\, l!\,(m-2l)!}\, x_j |x|^{2l}(x\cdot y)^{m-2l}|y|^{2l} \\
&\quad + \sum_{l=1}^{\lfloor (m+1)/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-2)}{(-1)^l\, 2^{l-1}\,(l-1)!\,(m+1-2l)!}\, y_j |x|^{2l}(x\cdot y)^{m+1-2l}|y|^{2l-2}\Bigg).
\end{aligned}$$

By Lemma 4.5,

$$\begin{aligned}
\frac{\partial}{\partial x_j}\left(\frac{X_m(x,y)}{|x|^{n+2m-2}}\right) &= \frac{|x|^{n+2m-2}\partial_j X_m - X_m(n+2m-2)|x|^{n+2m-4}x_j}{|x|^{2n+4m-4}} \\
&= \frac{|x|^2\partial_j X_m - (n+2m-2)x_j X_m}{|x|^{n+2m}}.
\end{aligned}$$

After taking the Kelvin transform, the right side equals

$$\begin{aligned}
&(m+1)\left(x_j X_m - \frac{1}{n+2m-2}\,|x|^2\frac{\partial X_m}{\partial x_j}\right) \\
&= \frac{m+1}{A_m}\Bigg((n+2m-2)\sum_{l=0}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-4)}{(-1)^l\, 2^l\, l!\,(m-2l)!}\, x_j |x|^{2l}(x\cdot y)^{m-2l}|y|^{2l} \\
&\quad - \sum_{l=0}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-4)}{(-1)^l\, 2^l\, l!\,(m-2l)!}\, 2l\, x_j |x|^{2l}(x\cdot y)^{m-2l}|y|^{2l} \\
&\quad - \sum_{l=0}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-4)}{(-1)^l\, 2^l\, l!\,(m-2l-1)!}\, y_j |x|^{2l+2}(x\cdot y)^{m-2l-1}|y|^{2l}\Bigg) \\
&= \frac{m+1}{A_m}\Bigg(\sum_{l=0}^{\lfloor m/2\rfloor} \frac{n(n+2)\cdots(n+2m-2l-2)}{(-1)^l\, 2^l\, l!\,(m-2l)!}\, x_j |x|^{2l}(x\cdot y)^{m-2l}|y|^{2l}
\end{aligned}$$

$$+ \sum_{l=1}^{\lfloor m/2\rfloor+1} \frac{n(n+2)\cdots(n+2m-2l-2)}{(-1)^l\, 2^{l-1}\,(l-1)!\,(m+1-2l)!}\, y_j |x|^{2l} (x\cdot y)^{m+1-2l} |y|^{2l-2}\Bigg).$$

Since the coefficients multiplying the sums are equal, the two sides are equal except perhaps in the upper limits of the sums. Let m be even. The upper limit on the left is $l = \lfloor (m+1)/2\rfloor = m/2 = \lfloor m/2\rfloor$. In the sum with x_j, this is also the upper limit on the right. The sum with y_j on the right ends with $l = m/2+1$, but then $m+1-2l = -1 < 0$, so this term is not really there. Next let m be odd. The upper limit on the left is $l = \lfloor (m+1)/2\rfloor = (m+1)/2 = \lfloor m/2\rfloor + 1$. In the sum with y_j, this is also the upper limit on the right. The sum with x_j on the right ends with $l = (m-1)/2$, so it seems as if the term on the left with $l = (m+1)/2$ is extra, but then $m - 2l = -1 < 0$, so this term is not really there, either. □

Example 4.9. Let's compute the action of shifts on a very simple harmonic function. Let's find $S_1 u$ and $S_2 u$ for $u(x) = x_1$. We apply Theorem 4.4 and follow the recipe in Theorem 3.2 separately for $n \geq 3$ and $n = 2$. Straightforward computations yield that $S_1 x_1 = x_1^2 - |x|^2/n$ for any $n \geq 2$. On the other hand, simply $S_2 x_1 = x_2 x_1$ since this product is already harmonic.

5. Harmonic Hilbert function spaces

We are inspired by a few earlier works in defining new reproducing kernels with desired properties. In [16], families of weighted symmetric Fock spaces of holomorphic functions that include the Drury-Arveson space are studied following [6]. In [12], Bergman-Besov kernels are defined as weighted infinite sums of zonal harmonics much like the Poisson kernel. And we have already noted that the right tool is the xonal harmonics rather than the zonal harmonics.

Definition 5.1. Let $\beta := \{\, \beta_m > 0 : \beta_0 = 1,\ m = 0, 1, 2, \dots \}$ be a sequence satisfying

$$\limsup_{m\to\infty} \left(\frac{\delta_m}{A_m} \beta_m \right)^{1/m} \leq 1. \tag{16}$$

We define positive definite kernels by

$$G_\beta(x, y) := \sum_{m=0}^{\infty} \beta_m\, X_m(x, y) \qquad (x, y \in \mathbb{B})$$

and spaces $\mathcal{G}_\beta$ as the reproducing kernel Hilbert spaces generated by these kernels.

We can also write

$$G_\beta(x, y) = \sum_{m=0}^{\infty} \sum_{k=1}^{\delta_m} \frac{\beta_m}{A_m}\, Y_{mk}(x)\, \overline{Y_{mk}(y)} \qquad (x, y \in \mathbb{B})$$

by (10). Nothing about the boundedness, summability, or monotonicity of β is assumed at this point. But the condition (16), via (13), ensures that the series defining the kernels G_β converge absolutely and uniformly on compact subsets of $\mathbb{B} \times \mathbb{B}$ and hence define harmonic functions of $x, y \in \mathbb{B}$. For any β,

$$G_\beta(0, y) = G_\beta(x, 0) = 1 \tag{17}$$

by (5) since $\beta_0 = 1$, $G_\beta(x, y) = G_\beta(y, x)$, and G_β is real-valued. The G_β depend on x and y via $x \cdot y$ since the Z_m are constant multiples of Gegenbauer polynomials of $x \cdot y$; see [12, (14.8)]

Theorem 5.2. *The elements of $\mathcal{G}_\beta$ are harmonic functions on $\mathbb{B}$.*

Proof. This is by [5, p. 43]; the result there is stated for sesquiholomorphic kernels, but works equally well for harmonic kernels since both function classes have the same topology, the topology of uniform convergence on compact subsets. The hypotheses there are satisfied, because each $G_\beta(x, y)$ is locally bounded by (13) and a harmonic function in each variable on $\mathbb{B}$. $\square$

Also every $G_\beta - \beta_m X_m = \sum_{l \neq m}^{\infty} \beta_l X_l$ is positive definite. Then by [5, Theorem II.1.2], every $\mathcal{H}_m$ is continuously imbedded in each $\mathcal{G}_\beta$.

Theorem 5.3. *The space $\mathcal{G}_\beta$ coincides with the space of harmonic functions u on $\mathbb{B}$ with expansions as in Theorem 3.3 for which*

$$\|u\|^2_{\mathcal{G}_\beta} := \sum_{m=0}^{\infty} \|u_m\|^2_{\mathcal{G}_\beta} := \sum_{m=0}^{\infty} \frac{A_m}{\beta_m} \|u_m\|^2_{L^2} < \infty \tag{18}$$

equipped with the inner product

$$[u, v]_{\mathcal{G}_\beta} := \sum_{m=0}^{\infty} [u_m, v_m]_{\mathcal{G}_\beta} := \sum_{m=0}^{\infty} \frac{A_m}{\beta_m} [u_m, v_m]_{L^2}. \tag{19}$$

Moreover,

$$\left\{ W^\beta_{mk} := \sqrt{\frac{\beta_m}{A_m}}\, Y_{mk} : k = 1, \ldots, \delta_m,\ m = 0, 1, 2, \ldots \right\}$$

is an orthonormal basis for $\mathcal{G}_\beta$.

Proof. We adapt the proof of [5, Theorem III.3.1] to our situation mainly to show how the L^2 norm comes in. Consider the space of $u \in h(\mathbb{B})$ satisfying the finiteness condition in the statement of the theorem. Then this space has the inner product in the statement of the theorem. Since

$$\begin{aligned}|u(x)| &= \left|\sum_{m=0}^{\infty}\sum_{k=1}^{\delta_m}\sqrt{\frac{A_m}{\beta_m}}\,u_{mk}\,\sqrt{\frac{\beta_m}{A_m}}\,Y_{mk}(x)\right| \\ &\leq \left(\sum_{m=0}^{\infty}\sum_{k=1}^{\delta_m}\frac{A_m}{\beta_m}\,|u_{mk}|^2\right)^{1/2}\left(\sum_{m=0}^{\infty}\sum_{k=1}^{\delta_m}\frac{\beta_m}{A_m}\,|Y_{mk}(x)|^2\right)^{1/2} \\ &= \left(\sum_{m=0}^{\infty}\frac{A_m}{\beta_m}\,\|u_m\|_{L^2}^2\right)^{1/2}\sqrt{G_\beta(x,x)} < \infty,\end{aligned}$$

point evaluations are bounded on this space, and norm convergence in the space implies convergence on compact subsets by (13). It follows that this space is complete, and it remains to find its reproducing kernel. It is clear that the set of normalized Y_{mk} in the statement of the theorem form an orthonormal basis for this space and if we add them up as in (10), we obtain G_β. □

Thus the spaces $\mathcal{G}_\beta$ are Hilbert harmonic function spaces. The series expansions of a $u \in \mathcal{G}_\beta$ given in Theorem 3.3, which can be recast as

$$u = \sum_{m=0}^{\infty}\sum_{k=1}^{\delta_m}\widehat{u}_{mk}\,W_{mk}^{\beta} \tag{20}$$

for some $\widehat{u}_{mk} \in \mathbb{C}$, converge both in $\|\cdot\|_{\mathcal{G}_\beta}$ and uniformly on compact subsets of $\mathbb{B}$,

Corollary 5.4. *Finite linear combinations of the collection*

$$\{\, X_m(\cdot, y) : y \in \mathbb{B},\ m = 0, 1, 2, \dots \}$$

and hence of harmonic polynomials are dense in norm in every $\mathcal{G}_\beta$.

Inner products of $\mathcal{G}_\beta$ and L^2 restricted to $\mathcal{H}_m$ are constant multiples of each other. Consequently, orthogonality in L^2 and in $\mathcal{G}_q$ are equivalent. The reproducing kernel of $\mathcal{H}_m$ with respect to $[\cdot,\cdot]_{\mathcal{G}_\beta}$ is $\beta_m X_m$. Hence

$$u_m(x) = \big[\, u_m(\cdot),\ \beta_m\, X_m(x,\cdot)\,\big]_{\mathcal{G}_\beta} \qquad (x \in \mathbb{B},\ u_m \in \mathcal{H}_m) \tag{21}$$

and

$$\|Z_m(\cdot, y)\|_{\mathcal{G}_\beta}^2 = \frac{A_m}{\beta_m}\, Z_m(y,y) \qquad (y \in \mathbb{B})$$

by (19).

In fact, (19) defines an inner product also on $\mathcal{P}_m$ for each β. With the help of Theorem 3.2 (b), these imply that the projection H_m is orthogonal with respect to each $[\,\cdot,\cdot\,]_{\mathcal{G}_\beta}$. Hence $\|H_m\| = 1$ when the same $\|\cdot\|_{\mathcal{G}_\beta}$ is used in both its domain and range.

We now discuss two particular families of harmonic kernels and associated Hilbert function spaces that have applications to function theory and operator theory.

Example 5.5. A general family of Banach spaces of harmonic functions are studied in [12] from the point of view of Bergman projections and function-theoretic properties; they are the Bergman-Besov spaces b^p_q with $q \in \mathbb{R}$ and $p \geq 1$. The Hilbert spaces in this family include the weighted Bergman spaces b^2_q with $q > -1$, the Dirichlet space b^2_{-n}, and the Hardy space $h^2 = b^2_{-1}$. These Hilbert spaces are all special cases of the $\mathcal{G}_\beta$ and their reproducing kernels $R_q(x,y) = \sum\limits_{m=0}^{\infty} \gamma_m(q)\, Z_m(x,y)$ are given by the coefficient sequence

$$\gamma_m(q) = \begin{cases} \dfrac{(1+n/2+q)_m}{(n/2)_m}, & \text{if } \ q > -(1+n/2), \\[2ex] \dfrac{(m!)^2}{(1-(n/2+q))_m (n/2)_m}, & \text{if } \ q \leq -(1+n/2). \end{cases}$$

Naturally $R_{-1} = P$, the Poisson kernel, which is the reproducing kernel of h^2. By (2), $\gamma_m(q) \sim m^{1+q}$ as $m \to \infty$ for any q. We rewrite R_q in terms of the X_m and notice that what replaces β_m here is $\widetilde{\gamma}_m(q) = A_m \gamma_m(q)$. If we calculate the $\limsup$ in (16) using (11), we see that it equals 1 for any q. All the spaces b^p_q have norms that are integrals with respect to weighted versions of ν on $\mathbb{B}$ of the pth power of either the functions or high enough derivatives of them; see [12, Theorem 1.1] and also [7, Theorem 6.13] for $q = -1$.

In this work we are interested in another family.

Definition 5.6. For $q \in \mathbb{R}$ and $m = 0, 1, 2, \ldots$, we set

$$\beta_m(q) := \begin{cases} \dfrac{(1+n/2+q)_m}{m!}, & \text{if } \ q > -(1+n/2), \\[2ex] \dfrac{m!}{(1-(n/2+q))_m}, & \text{if } \ q \leq -(1+n/2), \end{cases}$$

and denote by $G_q(x,y) := \sum\limits_{m=0}^{\infty} \beta_m(q)\, X_m(x,y)$ the reproducing kernel with coefficient sequence $\{\beta_m(q)\}$ and by $\mathcal{G}_q$ the Hilbert space generated by this kernel G_q.

Note that the $\beta_m(q)$ are the same as the coefficients of the holomorphic Bergman-Besov kernels in [16, Definition 4.1] under the identification $n = 2N$. By (2),

$$\beta_m(q) \sim m^{n/2+q} \qquad (m \to \infty) \tag{22}$$

for any q. Then calculating the lim sup in (16) using (11), we see that it equals 1/2 for any q. This has the added consequence that the series defining $G_q(x,y)$ for any q converges absolutely and uniformly on compact subsets of $2\mathbb{B}\times\mathbb{B}$ and is harmonic there. Then it follows from the proof of Theorem 5.3 or the density of the kernel functions in a reproducing kernel Hilbert space that every function $u\in\mathcal{G}_q$ is harmonic on $2\mathbb{B}$ and hence is bounded on $\overline{\mathbb{B}}$. Further, if $q<-n$, then the functions in $\mathcal{G}_q$ are bounded on $2\mathbb{B}$.

The growth rate of the $\beta_m(q)$ in (22) and the norms of the $\mathcal{G}_q$ given in (18) show that if $q_1<q_2$, then $\mathcal{G}_{q_1}\subset\mathcal{G}_{q_2}$ continuously.

By Definition 5.6 and (12), for $q>-(1+n/2)$ and $x,y\in\mathbb{B}$,

$$G_q(x,y)=\sum_{m=0}^{\infty}\frac{(1+n/2+q)_m}{m!}\big((x\cdot y)^m-\cdots\big)=\frac{1}{(1-x\cdot y)^{1+n/2+q}}-\cdots,$$

which is more reminiscent of the way some holomorphic Bergman-Besov kernels are written. Although $G_q(x,y)$ is defined for $x,y\in 2\mathbb{B}$, the last formula above is valid for $x,y\in\mathbb{B}$ only.

Definition 5.7. We define $\breve{G}:=G_{-n/2}$ for which all $\breve{\beta}_m:=\beta_m(-n/2)=1$. So

$$\breve{G}(x,y):=\sum_{m=0}^{\infty}\big((x\cdot y)^m-\cdots\big)=\frac{1}{1-x\cdot y}-\cdots\qquad(x,y\in\mathbb{B}).$$

We denote by $\breve{\mathcal{G}}$ the reproducing kernel Hilbert space generated by $\breve{G}$. We use $\breve{}$ to indicate any object connected with this kernel and space.

The space $\breve{\mathcal{G}}$ is our candidate for the harmonic version of the Drury-Arveson space. Our results in Sections 7 and 8 reveal some important extremal properties of it. For example, simply X_m is the reproducing kernel of $\mathcal{H}_m$ with respect to $[\cdot,\cdot]_{\breve{\mathcal{G}}}$. On the other hand, since the kernels and the spaces in Example 5.5 and Definition 5.6 are totally different, the harmonic Hardy space and the Poisson kernel are unrelated to $\breve{\mathcal{G}}$, even when $n=2$.

There is also the question whether the spaces $\mathcal{G}_q$ can have equivalent norms in the form of integrals of the functions or their derivatives with respect to some measures on $\overline{\mathbb{B}}$ like those of the b^p_q, and we answer it in the negative now following the technique used in [17, Corollary 7.2].

For $t\in\mathbb{R}$, let's define a tth order *radial derivative* $\mathcal{R}^t$ acting on the homogeneous expansion of a $u\in h(\mathbb{B})$ by

$$\mathcal{R}^t u:=\sum_{m=0}^{\infty}(m+1)^t\,u_m.$$

First, if $u\in\mathcal{G}_q$, then $\mathcal{R}^t u\in\mathcal{G}_{q+2t}$ by comparing the coefficients in (18), hence for any t, $\mathcal{R}^t u\in h(2\mathbb{B})$, hence $\mathcal{R}^t u$ is bounded on $\overline{\mathbb{B}}$ for any t. Second, there are harmonic functions

u on $\mathbb{B}$ for which all $\mathcal{R}^t u$ are bounded on $\mathbb{B}$ and that do not lie in any $\mathcal{G}_q$; we can just take an element of a new $\mathcal{G}_{\widetilde{\beta}}$ produced by a sequence $\{\widetilde{\beta}_m\}$ for which the lim sup in (16) is, say, 2/3 instead of the 1/2 obtained for all $\{\beta_m(q)\}$.

Proposition 5.8. *Let $\kappa : [0, \infty) \to \mathbb{R}$ be an increasing function with $\kappa(0) = 0$, and let μ be a positive measure with support in $\overline{\mathbb{B}}$. Define $\mathcal{E}_{t\kappa\mu}$ as the set of all $u \in h(\mathbb{B})$ for which*

$$\limsup_{r\to 1^-} \int_{\overline{\mathbb{B}}} \kappa\big(|\mathcal{R}^t u(rx)|\big)\, d\mu(x) < \infty.$$

Then $\mathcal{E}_{t\kappa\mu} \neq \mathcal{G}_q$ for any values of the parameters.

Proof. By the first point above, it suffices to take $t = 0$. Let q be given and suppose that $\mathcal{G}_q = \mathcal{E}_{0\kappa\mu}$ for some κ, μ. Applying the definition of $\mathcal{E}_{0\kappa\mu}$ with $u = 1 \in \mathcal{G}_q$ gives $\kappa(1)\mu(\overline{\mathbb{B}}) < \infty$; hence μ is a finite measure. Let's denote the sup norm on $\overline{\mathbb{B}}$ by $\|\cdot\|_\infty$. If u is a harmonic function on $\mathbb{B}$ with $\|u\|_\infty < \infty$, then the integral in the definition of $\mathcal{E}_{0\kappa\mu}$ is dominated by $\kappa\big(\|u\|_\infty\big)\mu(\overline{\mathbb{B}}) < \infty$. This shows $u \in \mathcal{G}_q = \mathcal{E}_{0\kappa\mu}$. But not all such u belongs to $\mathcal{G}_q$ by the second point above. $\square$

The same proof shows that it is impossible to find a measure with support on a compact subset K of $2\mathbb{B}$ for the $\mathcal{G}_q$. The same result applies to any $\mathcal{G}_\beta$ for which the lim sup in (16) is strictly less than 1.

6. Adjoints

Having families of harmonic Hilbert function spaces at hand, we now investigate the action of shift operators on them and of their adjoints. When we extend the S_j to all of $\mathcal{G}_\beta$ and $\mathcal{G}_q$ by linearity and density, we name them S_{β_j} and S_{q_j}.

We start by obtaining the adjoints, the backward shifts, $S^*_{\beta_j} : \mathcal{H}_m \to \mathcal{H}_{m-1}$ now with respect to $[\,\cdot,\cdot\,]_{\mathcal{G}_\beta}$ using (21) this time. Repeating the computation that leads to (14), we similarly obtain $S^*_{\beta_j}(1) = 0$ and

$$S^*_{\beta_j} u_m = \frac{1}{m}\,\frac{\beta_{m-1}}{\beta_m}\,\partial_j u_m \qquad (u_m \in \mathcal{H}_m,\ m \geq 1). \tag{23}$$

We then extend both the S_j and the $S^*_{\beta_j}$ to all of $\mathcal{G}_\beta$ and $\mathcal{G}_q$ using linearity and the density stated in Corollary 5.4, and also use the notation S_{β_j} and S_{q_j} in place of S_j. So for $u_m \in \mathcal{H}_m$,

$$S^*_{q_j} u_m = \begin{cases} \dfrac{1}{n/2 + q + m}\,\partial_j u_m, & \text{if } \ q > -(1 + n/2), \\[2ex] \dfrac{-(n/2 + q) + m}{m^2}\,\partial_j u_m, & \text{if } \ q \leq -(1 + n/2). \end{cases} \tag{24}$$

Notice the similarity of (23) to [16, (24)] and of (24) to [16, (26)].

The action of the operator tuple $T = (T_1, \dots, T_n) : H \oplus \cdots \oplus H \to H$ is defined by $T(v_1, \dots, v_n) = T_1 v_1 + \cdots + T_n v_n$. Then its adjoint $T^* : H \to H \oplus \cdots \oplus H$ is given by $T^* v = (T_1^* v, \dots, T_n^* v)$. It follows that $TT^* = T_1 T_1^* + \cdots + T_n T_n^*$. We also use the notation

$$T \cdot U := T_1 U_1 + \cdots + T_n U_n$$

with another tuple U; so $T \cdot T^* = TT^*$.

For $m \geq 1$ and $u_m \in \mathcal{H}_m$, by (23), Theorem 4.4, and (3), we have

$$S_\beta S_\beta^* u_m = \sum_{j=1}^n S_{\beta_j} \frac{1}{m} \frac{\beta_{m-1}}{\beta_m} \partial_j u_m = \frac{1}{m} \frac{\beta_{m-1}}{\beta_m} H_m(x \cdot \partial) u_m = \frac{\beta_{m-1}}{\beta_m} u_m \tag{25}$$

even without using the projection H_m. Hence

$$S_\beta S_\beta^* u = \sum_{m=1}^\infty \frac{\beta_{m-1}}{\beta_m} u_m \qquad (u \in \mathcal{G}_\beta) \tag{26}$$

and

$$(I - S_\beta S_\beta^*) u = u_0 + \sum_{m=1}^\infty \left(1 - \frac{\beta_{m-1}}{\beta_m}\right) u_m \qquad (u \in \mathcal{G}_\beta). \tag{27}$$

Specializing to $u \in \mathcal{G}_q$,

$$S_q S_q^* u = \begin{cases} \displaystyle\sum_{m=1}^\infty \frac{m}{n/2 + q + m} u_m, & \text{if } q > -(1 + n/2), \\ \displaystyle\sum_{m=1}^\infty \frac{-(n/2 + q) + m}{m} u_m, & \text{if } q \leq -(1 + n/2), \end{cases} \tag{28}$$

and

$$(I - S_q S_q^*) u = \begin{cases} u_0 + \displaystyle\sum_{m=1}^\infty \frac{n/2 + q}{n/2 + q + m} u_m, & \text{if } q > -(1 + n/2), \\ u_0 + \displaystyle\sum_{m=1}^\infty \frac{n/2 + q}{m} u_m, & \text{if } q \leq -(1 + n/2). \end{cases} \tag{29}$$

Now let

$$\theta_\beta = \sup_m \frac{\beta_m}{\beta_{m+1}} \qquad \text{and} \qquad \theta_q = \sup_m \frac{\beta_m(q)}{\beta_{m+1}(q)}.$$

Theorem 6.1. *The operator $S_\beta : \mathcal{G}_\beta \to \mathcal{G}_\beta$ is bounded and $\|S_\beta\| = \sqrt{\theta_\beta}$ if and only if $\theta_\beta < \infty$, in which case each S_{β_j} is bounded with $\|S_{\beta_j}\| \leq \sqrt{\theta_\beta}$.*

Proof. (26) shows that $S_\beta S_\beta^*$ is a diagonal operator with the same coefficient sequence $\{\beta_m/\beta_{m-1}\}$ not only on the homogeneous expansion of $u \in \mathcal{G}_\beta$, but also on its orthonormal expansion in (20). Then $\|S_\beta\|^2 = \|S_\beta S_\beta^*\| = \theta_\beta$. $\square$

In particular, when $\beta_m = \beta_m(q)$, the coefficients of u_m in (28) are increasing in m if $m \geq -n/2$ and decreasing if $m \leq -n/2$. Then

$$\|S_q\| = \sqrt{\|S_q S_q^*\|} = \begin{cases} 1, & \text{if} \quad q > -n/2, \\ 1/\sqrt{1+n/2+q}, & \text{if} \quad -(1+n/2) < q \leq -n/2, \\ \sqrt{1-(n/2+q)}, & \text{if} \quad q \leq -(1+n/2). \end{cases}$$

The last two cases can be combined as $\|S_q\| = 1/\sqrt{\beta_1(q)} \geq 1$ for $q \leq -n/2$. Thus $S_q : \mathcal{G}_q \to \mathcal{G}_q$ is bounded for all q. Compare these formulas with those for shift operators on holomorphic Dirichlet spaces in [16, Section 6].

For the norms of the individual S_{q_j}, first note that $\|S_{q_j}\| \leq \|S_q\|$. We use the action $S_{q_j} 1 = x_j = Y_{1j}/\sqrt{n}$. By Theorem 5.3, $\|x_j\|_{\mathcal{G}_q} = 1/\sqrt{\beta_1(q)}$ and $\|1\|_{\mathcal{G}_q} = 1$. Then $\|S_{q_j}\| = \|S_q\|$ for $q \leq -n/2$. For $q > -n/2$, we stop at $1 \geq \|T_j\| \geq 1/\sqrt{\beta_1(q)}$, where the last value is less than 1.

In the important special case $q = -n/2$, first $\|\breve{S}\| = 1$. Also all the $\breve{\beta}_m = 1$ and

$$\breve{S}_j^* u_m = \frac{1}{m}\, \partial_j u_m. \tag{30}$$

The homogeneity of u_m shows that

$$\int_0^1 (\partial_j u_m)(tx)\, dt = \int_0^1 t^{m-1} (\partial_j u_m)(x)\, dt = \frac{1}{m} (\partial_j u_m)(x).$$

In fact, the integral form is independent of $\mathcal{H}_m$ and it holds that

$$\breve{S}_j^* u(x) = \int_0^1 (\partial_j u)(tx)\, dt \qquad (u \in \breve{\mathcal{G}}),$$

which is the exact formula used in [2, p. 278] and [4, Definition 2.5] for the backward shift operators on Drury-Arveson spaces in holomorphic and quaternionic settings derived from solutions of Gleason problems. Further,

$$\breve{S}\breve{S}^* u = \sum_{m=1}^{\infty} u_m \qquad \text{and} \qquad I - \breve{S}\breve{S}^* = 1 \otimes 1, \tag{31}$$

the second being the only finite-rank case, which is common for shift operators on Drury-Arveson spaces starting with [6, Lemma 2.8].

Combining Definition 4.1 and (30) produces the interesting result

$$S_j^x X_m(x,y) = (\breve{S}_j^y)^* X_{m+1}(x,y).$$

It follows from this and (31) that

$$\sum_{j=1}^{n} S_j^y S_j^x X_m(x,y) = \sum_{j=1}^{n} \breve{S}_j^y (\breve{S}_j^y)^* X_{m+1}(x,y) = \sum_{\ell=1}^{\infty} (X_{m+1})_\ell(x,y) = X_{m+1}(x,y)$$

since $S_j^y = \breve{S}_j^y$ by definition. The holomorphic counterpart of this is the obvious $\sum\limits_{j=1}^{N} z_j \overline{w}_j \langle z,w\rangle^m = \langle z,w\rangle^{m+1}$.

We finally consider in this section the action of the shifts on the Bergman-Besov spaces b_q^2 of Example 5.5. Using the $\widetilde{\gamma}_m(q) = A_m \gamma_m(q)$ there, we see that the shift is bounded on all the b_q^2 and obtain

$$S_{\widetilde{\gamma}_j}^* = \frac{1}{2}\, S_{q_j}^*, \qquad S_{\widetilde{\gamma}_q} S_{\widetilde{\gamma}_q}^* = \frac{1}{2}\, S_q^* S_q^*, \qquad \text{and} \qquad \|S_{\widetilde{\gamma}_q}\| = \frac{1}{\sqrt{2}}\, \|S_q\|,$$

because of the factor 2^m in A_m. This shows why the spaces b_q^2 are not suitable for our purposes, because we seek a space on which the norm of the shift is maximal.

7. Row contractions

Our aim in this section is to show that the norm of $\breve{\mathcal{G}}$ is maximal among all contractive Hilbert norms. We start by recalling the necessary terms.

Definition 7.1. A commuting operator tuple $T = (T_1, \ldots, T_n) : H \oplus \cdots \oplus H \to H$ on a Hilbert space H is called a *row contraction* if $\|T\| \leq 1$, that is, if

$$\|T_1 u_1 + \cdots + T_n u_n\|_H^2 \leq \|u_1\|_H^2 + \cdots + \|u_n\|_H^2 \qquad (u_1, \ldots, u_n \in H).$$

We note that being commuting is part of the definition of a row contraction.

Proposition 7.2. *The shift S_β is a row contraction if and only if $\{\beta_m\}$ is an increasing sequence, and S_q is a row contraction if and only if $q \geq -n/2$.*

Proof. It is straightforward that T is a row contraction if and only if $I - TT^* \geq 0$. Since $I - S_\beta S_\beta^*$ is a diagonal operator, this happens when the coefficients of all the u_m in (27) and (29) are positive. □

So $\breve{\mathcal{G}}$ is the smallest space in the family $\{\mathcal{G}_q\}$ on which the shift operator is a row contraction. We have a stronger version of this result below.

For a row contraction T, we call $D_T := (I - TT^*)^{1/2} : H \to H$, which is the unique positive square root, the *defect operator* of T. For $T = S_q$, diagonality yields that

$$D_{S_q} u = u_0 + \sum_{m=1}^{\infty} \sqrt{\frac{n/2+q}{n/2+q+m}}\, u_m, \qquad (q \geq -n/2,\ u \in \mathcal{G}_q).$$

In particular, $D_{\breve{S}} u = u_0$.

We introduce below a sequence of maps on $\mathcal{B}(H)$ depending on an operator tuple $T = (T_1, \dots, T_n)$ on a Hilbert space H much like the map

$$J_T(B) := T_1 B T_1^* + \cdots + T_n B T_n^* \qquad (B \in \mathcal{B}(H))$$

which is of great importance in the holomorphic setting for which

$$J_T^m(B) := J_T(J_T^{m-1}(B)) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} T^\alpha B (T^*)^\alpha.$$

The last form of the above formula shows that $J_T^m(I) \geq 0$ for all $m = 1, 2, \dots$. If T is a row contraction, it is seen easily that $I \geq J_T^m(I) \geq J_T^{m+1}(I) \geq 0$.

Lemma 7.3. *If T is a row contraction, then $\lim_{m\to\infty} J_T^m(I) =: T_\infty$ exists in the strong operator topology and satisfies $0 \leq T_\infty \leq I$.*

Proof. This is contained in [14, Lemma 5.1.4]. □

A T for which $T_\infty = 0$ is called *pure*.

Definition 7.4. For $m = 0, 1, 2, \dots$, we define

$$V_T^m(B) := \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{A_{ml}}{A_m} (T \cdot T)^l J_T^{m-2l}(B) (T^* \cdot T^*)^l = J_T^m(B) - \cdots \qquad (B \in \mathcal{B}(H))$$

referring to the explicit formula (7) for the zonal harmonics.

We consider $J_T^m(I) = (TT^*)^m$ as a *hereditary polynomial* in the noncommuting variables T and T^* borrowing a term from [1] meaning that after we multiply out $(T_1 T_1^* + \cdots + T_n T_n^*)^m$, we collect in each term all the T_j on the left and all the T_j^* on the right. Both $V_T^0 = J_T^0 = I$ and $V_T^1 = J_T^1$. Also

$$\begin{aligned} V_T^m(I) &= \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{A_{ml}}{A_m} (T \cdot T)^l (TT^*)^{m-2l} (T^* \cdot T^*)^l = (TT^*)^m - \cdots \\ &= X_m(T, T^*) = \sum_{k=1}^{\delta_m} \breve{W}_{mk}(T)\, \overline{\breve{W}_{mk}}(T^*) \end{aligned} \tag{32}$$

as hereditary polynomials, where conjugation applies only to coefficients. The last expression in the above formula holds because $\breve{\beta}_m = 1$ for all m and it shows that $V_T^m(I) \geq 0$ for all $m = 1, 2, \ldots$.

It does not seem tractable to obtain the equivalent of Lemma 7.3 for $V_T^m(I)$ for a general row contraction T. Therefore we are obliged to restrict ourselves to certain subclasses of T in which $V_T^m(I)$ reduces to a single power.

Here is a crucial observation that directs us to our most important subclass. If $u_m \in \mathcal{H}_m$, then by the very definition of harmonicity,

$$(S_\beta^* \cdot S_\beta^*)u_m = \frac{1}{m}\,\frac{\beta_{m-1}}{\beta_m} \sum_{j=1}^n S_{\beta_j}^* \partial_j u_m = \frac{1}{m(m-1)}\,\frac{\beta_{m-2}}{\beta_m} \sum_{j=1}^n \partial_j^2 u_m = 0, \tag{33}$$

which extends to all $u \in \mathcal{G}_\beta$ by Corollary 5.4. Compare this to Lemma 4.6.

Definition 7.5. We call an operator tuple $T = (T_1, \ldots, T_n)$ on a Hilbert space *harmonic type* if $T^* \cdot T^* = 0$, or equivalently, if $T \cdot T = 0$.

If T is a harmonic-type operator tuple on a Hilbert space, then

$$V_T^m = J_T^m \qquad \text{and} \qquad V_T^m(V_T^1) = V_T^{m+1}; \tag{34}$$

if T is also a row contraction, then Lemma 7.3 implies $\lim\limits_{m\to\infty} V_T^m(I) = T_\infty$ exists and $0 \leq T_\infty \leq I$.

Shifts acting on harmonic Hilbert function spaces are harmonic type in particular. A straightforward induction using (25) also shows

$$V_{S_\beta}^m(I)u = J_{S_\beta}^m(I)u = \sum_{l=m}^\infty \frac{\beta_{l-m}}{\beta_l}\, u_l \qquad (u \in \mathcal{G}_\beta).$$

Now let $\{\beta_m\}$ be an increasing sequence, that is, let S_β be a row contraction. Then

$$\|V_{S_\beta}^m(I)u\|_{\mathcal{G}_\beta}^2 = \sum_{l=m}^\infty \frac{\beta_{l-m}^2}{\beta_l^2}\, \|u_l\|_{\mathcal{G}_\beta}^2 \leq \sum_{l=m}^\infty \|u_l\|_{\mathcal{G}_\beta}^2 \to 0 \qquad (m \to \infty)$$

by Theorem 5.3. Hence

$$(S_\beta)_\infty = \lim_{m\to\infty} V_{S_\beta}^m(I) = 0$$

in the strong operator topology of $\mathcal{G}_\beta$, in other words, every such S_β is pure. In particular, S_q is pure when $q \geq -n/2$; and when $q = -n/2$, all the $\breve{\beta}_m = 1$ and

$$V_{\breve{S}}^m(I)u = \sum_{l=m}^\infty u_l = u - (u_0 + u_1 + \cdots + u_{m-1}).$$

The following dilation-type result is important for our main results on $\breve{\mathcal{G}}$. We follow the exposition in [9, Section 6.1] in its proof.

Theorem 7.6. *Let T be a harmonic-type row contraction on a Hilbert space H. Then there exists a unique bounded linear operator $L : \breve{\mathcal{G}} \otimes \overline{D_T H} \to H$ satisfying $\|L\| \le 1$ and*

$$L(u \otimes \zeta) = u(T)\, D_T \zeta \qquad (u \in \mathcal{H},\ \zeta \in \overline{D_T H}).$$

In particular, $L(1 \otimes \zeta) = D_T \zeta$. If T is pure, then L is a coisometry.

Recall that $\mathcal{H}$ denotes the harmonic polynomials.

Proof. Let $E = \breve{\mathcal{G}} \otimes \overline{D_T H}$. For $v \in H$, define for $m = 0, 1, 2, \ldots$,

$$w_{mk} = \breve{W}_{mk} \otimes D_T \breve{W}_{mk}(T^*) v \qquad \text{and} \qquad w_m = \sum_{k=1}^{\delta_m} w_{mk} \in E.$$

Then by Theorem 5.3 and (32),

$$\begin{aligned}
\|w_m\|_E^2 &= \sum_{k=1}^{\delta_m} \big\|\breve{W}_{mk}\big\|_{\breve{\mathcal{G}}}^2 \big\|D_T \breve{W}_{mk}(T^*) v\big\|_H^2 \\
&= \sum_{k=1}^{\delta_m} \big[D_T \breve{W}_{mk}(T^*) v,\, D_T \breve{W}_{mk}(T^*) v\big]_H \\
&= \sum_{k=1}^{\delta_m} \big[\breve{W}_{mk}(T)(1 - TT^*) \breve{W}_{mk}(T^*) v,\, v\big]_H \\
&= \big[X_m(T, T^*) v,\, v\big]_H - \sum_{k=1}^{\delta_m} \big[\breve{W}_{mk}(T)(TT^*) \breve{W}_{mk}(T^*) v,\, v\big]_H \\
&= \big[V_T^m(I) v,\, v\big]_H - \big[(T_1 V_T^m(I) T_1^* + \cdots + T_n V_T^m(I) T_n^*) v,\, v\big]_H.
\end{aligned}$$

Since T is harmonic type, by (34),

$$\|w_m\|_E^2 = \big[J_T^m(I) v,\, v\big]_H - \big[J_T(J_T^m(I)) v,\, v\big]_H = \big[\big(J_T^m(I) - J_T^{m+1}(I)\big) v,\, v\big]_H.$$

Define $N : H \to E$ by $N(v) = w = \sum_{m=0}^{\infty} w_m$. Then by a telescoping sum and Lemma 7.3,

$$\begin{aligned}
\big\|N(v)\big\|_E^2 = \sum_{m=0}^{\infty} \|w_m\|_E^2 &= \sum_{m=0}^{\infty} \big[\big(J_T^m(I) - J_T^{m+1}(I)\big) v,\, v\big]_H \\
&= \|v\|_H^2 - \big[T_\infty v,\, v\big]_H.
\end{aligned}$$

Thus $\|N\| \leq 1$; equality holds and N is an isometry if T is pure.

Now define $L = N^*$. Then for $k = 1, \dots, \delta_m$, by the orthogonality in $\breve{\mathcal{G}}$,

$$\begin{aligned}\left[L(\breve{W}_{mk} \otimes \zeta),\, v\right]_H &= \left[\breve{W}_{mk} \otimes \zeta,\, Nv\right]_E = \left[\breve{W}_{mk} \otimes \zeta,\, w_{mk}\right]_E \\ &= \left[\breve{W}_{mk} \otimes \zeta,\, \breve{W}_{mk} \otimes D_T \breve{W}_{mk}(T^*)v\right]_E \\ &= \left\|\breve{W}_{mk}\right\|^2_{\breve{\mathcal{G}}} \left[\zeta,\, D_T \breve{W}_{mk}(T^*)v\right]_H \\ &= \left[\breve{W}_{mk}(T) D_T \zeta,\, v\right]_H \qquad (v \in H).\end{aligned}$$

Therefore $L(u_m \otimes \zeta) = u_m(T) D_T \zeta$ for $u_m \in \mathcal{H}_m$ since $\{\breve{W}_{mk}\}$ is a basis for $\mathcal{H}_m$ by Theorem 5.3, which also gives the uniqueness of L. Passing to a general $u \in \mathcal{H}$ is by linearity. $\square$

Another subclass of $T = (T_1, \dots, T_n)$ for which $V_T^m(I)$ simplifies is that of *self-adjoint* tuples by which we mean $T_j^* = T_j$ for all $j = 1, \dots, n$. For a self-adjoint T, by (9),

$$V_T^m(I) = \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{A_{ml}}{A_m} (T \cdot T)^m = \frac{\delta_m}{A_m} (T \cdot T)^m = \frac{\delta_m}{A_m} J_T^m(I), \tag{35}$$

complementing (34).

Lemma 7.7. *Let* $\kappa_m = \dfrac{\delta_m}{A_m} - \dfrac{\delta_{m+1}}{A_{m+1}}$. *Then* $\kappa_0 = 0$, $\kappa_m > 0$ *for* $m \geq 1$, *and* $\kappa = \sum_{m=1}^{\infty} \kappa_m = 1$.

Proof. Writing out the explicit forms using (4) and the line following it, and (8), we see that $\kappa_0 = 0$, $\kappa_m = \dfrac{1}{2^m}$ for $m \geq 1$ when $n = 2$, and $\kappa_m = \dfrac{1}{n-2} \dfrac{(n-2)_m}{(n/2)_m} \dfrac{m}{2^m}$ for $m \geq 1$ when $n \geq 3$. Then the positivity of κ_m for any n and that $\kappa = 1$ for $n = 2$ are obvious.

For $n \geq 3$, the hypergeometric function ${}_2F_1\left(n-2, 1; \frac{n}{2}; x\right) = \sum_{m=0}^{\infty} \dfrac{(n-2)_m}{(n/2)_m} x^m$ satisfies

$$\frac{x}{n-2}\, {}_2F_1'\left(n-2, 1; \frac{n}{2}; x\right)\bigg|_{x=1/2} = \kappa.$$

But by [18, 15.5.1],

$${}_2F_1'\left(n-2, 1; \frac{n}{2}; x\right) = \frac{n-2}{n/2}\, {}_2F_1\left(n-1, 2; \frac{n}{2}+1; x\right),$$

and by [18, 15.4.28],

$${}_2F_1\left(n-1, 2; \frac{n}{2}+1; \frac{1}{2}\right) = \sqrt{\pi}\, \frac{\Gamma(n/2+1)}{\Gamma(n/2)\, \Gamma(3/2)} = n.$$

Collecting together, $\kappa = 1$ again. $\square$

The following lemma complements Lemma 7.3. Its proof too is contained in [14, Lemma 5.1.4] by Lemma 7.7.

Lemma 7.8. *Let T be a row contraction and define $R_M = \sum_{m=1}^{M} \kappa_m J_T^{m+1}(I)$. Then $\lim_{M\to\infty} R_M =: R_\infty$ exists in the strong operator topology and satisfies $0 \le R_\infty \le I$.*

With this preparation, we can now prove the counterpart of Theorem 7.6 for self-adjoint row contractions.

Theorem 7.9. *Let T be a self-adjoint row contraction on a Hilbert space H. Then there exists a unique bounded linear operator $L : \breve{\mathcal{G}} \otimes \overline{D_T H} \to H$ satisfying $\|L\| \le 1$ and*

$$L(u \otimes \zeta) = u(T)\, D_T \zeta \qquad (u \in \mathcal{H},\ \zeta \in \overline{D_T H}).$$

In particular, $L(1 \otimes \zeta) = D_T \zeta$. If $R_\infty = 0$, then L is a coisometry.

Proof. The first and third paragraphs of the proof of Theorem 7.6 carry through without change. Only the second paragraph needs to be modified as follows.

Since T is self-adjoint, by (35),

$$\begin{aligned}\|w_m\|_E^2 &= \frac{\delta_m}{A_m}\Big(\big[J_T^m(I)v,\, v\big]_H - \big[J_T(J_T^m(I))v,\, v\big]_H\Big)\\ &= \frac{\delta_m}{A_m}\big[\big(J_T^m(I) - J_T^{m+1}(I)\big)v,\, v\big]_H.\end{aligned}$$

Again define $N : H \to E$ by $N(v) = w = \sum_{m=0}^{\infty} w_m$. Then by Lemma 7.8,

$$\begin{aligned}\big\|N(v)\big\|_E^2 &= \sum_{m=0}^{\infty} \|w_m\|_E^2 = \sum_{m=0}^{\infty} \frac{\delta_m}{A_m}\big[\big(J_T^m(I) - J_T^{m+1}(I)\big)v,\, v\big]_H\\ &= \|v\|_H^2 - \sum_{m=1}^{\infty} \kappa_m \big[J_T^{m+1}(I)v,\, v\big]_H = \|v\|_H^2 - \big[R_\infty v,\, v\big]_H.\end{aligned}$$

Thus $\|N\| \le 1$; equality holds and N is an isometry if $R_\infty = 0$. □

8. Von Neumann inequality

Definition 8.1. A norm on $\mathcal{H}$ derived from an inner product that respects the orthogonality in L^2 is called *contractive* if the shift operator is a row contraction in this norm.

We are now ready to prove Theorem 1.3 which we restate.

Theorem 8.2. *Let $\|\cdot\|$ be a contractive norm on $\mathcal{H}$. Then*

$$\|u\| \leq \|u\|_{\breve{\mathcal{G}}}\,\|1\| \qquad (u \in \mathcal{H}).$$

Proof. Let H be the Hilbert space that is the completion of $\mathcal{H}$ in the norm $\|\cdot\|$. Let also $S = (S_1, \ldots, S_n)$ be the shift on H and S^* its adjoint with respect to the inner product $[\cdot,\cdot]$ of H. We have $[S_j^*1, v] = [1, S_j v] = 0$ for all $v \in H$ by hypothesis, so $S_j^*1 = 0$, $j = 1, \ldots, n$. Then $\|D_S 1\|^2 = [(1 - SS^*)1, 1] = \|1\|^2$. Since clearly $0 \leq D_S \leq I$, Lemma 8.3 below implies $D_S 1 = 1$. The orthogonality condition on $[\cdot,\cdot]$ implies that S_j^* has a form involving ∂_j like the one in (23) which is derived from (14). Then S is harmonic type and pure. Now Theorem 7.6 applied with $T = S$ yields $L(u \otimes 1) = u(S)1 = u$ for $u \in \mathcal{H}$ by Proposition 4.2. Therefore $\|u\| \leq \|L\|\,\|u\|_{\breve{\mathcal{G}}}\|1\| = \|u\|_{\breve{\mathcal{G}}}\,\|1\|$. □

Moreover, as discussed in [9, p. 130], all Hilbert spaces mentioned in the statement of this theorem contain $\breve{\mathcal{G}}$ continuously. This result strengthens what we have above for the family $\mathcal{G}_q$.

Lemma 8.3. *Suppose H is Hilbert space and $R : H \to H$ satisfies $0 \leq R \leq I$ and $\|Re\| = \|e\|$ for some $e \in H$. Then $Re = e$.*

Proof. The operator inequalities on R yield $0 \leq [Rv, v] \leq \|v\|^2$. By the normality of R, $\|R\| = \sup_{\|v\|\leq 1} [Rv, v] \leq 1$. It is routine to check that $R^2 \leq R$. On the other hand, we have $[(R - R^2)e, e] = [(I - R)e, Re] = [e, Re] - \|Re\|^2 = [e, Re] - \|e\|^2 \leq 0$. Hence $[R^2 e, e] = \|Re\|^2 = \|e\|^2 \geq [Re, e] \geq [R^2 e, e]$ and consequently $[Re, e] = \|e\|^2$. Then it follows from $\|Re - e\|^2 = \|Re\|^2 - 2[Re, e] + \|e\|^2 = 0$ that $Re = e$. □

Similar to the passage from (7) to Definition 7.4, for a harmonic-type operator tuple T, we also have

$$X_m(T, y) = \sum_{l=0}^{\lfloor m/2 \rfloor} \frac{A_{ml}}{A_m}\,(T \cdot T)^l\,(y \cdot T)^{m-2l}\,|y|^{2l} = (y \cdot T)^m. \tag{36}$$

Then for $p_m \in \mathcal{P}_m$, by (36),

$$\begin{aligned} p_m(\partial_y)(X_{m+\ell})(T, y) &= p_m(\partial_y)((y \cdot T)^{m+\ell}) \\ &= (m + \ell) \cdots (1 + \ell)\,(y \cdot T)^{\ell}\,p_m(T) \\ &= (1 + \ell)_m\,p_m(T)\,X_l(T, y). \end{aligned} \tag{37}$$

Finally, we prove Theorem 1.4 which we restate.

Theorem 8.4. *Let T be a harmonic-type row contraction on a Hilbert space H. If u is a harmonic polynomial, then $\|u(T)\| \leq \|u(\breve{S})\|$.*

Proof. It suffices to consider $u = u_m \in \mathcal{H}_m$. By Theorem 7.6, there is a bounded operator $L : \breve{\mathcal{G}} \otimes \overline{D_T H} \to H$ with $\|L\| \leq 1$ and satisfying $L(u_m \otimes \zeta) = u_m(T)\, D_T \zeta$, which can be written as $L(u_m(\breve{S}) \otimes I)(1 \otimes \zeta) = u_m(T)\, L(1 \otimes \zeta)$ by Proposition 4.2 and Theorem 7.6 again. Now we replace 1 by $X_\ell(x, y)$ treating y as a parameter. Using Proposition 4.3, Theorem 7.6 twice, and (37), we compute

$$\begin{aligned} L(u_m(\breve{S}^x) \otimes I)(X_\ell(x,y) \otimes \zeta) &= L\left(\frac{1}{(1+\ell)_m}\, u_m(\partial_y)(X_{m+\ell})(x,y) \otimes \zeta\right) \\ &= \frac{1}{(1+\ell)_m}\, u_m(\partial_y)(X_{m+\ell})(T,y)\, D_T \zeta \\ &= u_m(T)\, X_\ell(T,y)\, D_T\zeta \\ &= u_m(T)\, L(X_\ell(x,y) \otimes \zeta). \end{aligned}$$

Thus $L(u_m(\breve{S}) \otimes I) = u_m(T)L$ by the density of the $X_l(\cdot, y)$ in $\mathcal{H}_l$.

For the end part, first we consider the case that T is pure. Then L is a coisometry and applying L^* on the right gives $L(u_m(\breve{S}) \otimes I)L^* = u_m(T)$. Taking norms yields us $\|u_m(T)\| \leq \|L\|^2\, \|u_m(\breve{S})\|\, \|I\| \leq \|u_m(\breve{S})\|$, which is what we want. In the case T is not pure, we apply the above to rT for $0 < r < 1$ for which $\|rT\| \leq r < 1$ and $(rT)_\infty = 0$. We obtain $\|u_m(rT)\| \leq \|u_m(\breve{S})\|$. Since this is true for all $0 < r < 1$, the desired result follows. □

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