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The Schur algorithm and reproducing kernel Hilbert spaces in the ball☆

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Abstract

Using reproducing kernel Hilbert spaces methods we develop a Schur-type algorithm for a subclass of the functions analytic and contractive in the ball. We also consider the Nevanlinna–Pick interpolation problem in that class. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

A function s analytic in the open unit disk $\mathbb{D}$ is called a *Schur function* if it is bounded by 1 in modulus there: $\sup_{z\in\mathbb{D}} |s(z)| \leqslant 1$. Let s be such a function which is not equal to a constant of modulus 1. Then for any $a \in \mathbb{D}$, the function

$$\frac{s(z) - s(a)}{[(z-a)/(1-za^*)](1-s(z)s(a)^*)} \tag{1.1}$$

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is still a Schur function. This is the crucial step in Schur's algorithm (see [29]) which was applied by Nevanlinna (see [22]) to solve interpolation problems.

The Schur algorithm has extensions and applications to various settings; let us mention in particular the case of functions that may have poles in the disk (see [15], and [13] for applications to number theory) and the case of upper triangular operators (see [16]). It was studied using reproducing kernel Hilbert spaces methods in [4]. In the present paper we study the Schur algorithm for Schur multipliers of the ball (the definition is given in Section 2) using the reproducing kernel approach. We follow the analysis of [4] suitably adapted to the present setting.

We first recall that positive kernels $k(z, w)$ (in the sense of reproducing kernels) for which $1/k(z, w)$ has one positive square are called *complete Nevanlinna–Pick kernels* (see [18]) and have been characterized as those kernels for which the matrix version of Pick's theorem holds. This result originates with the work of Quiggin (see [23]), and quite a number of authors have studied these kernels; see for instance [1,10,12,18,20]. A particular example of such a kernel is given by the function

$$k(z, w) = \frac{1}{1 - \langle z, w\rangle}, \tag{1.2}$$

where $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ vary in the ball

$$\mathbb{B}_N = \left\{ (z_1, \ldots, z_N) \in \mathbb{C}^N \;\middle|\; \sum_1^N |z_j|^2 < 1 \right\}, \tag{1.3}$$

and where

$$\langle z, w\rangle = \sum_{j=1}^N z_j w_j^*.$$

Much of the analysis in the Hardy space of the open unit disk $\mathbb{D}$ goes through to the case of the reproducing kernel Hilbert space $\mathbf{H}(\mathbb{B}_N)$ of functions analytic in the ball and with reproducing kernel (1.2) as is illustrated in the above-mentioned works and in [8,9]. We recall that $\mathbf{H}(\mathbb{B}_N)$ is *contractively* included in the Hardy space of the ball. We also recall (see, e.g., [8]) that

$$\mathbf{H}(\mathbb{B}_N) = \left\{ f(z_1, z_2, \ldots, z_N) = \sum_{n=0}^{\infty} \sum_{\substack{(n_1, n_2, \ldots, n_N) \in \mathbb{N}^N \\ n_1 + n_2 + \cdots + n_N = n}} a_{n_1, n_2, \ldots, n_N} z_1^{n_1} z_2^{n_2} \cdots z_N^{n_N} \right\} \tag{1.4}$$

with norm

$$\|f\|_{\mathbf{H}(\mathbb{B}_N)} = \sqrt{\sum_{n=0}^{\infty} \sum_{\substack{(n_1,n_2,\dots,n_N)\in\mathbb{N}^N \\ n_1+n_2+\cdots+n_N=n}} \frac{|a_{n_1,n_2,\dots,n_N}|^2}{\left(\frac{n!}{n_1!n_2!\cdots n_N!}\right)}}. \tag{1.5}$$

These facts follow easily from the series expansion

$$\begin{aligned}\frac{1}{1-zw^*} &= \sum_{n=0}^{\infty}(zw^*)^n \\ &= \sum_{n=0}^{\infty} \sum_{\substack{(n_1,n_2,\dots,n_N)\in\mathbb{N}^N \\ n_1+n_2+\cdots+n_N=n}} \frac{n!}{n_1!n_2!\cdots n_N!} \\ &\qquad\qquad \times (z_1w_1^*)^{n_1}(z_2w_2^*)^{n_2}\cdots(z_Nw_N^*)^{n_N}.\end{aligned}$$

The paper consists of nine sections besides the introduction. In Section 2 we review Pick's theorem and some results on the space $\mathbf{H}(\mathbb{B}_N)$. In Section 3 we prove a version of Leech's theorem in the setting of the ball, while Section 4 is devoted to reproducing kernel Hilbert spaces with reproducing kernels of the form $[I_p - S(z)S(w)^*]/[1-\langle z, w\rangle]$. In Section 5 we study certain linear fractional transformations, while in Section 6 we prove a structure theorem for a family of one-dimensional reproducing kernel Hilbert spaces. The Schur algorithm is presented in Section 7. In Section 8 we consider a general family of finite-dimensional spaces of rational functions in the ball, while Section 9 deals with the Nevanlinna–Pick interpolation problem solved using Potapov's method of fundamental matrix inequalities. In Section 10 we consider interpolation in the space $\mathbf{H}(\mathbb{B}_N)$.

Finally a word on notation. For $\mathscr{H}$ a Hilbert space, the symbol $\mathscr{H}^{p\times q}$ will denote the Hilbert space of $p\times q$ matrices $F=(F_{\ell j})$ with entries in $\mathscr{H}$ and with norm defined by

$$\|F\|^2_{\mathscr{H}^{p\times q}} = \sum_{\ell,j} \|F_{\ell j}\|^2_{\mathscr{H}}.$$

When $q=1$, we will also use the notation $\mathscr{H}^p$ for $\mathscr{H}^{p\times 1}$.

The symbol $\mathbb{S}$ stands for the sphere

$$\left\{(z_1,\dots,z_N)\in\mathbb{C}^N \;\middle|\; \sum_1^N |z_j|^2 = 1\right\},$$

while $(\nu_+(M), \nu_-(M), \nu_0(M))$ is the signature of a hermitian matrix M, that is, the number of strictly positive, strictly negative and zero eigenvalues (counting multiplicities), respectively.

2. Pick's theorem and some preliminaries

A function $S : \mathbb{B}_N \to \mathbb{C}^{p\times q}$ is called a Schur multiplier if the operator M_S of multiplication by S on the left given by

$$F \mapsto SF$$

is a contraction from $(\mathbf{H}(\mathbb{B}_N))^q$ into $(\mathbf{H}(\mathbb{B}_N))^p$. When $N = 1$, Schur multipliers are exactly the $\mathbb{C}^{p\times q}$-valued functions analytic and contractive in the disk. For S a Schur multiplier and $\xi \in \mathbb{C}^q$, one has

$$M_S^*\left(\frac{\xi}{1-\langle z, w\rangle}\right) = \frac{S(w)^*\xi}{1-\langle z, w\rangle}.$$

It follows that an equivalent characterization of a Schur multiplier is that the kernel

$$K_S(z, w) = \frac{I_p - S(z)S(w)^*}{1-\langle z, w\rangle} \tag{2.1}$$

is positive in $\mathbb{B}_N$. The proofs are as in the case $N = 1$. See, e.g., [4, Example 1, p. 95] and [5, Section 3.2].

It follows that a Schur multiplier has contractive values in the ball (i.e., a Schur function). On the other hand, there are Schur functions which are not Schur multipliers. Examples are given in [8]. We now recall the results of [8] where a whole family of Schur functions that are not Schur multipliers are constructed using the following idea due to Rudin (see [26, p. 164]). Define numbers c_j via $1-\sqrt{1-t} = \sum_j c_j t^j$, where $|t| < 1$. Then all $c_j > 0$, and

$$p_m(z_1, \ldots, z_N) = z_1 + c_1 z_2^2 + c_2 z_2^4 + \cdots + c_m z_2^{2m}$$

are Schur functions. Now $\|1\|_{\mathbf{H}(\mathbb{B}_N)} = 1$ and so the norm of the operator of multiplication by p_m on $\mathbf{H}(\mathbb{B}_N)$ is at least $\|p_m\|_{\mathbf{H}(\mathbb{B}_N)}$. But in view of (1.5),

$$\begin{aligned}\|p_m\|^2_{\mathbf{H}(\mathbb{B}_N)} &= \|z_1\|^2_{\mathbf{H}(\mathbb{B}_N)} + c_1^2\|z_2^2\|^2_{\mathbf{H}(\mathbb{B}_N)} + \cdots + c_m^2\|z_2^{2m}\|^2_{\mathbf{H}(\mathbb{B}_N)}\\ &= 1 + c_1^2 + \cdots + c_m^2 > 1.\end{aligned}$$

Thus the norm of the operator of multiplication by p_m on $\mathbf{H}(\mathbb{B}_N)$ is strictly bigger than 1. The case $m = 1$ of these examples reduces to the example of Misra [21, Example 4.4, p. 834], who first saw the difference between the two classes using different methods. Another family of Schur functions which are not Schur multipliers consist of the inner functions of the ball (see [7]).

We consider the following tangential Nevanlinna–Pick problem, which we call **NP**:

Given points $w^{(1)}, \ldots, w^{(m)} \in \mathbb{B}_N$ and given vectors $\xi_1, \ldots, \xi_m \in \mathbb{C}^p$ and $\eta_1, \ldots, \eta_m \in \mathbb{C}^q$, find the necessary and sufficient condition for a $\mathbb{C}^{p\times q}$-valued Schur multiplier to exist such that

$$S(w^{(\ell)})^*\xi_\ell = \eta_\ell, \quad \ell = 1, \ldots, m, \tag{2.2}$$

and describe the set of all solutions.

The following result is due to Pick in the case of the disk and in the scalar case. In the case of the ball, the characterization of kernels for which Pick's theorem holds is due, as already mentioned, to Quiggin. For the case of matrix-valued functions we refer to [12, Theorem 4.1, p. 107].

Theorem 2.1. *Problem* **NP** *has a solution if and only the $m \times m$ hermitian matrix*

$$\mathbf{K} = \left(\frac{\xi_\ell^* \xi_j - \eta_\ell^* \eta_j}{1 - \langle w^{(\ell)}, w^{(j)} \rangle} \right)_{\ell, j=1}^{m} \tag{2.3}$$

is positive semidefinite.

We need the following results, taken from the preprint [8] (see also [9]).

Proposition 2.2. *Let $a \in \mathbb{B}_N$. Then the $\mathbb{C}^{1\times N}$-valued function*

$$b_a(z) = \frac{(1 - \langle a, a \rangle)^{1/2}}{1 - \langle z, a \rangle} (z_1 - a_1 \quad \cdots \quad z_N - a_N)(I_N - a^* a)^{-1/2} \tag{2.4}$$

satisfies

$$\frac{1 - b_a(z) b_a(w)^*}{1 - \langle z, w \rangle} = \frac{1 - \langle a, a \rangle}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}, \quad z, w \in \mathbb{B}_N. \tag{2.5}$$

In particular,

$$b_a(z) b_a(z)^* < 1 \quad \text{if } z \in \mathbb{B}_N \quad \text{and} \quad b_a(z) b_a(z)^* = 1 \quad \text{if } z \in \mathbb{S}. \tag{2.6}$$

Lastly, b_a belongs to $\mathbf{H}(\mathbb{B}_N)^{1\times N}$ and the entries of $b_a(z)$ are multipliers of $\mathbf{H}(\mathbb{B}_N)$.

Formula (2.5) appears in Rudin's book on the ball; see [26, Theorem 2.2.2, p. 26] (with an apparently different choice of b_a but in fact, up to a sign, the same; see [9]). It expresses the fact that the one-dimensional vector space spanned by the function $z \mapsto 1/[1 - \langle z, a \rangle]$ endowed with the metric of $\mathbf{H}(\mathbb{B}_N)$ has reproducing kernel of the form $[1 - b_a(z) b_a(w)^*]/[1 - \langle z, w \rangle]$.

3. Leech's theorem

We will need the following result that relates factorization and positivity, which was first proved in the setting of the disk by Leech; see [19] and [24, p. 107].

Theorem 3.1. *Let A and B be two analytic functions from $\mathbb{B}_N$ to $\mathbb{C}^{k\times p}$ and $\mathbb{C}^{k\times q}$, respectively, and assume that the kernel*

$$\frac{A(z)A(w)^* - B(z)B(w)^*}{1 - \langle z, w\rangle}$$

is positive in $\mathbb{B}_N$. *Then there exists a Schur multiplier* $S : \mathbb{B}_N \to \mathbb{C}^{p\times q}$ *such that* $B = AS$.

Proof. Pick up an integer M and M points $w^{(1)}, \ldots, w^{(M)} \in \mathbb{B}_N$ and M vectors $c_1, \ldots, c_M \in \mathbb{C}^k$. The $M \times M$ hermitian matrix with ℓ, j entry equal to

$$\frac{(c_\ell^* A(w^{(\ell)}))(c_j^* A(w^{(j)}))^* - (c_\ell^* B(w^{(\ell)}))(c_j^* B(w^{(j)}))^*}{1 - \langle w^{(\ell)}, w^{(j)}\rangle}$$

is positive semidefinite. Since Pick's theorem holds in the space $\mathbf{H}(\mathbb{B}_N)$ there exists a Schur multiplier $S_{M,c_1,\ldots,c_M,w^{(1)},\ldots,w^{(M)}}(z)$ that depends on the given interpolation data and is such that

$$\left(S_{M,c_1,\ldots,c_M,w^{(1)},\ldots,w^{(M)}}(w^{(\ell)})\right)^* \left(c_\ell^* A(w^{(\ell)})\right)^* = \left(c_\ell^* B(w^{(\ell)})\right)^*.$$

We let M increase to infinity and the w_ℓ in such a way that $\{w_1, w_2, \ldots\}$ becomes a dense set of the ball. The functions $S_{M,c_1,\ldots,c_M,w^{(1)},\ldots,w^{(M)}}$ are in particular bounded by 1 in modulus in the ball, and we can use Montel's theorem to find an analytic function S such that, maybe via a subsequence,

$$S(z) = \lim_{M\to\infty} S_{M,c_1,\ldots,c_M,w^{(1)},\ldots,w^{(M)}}(z).$$

The function S satisfies $B(z) = A(z)S(z)$ on a dense set and, hence, everywhere in the ball by continuity. Furthermore, it is a Schur multiplier. Indeed, set for simplicity $S_M = S_{M,c_1,\ldots,c_M,w^{(1)},\ldots,w^{(M)}}$ and take points $v^{(1)}, \ldots, v^{(t)} \in \mathbb{B}_N$ and vectors $d_1, \ldots, d_t \in \mathbb{C}^k$. The $t \times t$ hermitian matrix with ℓj entry equal to

$$\frac{d_\ell^* d_j - d_\ell^* S_M(v^{(\ell)}) S_M(v^{(j)})^* d_j}{1 - \langle v^{(\ell)}, v^{(j)}\rangle}$$

is positive semidefinite. Letting $M \to \infty$ we get that the same conclusion holds for S, and hence S is a Schur multiplier of the ball. □

4. $\mathscr{H}(S)$ spaces

We will denote by $\mathscr{H}(S)$ the reproducing kernel Hilbert space of $\mathbb{C}^p$-valued functions analytic in the ball and with reproducing kernel (2.1). As in the case $N = 1$, it follows from the decomposition

$$\frac{I_p}{1 - \langle z, w\rangle} = \frac{I_p - S(z)S(w)^*}{1 - \langle z, w\rangle} + \frac{S(z)S(w)^*}{1 - \langle z, w\rangle} \tag{4.1}$$

that the space $\mathscr{H}(S)$ is contractively included in $(\mathbf{H}(\mathbb{B}_N))^p$ and that

$$\mathscr{H}(S) = \left\{ F \in (\mathbf{H}(\mathbb{B}_N))^p \;\middle|\; \sup_{u \in (\mathbf{H}(\mathbb{B}_N))^q} \|F + Su\|^2_{(\mathbf{H}(\mathbb{B}_N))^p} - \|u\|^2_{(\mathbf{H}(\mathbb{B}_N))^q} < \infty \right\}. \tag{4.2}$$

See [14] for the disk case.

Proposition 4.1. *Let S be a $\mathbb{C}^{p\times q}$-valued Schur multiplier of the ball. The corresponding space $\mathscr{H}(S)$ is reduced to $\{0\}$ if and only if S is constant and coisometric.*

Proof. This is just the corollary of Theorem 4.3 in [4], proved there for $N = 1$. The proof goes through here and relies on the fact that the set $\mathscr{N}$ of vectors of the form

$$\begin{pmatrix} c \\ S(z)^* c \end{pmatrix}, \quad z \in \mathbb{B}_N, \quad c \in \mathbb{C}^p,$$

is a neutral subspace of $\mathbb{C}^{p+q}$ endowed with the inner product $[u, v]_{\mathbb{C}_J} \overset{\text{def.}}{=} v^* J u$, where $u, v \in \mathbb{C}^{p+q}$ (i.e., $[u, v]_{\mathbb{C}_J} = 0$ for all $u, v \in \mathscr{N}$). $\square$

The spaces $\mathscr{H}(S)$ can be used to solve interpolation problems as in the case $N = 1$. In the present work, we illustrate this point in Proposition 7.4. The Nevanlinna–Pick interpolation problem in Section 9 is solved using Potapov's method and not the reproducing kernel method.

The general theory of $\mathscr{H}(S)$ spaces for $N > 1$ will be investigated in a future publication.

5. Linear fractional transformations

Let

$$J = J_{pq} \overset{\text{def.}}{=} \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}. \tag{5.1}$$

A matrix $\Theta \in \mathbb{C}^{(p_2+q)\times(p_1+q)}$ is called (J_{p_1q}, J_{p_2q})-contractive if

$$\Theta J_{p_1q} \Theta^* \leqslant J_{p_2q}. \tag{5.2}$$

Lemma 5.1. *Let Θ be (J_{p_1q}, J_{p_2q})-contractive and let*

$$\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} : \begin{pmatrix} \mathbb{C}^{p_1} \\ \mathbb{C}^q \end{pmatrix} \to \begin{pmatrix} \mathbb{C}^{p_2} \\ \mathbb{C}^q \end{pmatrix} \tag{5.3}$$

be its decomposition into four blocks of indicated sizes. Then

$$\det \theta_{22} \neq 0, \qquad \|\theta_{22}^{-1}\theta_{21}\| < 1,$$

and the map

$$X \mapsto T_\Theta(X) \stackrel{\text{def.}}{=} (\theta_{11}X + \theta_{12})(\theta_{21}X + \theta_{22})^{-1} \tag{5.4}$$

sends the unit ball of $\mathbb{C}^{p_2\times q}$ *into the unit ball of* $\mathbb{C}^{p_1\times q}$. *Finally,*

$$\Theta^* J_{p_2q}\Theta \leqslant J_{p_1q}. \tag{5.5}$$

Proof. First we note that under assumptions of the lemma, Θ is a contraction between two Pontryagin spaces with the same negativity index, and (5.5) expresses the classical result that the adjoint of a contraction between Pontryagin spaces of the same negativity index is still a contraction; see [3, Corollary 1.3.5, p. 26] for a proof and references.

Upon multiplying inequality (5.2) by the matrix $\begin{pmatrix} 0_{q\times p_2} & I_q \end{pmatrix}$ on the left and by its adjoint on the right and making use of the block decomposition (5.3), we get

$$\theta_{21}\theta_{21}^* - \theta_{22}\theta_{22}^* \leqslant -I_q.$$

Therefore θ_{22} is invertible, and rewriting the last inequality in the following equivalent form

$$I_q - \theta_{22}^{-1}\theta_{21}\theta_{21}^*\theta_{22}^{-*} \geqslant \theta_{22}^{-1}\theta_{22}^{-*},$$

we conclude that $\|\theta_{22}{}^{-1}\theta_{21}\| < 1$. Therefore the matrix

$$(\theta_{21}X + \theta_{22}) = \theta_{22}(\theta_{22}^{-1}\theta_{21}X + I_q)$$

is invertible for every $X \in \mathbb{C}^{p_2\times q}$ with $\|X\| \leqslant 1$, which means that the linear fractional transformation $T_\Theta(X)$ is well defined on the unit ball of $\mathbb{C}^{p_2\times q}$. Finally, it is readily seen that

$$\begin{aligned} &I_{p_1} - T_\Theta(X)^* T_\Theta(X) \\ &\quad = -(\theta_{21}X + \theta_{22})^{-*}(X^* I_q)\Theta^* J_{p_2q}\Theta \begin{pmatrix} X \\ I_q \end{pmatrix} (\theta_{21}X + \theta_{22})^{-1}, \end{aligned}$$

and since, by (5.5),

$$-(X^* I_q)\Theta^* J_{p_2q}\Theta \begin{pmatrix} X \\ I_q \end{pmatrix} \geqslant (X^* I_q) J_{p_1q} \begin{pmatrix} X \\ I_q \end{pmatrix} = I_q - X^*X,$$

we conclude from the two last relations that $\|T_\Theta(X)\| \leqslant 1$ whenever $\|X\| \leqslant 1$. This completes the proof. □

Lemma 5.2. *Let* $\Theta_1 \in \mathbb{C}^{(p_1+q)\times(p_2+q)}$ *and* $\Theta_2 \in \mathbb{C}^{(p_2+q)\times(p_3+q)}$ *and assume that*

$$\Theta_1 J_{p_1q}\Theta_1^* \leqslant J_{p_2q}, \qquad \Theta_2 J_{p_2q}\Theta_2^* \leqslant J_{p_3q}.$$

Then $T_{\Theta_1\Theta_2}$ *sends the unit ball of* $\mathbb{C}^{p_3\times q}$ *into the unit ball of* $\mathbb{C}^{p_1\times q}$ *and the semigroup property*

$$T_{\Theta_1\Theta_2}(X) = T_{\Theta_1}\big(T_{\Theta_2}(X)\big) \tag{5.6}$$

holds.

The proof is straightforward and is omitted.

6. A one-dimensional structure theorem

First a definition and a lemma. Let $J \in \mathbb{C}^{n\times n}$ be a signature matrix, that is, a matrix which is both self-adjoint and unitary. We will denote by $\mathbf{H}_J(\mathbb{B}_N)$ the space $(\mathbf{H}(\mathbb{B}_N))^n$ endowed with the indefinite inner product

$$[F, G]_{\mathbf{H}_J(\mathbb{B}_N)} = \langle F, JG\rangle_{\mathbf{H}(\mathbb{B}_N)}\,. \tag{6.1}$$

The space $\mathbf{H}_J(\mathbb{B}_N)$ is a Krein space.

Lemma 6.1. *Let $J \in \mathbb{C}^{n\times n}$ be a signature matrix and let $c \in \mathbb{C}^n$ be such that $c^*Jc > 0$. Let*

$$M = J - \frac{cc^*}{c^*Jc}.$$

Then,
1. $\ker M = \operatorname{span}\{Jc\}$,
2. $\nu_+(M) = \nu_+(J) - 1$,
3. $\nu_-(M) = \nu_-(J)$.

Proof. Without loss of generality, we assume that $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and $q > 0$. When $J = I_p$,

$$M = I_p - \frac{cc^*}{c^*c}$$

is an orthogonal projection and the conclusions of the lemma are easily derived; details are left to the reader.

We will also assume that $c^*Jc = 1$ (this amounts to replacing c by $c/\sqrt{c^*Jc}$). We write $c = \binom{c_1}{c_2}$, where $c_1 \in \mathbb{C}^p$ and $c_2 \in \mathbb{C}^q$. Using formula

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} I_p & \beta\delta^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} \alpha - \beta\delta^{-1}\gamma & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} I_p & 0 \\ \delta^{-1}\gamma & I_q \end{pmatrix},$$

we conclude that the signature of the matrix M is equal to the signature of the matrix

$$\begin{pmatrix} I_p - c_1(1 - c_2^*(I_q + c_2c_2^*)^{-1}c_2)c_1^* & 0 \\ 0 & -(I_q + c_2c_2^*) \end{pmatrix}.$$

Taking into account that $c^*Jc = 1$ (that is, $c_1^*c_1 = 1 + c_2^*c_2$), we can rewrite the upper left top block as

$$I_p - c_1(1 - c_2^*(I_q + c_2c_2^*)^{-1}c_2)c_1^* = I_p - \frac{c_1c_1^*}{c_1^*c_1}.$$

The conclusions follow since

$$\operatorname{rank}\left(I_p - \frac{c_1c_1^*}{c_1^*c_1}\right) = p - 1 \quad \text{and} \quad \ker\left(I_p - \frac{c_1c_1^*}{c_1^*c_1}\right) = \operatorname{span}\{c_1\}\,. \qquad \square$$

Theorem 6.2. *Let $J \in \mathbb{C}^{n\times n}$ be a signature matrix and let $c_0 \in \mathbb{C}^n$ be such that $c_0^* J c_0 > 0$. Let $w_0 \in \mathbb{B}_N$. Then the one-dimensional space $\mathscr{M}$ spanned by the function $f(z) = c_0/[1 - \langle z, w_0\rangle]$ endowed with the $\mathbf{H}_J(\mathbb{B}_N)$ inner product is a reproducing kernel Hilbert space with reproducing kernel of the form*

$$\frac{J - \Theta(z)\tilde{J}\Theta(w)^*}{1 - \langle z, w\rangle}, \tag{6.2}$$

where $\tilde{J} \in \mathbb{C}^{(n+N-1)\times(n+N-1)}$ is a signature matrix satisfying

$$\nu_+(\tilde{J}) = \nu_+(J) + N - 1, \qquad \nu_-(\tilde{J}) = \nu_-(J),$$

and where the function Θ is $\mathbb{C}^{m\times(n+N-1)}$-valued and satisfies

$$\Theta(z)\tilde{J}\Theta(z) = J, \quad z \in \mathbb{S}. \tag{6.3}$$

Proof. By the previous lemma, we can write

$$J - \frac{c_0 c_0^*}{c_0^* J c_0} = \alpha J_1 \alpha^*, \tag{6.4}$$

where $\alpha \in \mathbb{C}^{n\times(n-1)}$ (remark that $\operatorname{rank}(J - c_0c_0^*/c_0^*Jc_0) = n - 1$), and J_1 is an $(n-1)\times(n-1)$ signature matrix such that

$$\nu_+(J_1) = \nu_+(J) - 1, \qquad \nu_-(J_1) = \nu_-(J). \tag{6.5}$$

Moreover, $\mathscr{M}$ is one-dimensional, and by a well-known formula (see, e.g., [17, p. 24]), its reproducing kernel is given by

$$\begin{aligned} K(z, w) &= f(z)\left(\langle f, f\rangle_{\mathbf{H}_J(\mathbb{B}_N)}\right)^{-1} f(w)^* \\ &= \frac{c_0 c_0^*}{(1 - \langle z, w_0\rangle)(1 - \langle w_0, w\rangle)} \frac{1 - \langle w_0, w_0\rangle}{c_0^* J c_0}. \end{aligned}$$

Making use of the function b_{w_0} defined via (2.4) and of relation (2.5), we get

$$\begin{aligned} K(z, w) &= c_0 \frac{1 - b_{w_0}(z) b_{w_0}(w)^*}{1 - \langle z, w\rangle} \frac{c_0^*}{c_0^* J c_0} \\ &= \frac{J - (J - c_0c_0^*/c_0^*Jc_0) + c_0 b_{w_0}(z) b_{w_0}(w)^* c_0^*/c_0^*Jc_0}{1 - \langle z, w\rangle}, \end{aligned}$$

which, in view of (6.4) and (6.5), is of the form (6.2) with

$$\Theta(z) = \begin{pmatrix} \alpha & \frac{c_0 b_{w_0}(z)}{\sqrt{c_0^* J c_0}} \end{pmatrix} \quad \text{and} \quad \tilde{J} = \begin{pmatrix} J_1 & 0 \\ 0 & I_N \end{pmatrix}. \tag{6.6}$$

Further, on the sphere we have $b_{w_0}(z) b_{w_0}(z)^* = 1$, and so for $z \in \mathbb{S}$ we have

$$\Theta(z)\tilde{J}\Theta(z) = \alpha^* J_1 \alpha + \frac{c_0 c_0^*}{c_0^* J c_0} = J. \qquad \square$$

Corollary 6.3. *Let Θ be as in Theorem* 6.2 *and assume that*

$$J = J_{pq} \quad \text{and} \quad \tilde{J} = J_{p+N-1,q}.$$

Let $\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ *be the block decomposition of Θ with θ_{11} being $\mathbb{C}^{p\times(p+N-1)}$-valued. Then* $\det\theta_{22}(z) \not\equiv 0$ *in $\mathbb{B}_N$ and*

$$\|\theta_{22}(z)^{-1}\theta_{21}(z)\| < 1, \quad z \in \mathbb{B}_N. \tag{6.7}$$

Proof. From (6.3) it follows that

$$\theta_{21}(z)\theta_{21}(z)^* - \theta_{22}(z)\theta_{22}(z)^* = -I_q, \quad z \in \mathbb{S}.$$

Hence $\theta_{22}(z)^*$ has a zero kernel, and the result then follows since θ_{22} is $\mathbb{C}^{q\times q}$-valued. The rest is as in the more classical case where Θ is square. □

7. The Schur algorithm

The Schur algorithm associates to a function analytic and contractive in the open unit disk $\mathbb{D}$ (a *Schur function*) a sequence, finite or infinite, of numbers in $\mathbb{D}$, and when the sequence is finite, a supplementary number of modulus 1. This sequence plays an important role in interpolation theory and in other topics such as filtering of stationary processes. We show the existence of a similar sequence in the setting of Schur multipliers of the ball. We begin with two preliminary lemmas.

Lemma 7.1. *Let S be a $\mathbb{C}^{p\times q}$-valued Schur multiplier of the ball and assume that the reproducing kernel Hilbert space $\mathscr{H}(S)$ does not reduce to* $\{0\}$. *Then there exists* $(\xi_0, w^{(0)}) \in \mathbb{C}^p \times \mathbb{B}_N$ *such that*

$$\xi_0^*\xi_0 > \xi_0^* S(w^{(0)})S(w^{(0)})^*\xi_0. \tag{7.1}$$

Proof. As for the case $N = 1$, assume that $\xi_0^*\xi_0 = \xi_0^* S(w^{(0)})S(w^{(0)})^*\xi_0$ for all choices of ξ_0 and $w^{(0)}$. Then, using the Cauchy–Schwartz inequality in the Hardy space $\mathscr{H}(S)$ we have for any function $f \in \mathscr{H}(S)$

$$\begin{aligned}
\left|\xi_0^* f(w^{(0)})\right| &= \left|\langle f, K_S(z, w^{(0)})\xi_0\rangle_{\mathscr{H}(S)}\right| \\
&\leqslant \|f\|_{\mathscr{H}(S)}\|K(z, w^{(0)})\xi_0\|_{\mathscr{H}(S)} \\
&= \|f\|_{\mathscr{H}(S)}\sqrt{\frac{\xi_0^*\xi_0 - \xi_0^* S(w^{(0)})S(w^{(0)})^*\xi_0}{1 - \langle w^{(0)}, w^{(0)}\rangle}} \\
&= 0,
\end{aligned}$$

and hence $f \equiv 0$. □

For the case $N = 1$ in the preceding lemma we refer to [4, Theorem 4.2, p. 108].

Lemma 7.2. *Let* $(\xi_0, w^{(0)}) \in \mathbb{C}^p \times \mathbb{B}_N$ *be such that* (7.1) *holds, and let*

$$\mathscr{M} = \operatorname{span}\left\{ \begin{pmatrix} \xi_0 \\ S(w_0)^*\xi_0 \end{pmatrix} \Big/ 1 - \langle z, w_0 \rangle \right\}$$

in the $\mathbf{H}_{J_{pq}}$*-inner product. Then the map* $F \mapsto \begin{pmatrix} I_p & -S \end{pmatrix}$ *is an isometry from* $\mathscr{M}$ *into* $\mathscr{H}(S)$.

Proof. The proof is by construction. We have

$$\begin{aligned} \|f\|^2_{\mathbf{H}_{J_{pq}}(\mathbb{B}_N)} &= \frac{\xi_0^*\xi_0 - \xi_0^* S(w^{(0)}) S(w^{(0)})^*\xi_0}{1 - \langle w^{(0)}, w^{(0)} \rangle} \\ &= \|K_S(z, w^{(0)})\xi_0\|^2_{\mathscr{H}(S)} \\ &= \left\| \begin{pmatrix} I_p & -S \end{pmatrix} f(z) \right\|^2_{\mathscr{H}(S)}. \qquad \square \end{aligned}$$

Theorem 7.3. *Let* S *be a* $\mathbb{C}^{p\times q}$*-valued Schur multiplier of the ball and let* $(\xi_0, w^{(0)}) \in \mathbb{C}^p \times \mathbb{B}_N$ *be such that* $\xi_0^*\xi_0 > \xi_0^* S(w^{(0)}) S(w^{(0)})^*\xi_0$. *Let* $c_0 = \begin{pmatrix} \xi_0 \\ S(w^{(0)})^*\xi_0 \end{pmatrix}$ *and let* Θ *be as in Theorem* 6.2. *Let* $\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ *be the block decomposition of* Θ *with* θ_{11} *being* $\mathbb{C}^{p\times(p+N-1)}$*-valued. Then there exists a* $\mathbb{C}^{(p+N-1)\times q}$*-valued Schur multiplier* S_0 *such that*

$$S(z) = (\theta_{11}(z)S_0(z) + \theta_{12}(z))(\theta_{12}(z)S_0(z) + \theta_{22}(z))^{-1}. \tag{7.2}$$

Proof. By Lemma 7.2, the kernel

$$\frac{\begin{pmatrix} I_p & -S(z) \end{pmatrix} \Theta(z)\tilde{J}\Theta(w)^* \begin{pmatrix} I_p \\ -S(w)^* \end{pmatrix}}{1 - \langle z, w \rangle}$$

is positive in $\mathbb{B}_N$. Using the above-mentioned block decomposition of Θ and writing $\tilde{J} = \begin{pmatrix} I_{n+N-1} & 0 \\ 0 & -I_q \end{pmatrix}$, we can rewrite this kernel as

$$\begin{aligned} &[(\theta_{11}(z) - S(z)\theta_{21}(z))(\theta_{11}(w) - S(w)\theta_{21}(w))^* \\ &\quad -(\theta_{12}(z) - S(z)\theta_{22}(z))(\theta_{12}(w) - S(w)\theta_{22}(w))^*]/1 - \langle z, w \rangle. \end{aligned}$$

Applying Theorem 3.1, we conclude that there is a $\mathbb{C}^{(p+N-1)\times q}$-valued Schur multiplier S_0 such that

$$(\theta_{11}(z) - S(z)\theta_{21}(z)) = -S_0(z)(\theta_{12}(z) - S(z)\theta_{22}(z)).$$

Thus,

$$S(z)(\theta_{21}(z)S_0(z) + \theta_{22}(z)) = \theta_{11}(z)S_0(z) + \theta_{12}(z).$$

In view of Corollary 6.3, we have $\det(\theta_{21}(z)S_0(z) + \theta_{22}(z)) \not\equiv 0$ and so $S = (\theta_{11}S_0 + \theta_{12})(\theta_{21}S_0 + \theta_{22})^{-1}$. $\square$

The process can be iterated; if the space $\mathscr{H}(S_0)$ does not reduce to the zero space, there is a pair $(\xi_1, w^{(1)}) \in \mathbb{C}^{(p+N-1)} \times \mathbb{B}_N$ such that $\xi_1^*\xi_1 > \xi_1^* S_0(w^{(1)})S_0(w^{(1)})^*\xi_1$. We can then apply Theorem 6.2 to the space $\mathscr{M}_1 \subset \mathbf{H}_{J_{p+N-1,q}}(\mathbb{B}_N)$ spanned by the function

$$\frac{\begin{pmatrix} \xi_1 \\ S_0(w^{(1)})^*\xi_1 \end{pmatrix}}{1 - \langle z, w^{(1)}\rangle}.$$

The reproducing kernel of $\mathscr{M}_1$ is of the form

$$\frac{J_{p+N-1,q} - \Theta_1(z) J_{p+2(N-1),q} \Theta_1(w)^*}{1 - \langle z, w\rangle},$$

where Θ_1 is $\mathbb{C}^{(p+(N-1)+q)\times(p+2(N-1)+q)}$-valued.

So we can characterize a $\mathbb{C}^{p\times q}$-valued Schur multiplier by a sequence, finite or infinite, of pairs

$$(\xi_k, w^{(k)}), \quad \text{where } w^{(k)} \in \mathbb{B}_N \text{ and } \xi^{(k)} \in \mathbb{C}^{(p+k(N-1))}.$$

Let $\Theta_0, \Theta_1, \ldots$ be the corresponding elementary factors obtained from Theorem 6.2 (the function $\Theta_k(z)$ is $\mathbb{C}^{(p+k(N-1)+q)\times(p+(k+1)(N-1)+q)}$-valued). We have, as long as the process can be iterated,

$$S(z) = T_{\Theta_0(z)\cdots\Theta_k(z)}(S_{k+1}(z)),$$

where $S_{k+1}(z)$ is a $\mathbb{C}^{(p+(k+1)(N-1)+q)\times q}$-valued Schur multiplier of the ball. The process stops at rank k if and only if the space $\mathscr{H}(S_{k+1})$ is equal to $\{0\}$, i.e., by Proposition 4.1, if and only if the multiplier $S_{k+1}(z)$ is constant and coisometric.

In the following proposition we study a relationship between the Schur algorithm and interpolation.

Proposition 7.4. *In the notation of Theorem* 7.3, *set* $c_0 = \binom{\xi_0}{\eta_0}$, *with* $\xi_0 \in \mathbb{C}^p$ *and* $\eta_0 \in \mathbb{C}^q$. *Then the linear fractional transformation* $S = (\theta_{11}S_0 + \theta_{12})(\theta_{21}S_0 + \theta_{22})^{-1}$ *describes the set of all Schur multipliers* S *such that* $S(w^{(0)})^*\xi_0 = \eta_0$, *where* S_0 *varies in the set of all* $\mathbb{C}^{(p+N-1)\times q}$*-valued Schur multipliers.*

Proof. Let S_0 be a $\mathbb{C}^{(p+N-1)\times q}$-valued Schur multiplier and let $d \in \mathbb{C}^q$. By formula (6.6) for Θ and Proposition 2.2, the entries of Θ are multipliers of $\mathbf{H}(\mathbb{B}_N)$. Hence, the function

$$z \mapsto \begin{pmatrix} S_0(z) \\ I_q \end{pmatrix} d$$

belongs to $(\mathbf{H}(\mathbb{B}_N))^{(p+N-1)}$ and so we have

$$\left\langle \Theta(z) \begin{pmatrix} S_0(z) \\ I_q \end{pmatrix} d, \frac{c_0}{1 - \langle z, w^{(0)}\rangle} \right\rangle_{\mathbf{H}_J(\mathbb{B}_N)} = 0,$$

that is,

$$\begin{pmatrix} \xi_0^* & -\eta_0^* \end{pmatrix} \Theta(w^{(0)}) \begin{pmatrix} S_0(w^{(0)}) \\ I_q \end{pmatrix} d = 0.$$

Since this equality holds for all $d \in \mathbb{C}^q$ we have

$$\begin{pmatrix} \xi_0^* & -\eta_0^* \end{pmatrix} \Theta(w^{(0)}) \begin{pmatrix} S_0(w^{(0)}) \\ I_q \end{pmatrix} = 0,$$

and thus

$$\xi_0^*(\theta_{11} S_0 + \theta_{12})(w^{(0)}) = \eta_0^*(\theta_{21} S_0 + \theta_{22})(w^{(0)}).$$

We have already noted that $\det(\theta_{21} S_0 + \theta_{22}) \neq 0$ in the ball; therefore

$$\xi_0^* \left(T_\Theta(S_0)\right)(w^{(0)}) = \eta_0.$$

For the converse claim, let S be a Schur multiplier such that $S(w^{(0)})^* \xi_0 = \eta_0$. The result follows directly from Theorem 7.3. □

Using this proposition one can solve recursively the interpolation problem **NP**. We will solve it in a different way in Section 9 using Potapov's method of fundamental matrix inequalities.

8. A general structure theorem

The following result is a generalization of Theorem 6.2.

Theorem 8.1. *Let $A_1, \ldots, A_N \in \mathbb{C}^{n\times n}$ be N matrices, let $C \in \mathbb{C}^{(p+q)\times n}$, let $J \in \mathbb{C}^{(p+q)\times(p+q)}$ be a signature matrix and set $\mathbf{J} = \begin{pmatrix} I_{nN} & 0 \\ 0 & J \end{pmatrix}$. Let $\mathbf{K} \in \mathbb{C}^{n\times n}$ be a solution of the matrix equation*

$$\mathbf{K} - \sum_{j=1}^{N} A_j^* \mathbf{K} A_j = C^* J C \tag{8.1}$$

that is positive definite. Then the function

$$\Theta(z) = (0, I_{p+q}) + C \left(I_n - \sum_{j=1}^{N} z_j A_j \right)^{-1} \times \mathbf{K}^{-1} \left((z_1 I_n - A_1^*)\mathbf{K}^{1/2}, \ldots, (z_N I_n - A_N^*)\mathbf{K}^{1/2}, -C^* J \right) \tag{8.2}$$

satisfies

$$\frac{J - \Theta(z)\mathbf{J}\Theta(w)^*}{1 - \langle z, w\rangle} = C\left(I_n - \sum_{j=1}^{N} z_j A_j\right)^{-1} \mathbf{K}^{-1} \left(I_n - \sum_{j=1}^{N} \bar{w}_j A_j^*\right)^{-1} C^* \tag{8.3}$$

for every choice of $z = (z_1, \ldots, z_N)$ *and* $w = (w_1, \ldots, w_N)$ *in* $\mathbb{B}_N$.

Proof. By (8.2),

$$J - \Theta(z)\mathbf{J}\Theta(w)^* = C\left(I_n - \sum_{j=1}^{N} z_j A_j\right)^{-1} \mathbf{K}^{-1} T(z, w)\mathbf{K}^{-1} \times \left(I_n - \sum_{j=1}^{N} \bar{w}_j A_j^*\right)^{-1} C^*, \tag{8.4}$$

where

$$T(z, w) = \left(I_n - \sum_{j=1}^{N} \bar{w}_j A_j^*\right)\mathbf{K} + \mathbf{K}\left(I_n - \sum_{j=1}^{N} z_j A_j\right) - \sum_{j=1}^{N} (z_j I - A_j^*)\mathbf{K}(\bar{w}_j I - A_j) - C^* J C.$$

Making use of (8.1), we get

$$T(z, w) = 2\mathbf{K} - \sum_{j=1}^{N} \left(z_j \bar{w}_j \mathbf{K} + A_j^* \mathbf{K} A_j\right) - C^* J C = (1 - \langle z, w\rangle)\mathbf{K},$$

which together with (8.4) implies (8.3). □

When $N = 1$, such a result is the finite-dimensional version of the disk version of a structure theorem of de Branges; see [6] for this case, and further discussion on the history of the theorem, which involves the work of Rovnyak [25] and the work of Ball [11]. The matrix functions Θ obtained in these various works are square. The sizes for Θ in (8.2) are far from optimal. In view of Theorem 6.2, we think, but

cannot prove, that Θ should be $\mathbb{C}^{(p+q)\times(p+q+n(N-1))}$-valued and therefore reduces to a $\mathbb{C}^{(p+q)\times(p+q)}$-valued function when $N = 1$.

9. Interpolation for Schur multipliers

Interpolation problems in the Schur class of the ball have been studied by Ball et al. in [12]. Here, we present an alternative proof of some of their results using Potapov's method of fundamental matrix inequalities. We note that the matrix function Θ which is obtained is bigger than the one that one would obtain by solving recursively the interpolation problem using Proposition 7.4.

Theorem 9.1. *Let S be a $\mathbb{C}^{p\times q}$-valued function analytic in $\mathbb{B}_N$. Then S is a solution to Problem* **NP** *if and only if the kernel*

$$\begin{pmatrix} \mathbf{K} & \Psi(w)^* \\ \Psi(z) & K_S(z,w) \end{pmatrix} \tag{9.1}$$

is positive on $\mathbb{B}_N$, where $\mathbf{K}$ is the matrix defined in (2.3), *K_S is the kernel from* (2.1) *and*

$$\Psi(z) = \left(\frac{\xi_1 - S(z)\eta_1}{1-\langle z, w^{(1)}\rangle} \cdots \frac{\xi_m - S(z)\eta_m}{1-\langle z, w^{(m)}\rangle} \right). \tag{9.2}$$

Proof. We first suppose that S is a Schur multiplier and satisfies (2.3). Then the kernel K_S is positive on $\mathbb{B}_N$ and therefore the kernel

$$\tilde{K}_S(z,w) = \begin{pmatrix} K_S(w^{(1)}, w^{(1)}) & \cdots & K_S(w^{(1)}, w^{(m)}) & K_S(w^{(1)}, w) \\ \vdots & & \vdots & \vdots \\ K_S(w^{(m)}, w^{(1)}) & \cdots & K_S(w^{(m)}, w^{(m)}) & K_S(w^{(m)}, w) \\ K_S(z, w^{(1)}) & \cdots & K_S(z, w^{(m)}) & K_S(z,w) \end{pmatrix}$$

is positive on $\mathbb{B}_N$. Let

$$T = \begin{pmatrix} \xi_1 & & 0 \\ & \ddots & \\ 0 & & \xi_m \end{pmatrix}. \tag{9.3}$$

Then clearly

$$\begin{pmatrix} T & 0 \\ 0 & I_p \end{pmatrix} \tilde{K}_S(z,w) \begin{pmatrix} T^* & 0 \\ 0 & I_p \end{pmatrix} \geqslant 0, \quad z, w \in \mathbb{B}_N. \tag{9.4}$$

Since also by (2.3) and (2.1),

$$\xi_\ell^* K(w^{(\ell)}, w^{(j)})\xi_j = \frac{\xi_\ell^*\xi_j - \eta_\ell^*\eta_j}{1-\langle w^{(\ell)}, w^{(j)}\rangle}, \qquad K(z, w^{(j)})\xi_j = \frac{\xi_j - S(z)\eta_j}{1-\langle z, w^{(j)}\rangle},$$
$$\ell, j = 1, \ldots, m,$$

it follows that

$$\begin{pmatrix} T & 0 \\ 0 & I_p \end{pmatrix} \tilde{K}_S(z, w) \begin{pmatrix} T^* & 0 \\ 0 & I_p \end{pmatrix} = \begin{pmatrix} \mathbf{K} & \Psi(w)^* \\ \Psi(z) & K_S(z, w) \end{pmatrix},$$

which, in view of (9.4), proves (9.1).

Conversely, let S be a $\mathbb{C}^{p\times q}$-valued function analytic in $\mathbb{B}_N$ for which the kernel (9.1) is positive on $\mathbb{B}_N$. Then, in particular, the kernel $K_S(z, w)$ is positive on $\mathbb{B}_N$, and thus, S is a Schur function. The positivity of the kernel (9.1) implies also (upon setting $w = z$) that the following matrices are positive semidefinite:

$$\begin{pmatrix} \dfrac{\xi_j^*\xi_j - \eta_j^*\eta_j}{1 - \langle w^{(j)}, w^{(j)}\rangle} & \dfrac{\xi_j^* - \eta_j^* S(z)^*}{1 - \langle w^{(j)}, z\rangle} \\ \dfrac{\xi_j - S(z)\eta_j}{1 - \langle z, w^{(j)}\rangle} & \dfrac{I_p - S(z)S(z)^*}{1 - \langle z, z\rangle} \end{pmatrix} \geqslant 0, \quad j = 1, \ldots, m.$$

Setting $z = w^{(j)}$ in the last inequality, we get

$$\begin{pmatrix} \xi_j^* & \eta_j^* \\ I_p & S(w^{(j)}) \end{pmatrix} \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \begin{pmatrix} \xi_j & I_p \\ \eta_j & S(w^{(j)})^* \end{pmatrix} \geqslant 0, \quad j = 1, \ldots, m.$$

The latter inequality means that the matrices

$$M_j = \begin{pmatrix} \xi_j^* & \eta_j^* \\ I_p & S(w^{(j)}) \end{pmatrix} \in \mathbb{C}^{(p+1)\times q}$$

are J_{pq}-nonnegative, where J_{pq} is the signature matrix defined in (5.1). Thus the rank of M_j is less than or equal to p. Due to the block I_p in M_j it follows that rank $M_j = p$. Thus, there exists $g_j \in \mathbb{C}^{1\times p}$ such that $(1\ g_j)M = 0$, i.e., such that

$$(1\ g_j) \begin{pmatrix} \xi_j^* & \eta_j^* \\ I_p & S(w^{(j)}) \end{pmatrix} = 0, \quad j = 1, \ldots, m.$$

But then $g_j = -\xi_j^*$ and $\eta_j^* = -g_j S(w^{(j)}) = \xi_j^* S(w^{(j)})$ for $j = 1, \ldots, m$, which are equivalent to (2.3). □

Theorem 9.2. *Let the Pick matrix* $\mathbf{K}$ *defined by* (2.3) *be positive definite*, *let*

$$B = \begin{pmatrix} \xi_1 & \cdots & \xi_m \\ \eta_1 & \cdots & \eta_m \end{pmatrix}, \qquad A_j = \begin{pmatrix} w_j^{(1)} & & 0 \\ & \ddots & \\ 0 & & w_j^{(m)} \end{pmatrix},$$
$$j = 1, \ldots, N, \tag{9.5}$$

where $w^{(j)} = (w_1^{(j)}, \ldots, w_N^{(j)}) \in \mathbb{B}_N$ *are the interpolating points*, *and let* $\Theta(z)$ *be the* $\mathbb{C}^{(p+q)\times(mN+p+q)}$*-valued function constructed via* (8.2). *Furthermore*, *let*

$$\Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} : \begin{pmatrix} \mathbb{C}^{mN+p} \\ \mathbb{C}^q \end{pmatrix} \to \begin{pmatrix} \mathbb{C}^p \\ \mathbb{C}^q \end{pmatrix} \tag{9.6}$$

be the block decomposition of Θ into four blocks. Then the set of all solutions to Problem **NP** *is parametrized by the linear fractional transformation*

$$S(z) = (\theta_{11}(z)\sigma(z) + \theta_{12}(z))\,(\theta_{21}(z)\sigma(z) + \theta_{22}(z))^{-1}\,, \tag{9.7}$$

where the parameter σ varies on the set of all $\mathbb{C}^{(mN+p)\times q}$-valued Schur multipliers in $\mathbb{B}_N$.

Proof. First we note that the matrices $\mathbf{K}$, B and A_j defined in (2.3) and (9.5) satisfy the Stein equation (8.1) and therefore, by Theorem 8.1, the function Θ defined via (8.2) satisfies (8.3) for every choice of z and w in $\mathbb{B}_N$. Using the same arguments as in the proof of Theorem 7.3, we conclude that the matrix

$$\theta_{21}(z)\sigma(z) + \theta_{22}(z) = \theta_{22}(z)(\theta_{22}(z)^{-1}\theta_{21}(z)\sigma(z) + I_q)$$

is invertible whenever $\|\sigma(z)\| \leqslant 1$. In particular, this matrix will be invertible at every point $z \in \mathbb{B}_N$ if σ is a Schur multiplier.

Next we note that the function Ψ in (9.2) can be expressed in terms of (9.5) as

$$\Psi(z) = (I_p - S(z))C\left(I_m - \sum_{j=1}^{N} z_j A_j\right)^{-1}. \tag{9.8}$$

By Theorem 9.1, S is a solution to Problem **NP** if and only if the kernel (9.1) is positive on $\mathbb{B}_N$, or, equivalently (since $\mathbf{K} > 0$), if and only if

$$K_S(z, w) - \Psi(z)\mathbf{K}^{-1}\Psi(w)^* \geqslant 0, \quad z, w \in \mathbb{B}_N.$$

Taking advantage of (9.8) and of the representation

$$K_S(z, w) = (I_p, -S(z))\frac{J_{pq}}{1 - \langle z, w\rangle}\begin{pmatrix} I_p \\ -S(w)^* \end{pmatrix},$$

we rewrite the last inequality as

$$(I_p - S(z))\left\{\frac{J_{pq}}{1 - \langle z, w\rangle} - C\left(I_m - \sum_{j=1}^{N} z_j A_j\right)^{-1} \times \mathbf{K}^{-1}\left(I_m - \sum_{j=1}^{N} \bar{w}_j A_j^*\right)^{-1} C^*\right\}\begin{pmatrix} I_p \\ -S(w)^* \end{pmatrix} \geqslant 0.$$

Making use of (8.3) we represent the last inequality as

$$(I_p - S(z))\frac{\Theta(z)\mathbf{J}\Theta(w)^*}{1 - \langle z, w\rangle}\begin{pmatrix} I_p \\ -S(w)^* \end{pmatrix} \geqslant 0,$$
$$\mathbf{J} = J_{mN+p,q} = \begin{pmatrix} I_{mN+p} & 0 \\ 0 & -I_q \end{pmatrix}. \tag{9.9}$$

Set

$$\begin{aligned}(p(z)-q(z)) &= (\theta_{11}(z)-S(z)\theta_{21}(z),\,\theta_{12}(z)-S(z)\theta_{22}(z))\\ &= (I_p-S(z))\Theta(z).\end{aligned} \tag{9.10}$$

Then inequality (9.9) is equivalent to

$$(p(z)-q(z))\frac{J_{mN+p,q}}{1-\langle z,w\rangle}\begin{pmatrix}p(w)^*\\ -q(w)^*\end{pmatrix} = \frac{p(z)p(w)^*-q(z)q(w)^*}{1-\langle z,w\rangle}\geqslant 0.$$

By Leech's theorem (Theorem 3.1), the last inequality is equivalent to the existence of a $\mathbb{C}^{(mN+p)\times q}$-valued Schur multiplier $\sigma(z)$ such that $q(z)=p(z)\sigma(z)$. Substituting this factorization into (9.10) we get

$$p(z)\left(I_{mN+p}-\sigma(z)\right) = (I_p-S(z))\Theta(z).$$

Upon multiplying both sides of this equality by the matrix $\left(\begin{smallmatrix}\sigma(z)\\ I_{mN+p}\end{smallmatrix}\right)$ on the right, we arrive at

$$(I_p-S(z))\begin{pmatrix}\theta_{11}(z) & \theta_{12}(z)\\ \theta_{21}(z) & \theta_{22}(z)\end{pmatrix}\begin{pmatrix}\sigma(z)\\ I_{mN+p}\end{pmatrix}=0,$$

which is the same as

$$S(z)\,(\theta_{21}(z)\sigma(z)+\theta_{22}(z)) = (\theta_{11}(z)\sigma(z)+\theta_{12}(z))$$

and is equivalent to the representation (9.7). □

10. Interpolation for $\mathbf{H}(\mathbb{B}_N)$ functions

In this section we consider the left-sided Nevanlinna–Pick interpolation problem for $\mathbf{H}(\mathbb{B}_N)$ functions H that are contractive multipliers from $\mathbb{C}^q$ to $\mathbf{H}^p(\mathbb{B}_N)$. That H is a contractive multiplier means that the kernel

$$K_H(z,w) = \frac{I_p}{1-\langle z,w\rangle} - H(z)H(w)^* \tag{10.1}$$

is positive on $\mathbb{B}_N$. We denote by $\mathscr{M}(\mathbf{H}(\mathbb{B}_N))$ the set of all contractive multipliers from $\mathbb{C}^q$ to $\mathbf{H}^p(\mathbb{B}_N)$. We note that the components of a multiplier are in $\mathbf{H}(\mathbb{B}_N)$.

Proposition 10.1. *Let*

$$H(z_1,z_2,\ldots,z_N) = \sum_{n=0}^{\infty}\sum_{\substack{(n_1,n_2,\ldots,n_N)\in\mathbb{N}^N\\ n_1+n_2+\cdots+n_N=n}} a_{n_1,n_2,\ldots,n_N} z_1^{n_1}z_2^{n_2}\cdots z_N^{n_N}$$

be in $\mathbf{H}(\mathbb{B}_N)^{p\times q}$ *with* $a_{n_1,n_2,\ldots,n_N}\in\mathbb{C}^{p\times q}$. *Then* H *is a contractive multiplier from* $\mathbb{C}^q$ *into* $\mathbf{H}^p(\mathbb{B}_N)$ *if and only if*

$$\sum_{n=0}^{\infty} \sum_{\substack{(n_1,n_2,\dots,n_N)\in\mathbb{N}^N \\ n_1+n_2+\cdots+n_N=n}} a^*_{n_1,n_2,\dots,n_N} a_{n_1,n_2,\dots,n_N} \frac{n!}{n_1!n_2!\cdots n_N!} \leqslant I_q. \tag{10.2}$$

Proof. It suffices to write that, for every $\xi \in \mathbb{C}^q$, the function $z \mapsto H(z)\xi$ belongs to $(\mathbf{H}(\mathbb{B}_N))^p$ and

$$\|H(z)\xi\|^2_{(\mathbf{H}(\mathbb{B}_N))^p} \leqslant \xi^*\xi.$$

In view of (1.5) we obtain the result. □

Problem 10.2. *Given points $w^{(1)}, \dots, w^{(m)} \in \mathbb{B}_N$ and given vectors $a_1, \dots, a_m \in \mathbb{C}^p$ and $c_1, \dots, c_m \in \mathbb{C}^q$, find all functions $H \in \mathscr{M}(\mathbf{H}(\mathbb{B}_N))$ such that*

$$a^*_\ell H(w^{(\ell)}) = c^*_\ell, \quad \ell = 1, \dots, m. \tag{10.3}$$

We follow the strategy of [2] and first prove a representation theorem for functions such that the kernel (10.1) is positive in terms of Schur functions. Such representations were first found by Sarason in the case of $N = 1$ and scalar functions; see [27,28].

Theorem 10.3. *A $\mathbb{C}^{p\times q}$-valued function H analytic in $\mathbb{B}_N$ belongs to $\mathscr{M}(\mathbf{H}(\mathbb{B}_N))$ if and only if it can be written as*

$$H(z) = S_0(z)\left(I_q - z_1S_1(z) - \cdots - z_NS_N(z)\right)^{-1} \tag{10.4}$$

for some $\mathbb{C}^{(p+Nq)\times q}$-valued Schur multiplier

$$S(z) = \begin{pmatrix} S_0(z) \\ S_1(z) \\ \vdots \\ S_N(z) \end{pmatrix}. \tag{10.5}$$

Proof. Let H admit a representation of the form (10.4) with a Schur multiplier S defined in (10.5). Setting

$$A(z) = \begin{pmatrix} I_p & z_1H(z) & \cdots & z_NH(z) \end{pmatrix} \tag{10.6}$$

we conclude from (10.4) and (10.5) that

$$H(z) = A(z)S(z), \tag{10.7}$$

which allows us to represent the kernel in (10.1) as

$$K_H(z,w) = A(z)\frac{I_p - S(z)S(w)^*}{1 - \langle z, w\rangle}A(w)^*.$$

Since S is a Schur multiplier, the last kernel is positive in $\mathbb{B}_N$, and then $H \in \mathscr{M}(\mathbf{H}(\mathbb{B}_N))$.

Conversely, let H be in $\mathscr{M}(\mathbf{H}(\mathbb{B}_N))$. Then the kernel K_H is positive and substituting (10.6) into (10.1), we get

$$K_H(z,w) = \frac{A(z)A(w)^* - H(z)H(w)^*}{1-\langle z,w\rangle} \geqslant 0, \quad z,w \in \mathbb{B}_N.$$

By Theorem 9.2, there exists a Schur multiplier S such that (10.7) holds. Finally, we get from (10.7) that

$$H(z) = S_0(z) + z_1 H(z) S_1(z) + \cdots + z_N H(z) S_N(z),$$

which implies (10.4). □

Lemma 10.4. *Let H and S be functions defined by* (10.4) *and* (10.5). *Then H satisfies* (10.3) *if and only if S satisfies the following interpolation conditions*:

$$\begin{pmatrix} a_\ell^* & w_1^{(\ell)} c_\ell^* & \cdots & w_N^{(\ell)} c_\ell^* \end{pmatrix} S(w^{(\ell)}) = c_\ell^*, \quad \ell = 1,\ldots,m. \tag{10.8}$$

Proof. Let H satisfy (10.3). Then for $A(z)$ defined in (10.6) we have

$$a_\ell^* A(w^{(\ell)}) = \begin{pmatrix} a_\ell^* & w_1^{(\ell)} c_\ell^* & \cdots & w_N^{(\ell)} c_\ell^* \end{pmatrix}, \quad \ell = 1,\ldots,m.$$

Multiplying (10.7) evaluated at $z = w^{(\ell)}$ by a_ℓ^* on the left and taking into account the last equalities we get (10.8).

Conversely, let S satisfy (10.8). Substituting the decomposition (10.5) into (10.8) we get

$$a_\ell^* S_0(w^{(\ell)}) + w_1^{(\ell)} c_\ell^* S_1(w^{(\ell)}) + \cdots + w_N^{(\ell)} c_\ell^* S_N(w^{(\ell)}) = c_\ell^*,$$

or, equivalently,

$$a_\ell^* S_0(w^{(\ell)}) \left(I_q - w_1^{(\ell)} S_1(w^{(\ell)}) - \cdots - w_N^{(\ell)} S_N(w^{(\ell)}) \right)^{-1} = c_\ell^*,$$

which coincides with (10.3) on account of (10.4). □

Combining Theorems 9.2 and 10.3 we can get a description of all solutions H to Problem 10.2. In the statement below, C denotes the matrix

$$C = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ w_1^{(1)} c_1 & w_1^{(2)} c_2 & \cdots & w_1^{(m)} c_m \\ \vdots & \vdots & & \vdots \\ w_N^{(1)} c_1 & w_N^{(2)} c_2 & \cdots & w_N^{(m)} c_m \\ c_1 & c_2 & \cdots & c_m \end{pmatrix}. \tag{10.9}$$

Theorem 10.5. *Let the matrix* $\mathbf{K}$ *given by* (2.3) *be positive definite. Then the set of all solutions to Problem* 10.2 *is parametrized by the linear fractional transformation*

$$(\Psi_{11}(z)\sigma(z) + \Psi_{12}(z))(\Psi_{21}(z)\sigma(z) + \Psi_{22}(z))^{-1}, \tag{10.10}$$

where

$$\begin{aligned}\Psi(z) &= \begin{pmatrix} \Psi_{11}(z) & \Psi_{12}(z) \\ \Psi_{21}(z) & \Psi_{22}(z) \end{pmatrix} \\ &= \begin{pmatrix} I_p & 0 & 0 & 0 & 0 \\ 0 & -z_1 I_q & \cdots & -z_N I_q & I_q \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{a_1}{1-\langle z, w^{(1)}\rangle} & \cdots & \frac{a_m}{1-\langle z, w^{(m)}\rangle} \\ c_1 & \cdots & c_m \end{pmatrix} \mathbf{K}^{-1} \\ &\quad \times \Big((z_1 I_m - A_1^*)\mathbf{K}^{1/2} \cdots (z_N I_m - A_N^*)\mathbf{K}^{1/2} - C^* J \Big) \end{aligned} \tag{10.11}$$

and σ is an arbitrary $\mathbb{C}^{p\times q}$-valued Schur multiplier.

Proof. Let Θ be the $\mathbb{C}^{(p+(N+1)q)\times(p+(N+1)q+Nm)}$-valued function defined via (8.2) with $\mathbf{K}$, A_j and C as in (2.3), (9.5) and (10.9), respectively. Furthermore, let

$$\Theta = \begin{pmatrix} \theta_{00} & \theta_{01} \\ \theta_{10} & \theta_{11} \\ \vdots & \vdots \\ \theta_{N+1,0} & \theta_{N+1,1} \end{pmatrix} : \begin{pmatrix} \mathbb{C}^{mN+qN+p} \\ \mathbb{C}^q \end{pmatrix} \to \begin{pmatrix} \mathbb{C}^p \\ \mathbb{C}^q \\ \vdots \\ \mathbb{C}^q \end{pmatrix}$$

be its decomposition in conformity with (10.9). By Theorem 9.2, all the $\mathbb{C}^{(p+Nq)\times q}$-valued Schur multipliers satisfying conditions (10.8) are parametrized by the linear fractional transformation (9.7):

$$\begin{pmatrix} S_0(z) \\ S_1(z) \\ \vdots \\ S_N(z) \end{pmatrix} = \begin{pmatrix} \theta_{00}(z)\sigma(z) + \theta_{01}(z) \\ \theta_{10}(z)\sigma(z) + \theta_{11}(z) \\ \vdots \\ \theta_{N0}(z)\sigma(z) + \theta_{N1}(z) \end{pmatrix} \big(\theta_{N+1,0}(z)\sigma(z) + \theta_{N+1,1}(z)\big)^{-1},$$

where the parameter σ varies on the class of $\mathbb{C}^{p\times q}$-valued Schur multipliers. Upon substituting linear fractional expressions for $S_0, \ldots, S_N$ into (10.4), we get

$$\begin{aligned} H(z) &= [\theta_{00}(z)\sigma(z) + \theta_{01}(z)][\theta_{N+1,0}(z)\sigma(z) + \theta_{N+1,1}(z)]^{-1} \\ &\quad \times [I_q - z_1(\theta_{10}(z)\sigma(z) + \theta_{11}(z))(\theta_{N+1,0}(z)\sigma(z) + \theta_{N+1,1}(z))^{-1} \\ &\quad - \cdots - z_N(\theta_{N0}(z)\sigma(z) + \theta_{N1}(z))(\theta_{N+1,0}(z)\sigma(z) \\ &\quad + \theta_{N+1,1}(z))^{-1}]^{-1} \\ &= [\theta_{00}(z)\sigma(z) + \theta_{01}(z)][\theta_{N+1,0}(z)\sigma(z) + \theta_{N+1,1}(z) \\ &\quad - z_1(\theta_{10}(z)\sigma(z) + \theta_{11}(z)) - \cdots - z_N(\theta_{N0}(z)\sigma(z) + \theta_{N1}(z))]^{-1} \\ &= (\Psi_{11}(z)\sigma(z) + \Psi_{12}(z))(\Psi_{21}(z)\sigma(z) + \Psi_{22}(z))^{-1}, \end{aligned}$$

where

$$\Psi(z) = \begin{pmatrix} \Psi_{11}(z) & \Psi_{12}(z) \\ \Psi_{21}(z) & \Psi_{22}(z) \end{pmatrix}$$
$$= \begin{pmatrix} I_p & 0 & 0 & 0 & 0 \\ 0 & -z_1 I_q & \cdots & -z_N I_q & I_q \end{pmatrix} \Theta(z). \tag{10.12}$$

Substituting (8.2) into (10.12) and taking into account that

$$\begin{pmatrix} I_p & 0 & 0 & 0 & 0 \\ 0 & -z_1 I_q & \cdots & -z_N I_q & I_q \end{pmatrix} C \left(I_n - \sum_{j=1}^{N} z_j A_j \right)^{-1}$$
$$= \begin{pmatrix} \frac{a_1}{1-\langle z, w^{(1)}\rangle} & \cdots & \frac{a_m}{1-\langle z, w^{(m)}\rangle} \\ c_1 & \cdots & c_m \end{pmatrix},$$

we get (10.11). □

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