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Kelvin-Möbius-invariant harmonic function spaces on the real unit ball

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ABSTRACT

We define the Kelvin-Möbius transform of a function harmonic on the unit ball of $\mathbb{R}^n$ and determine harmonic function spaces that are invariant under this transform. When $n \geq 3$, in the category of Banach spaces, the minimal Kelvin-Möbius-invariant space is the Bergman-Besov space $b^1_{-(1+n/2)}$ and the maximal invariant space is the Bloch space $b^\infty_{(n-2)/2}$. There exists a unique strictly Kelvin-Möbius-invariant Hilbert space, and it is the Bergman-Besov space b^2_{-2}. There is a unique Kelvin-Möbius-invariant Hardy space.

1. Introduction

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disc and $\operatorname{Aut}(\mathbb{D})$ be its automorphism group, that is, the group of holomorphic, bijective maps of $\mathbb{D}$. The Bloch space $\mathcal{B}$ consists of all holomorphic functions f on $\mathbb{D}$ such that the seminorm

$$\rho_{\mathcal{B}}(f) := \sup\{(1-|z|^2)|f'(z)| : z \in \mathbb{D}\}$$

is finite. The Bloch space is Möbius invariant in the sense that if $f \in \mathcal{B}$, then for every $\varphi \in \operatorname{Aut}(\mathbb{D})$, $f \circ \varphi$ is in $\mathcal{B}$ and $\rho_{\mathcal{B}}(f \circ \varphi) = \rho_{\mathcal{B}}(f)$. The quantity $(1-|z|^2)f'(z)$ is sometimes called the invariant derivative of f at z since its modulus is Möbius invariant in this particular sense.

More generally, let E be a linear space of holomorphic functions on $\mathbb{D}$ that is complete with respect to a seminorm ρ_E. Roughly speaking, E is called Möbius invariant if for every $f \in E$ and $\varphi \in \operatorname{Aut}(\mathbb{D})$, $f \circ \varphi$

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belongs to E and $\rho_E(f\circ\varphi)=\rho_E(f)$ (or the weaker condition $\rho_E(f\circ\varphi)\sim\rho_E(f)$ holds). We say "roughly speaking" because there should be a few technical restrictions on E; for details see [3], [5] and [25]. Among all Möbius-invariant spaces, the Bloch space $\mathcal{B}$ is the largest [25] and the Besov space B^1 is the smallest [5]. If one considers Hilbert spaces, the Dirichlet space is the unique Möbius-invariant Hilbert space [3]. Similar results with domains other than $\mathbb{D}$ are obtained in various sources, in [24] and [32] the domain is the unit ball of $\mathbb{C}^n$, in [21] the domain is the polydisc and [4] considers bounded symmetric domains.

The purpose of this paper is to study the *harmonic* analogue of this problem on the unit ball $\mathbb{B}$ of $\mathbb{R}^n$ with respect to the standard norm $|x|^2=x\cdot x=x_1^2+\cdots+x_n^2$. Let $h(\mathbb{B})$ be the Fréchet space of all complex-valued harmonic functions on $\mathbb{B}$ endowed with the topology of uniform convergence on compact subsets, and $\mathcal{M}(\mathbb{B})$ be the group of Möbius transformations of $\mathbb{B}$; see subsection 2.2. In the harmonic case the first problem one encounters is that if $f\in h(\mathbb{B})$ and $\varphi\in\mathcal{M}(\mathbb{B})$, then $f\circ\varphi$ need not be harmonic. To remedy this problem note that any $\varphi\in\mathcal{M}(\mathbb{B})$ can be written as a composition of an orthogonal transformation and an inversion. Composing a harmonic function with an orthogonal transformation preserves harmonicity. So, harmonicity is lost in $f\circ\varphi$ because of composition with an inversion. On the other hand, one can compose with an inversion and still preserve harmonicity provided one multiplies by a correction factor and this is called the Kelvin transform. Therefore we need to combine composition with φ with the Kelvin transform and this leads us to the following definition.

Definition 1.1. Let $f\in h(\mathbb{B})$ and $\varphi\in\mathcal{M}(\mathbb{B})$. The Kelvin-Möbius transform $\mathcal{K}_\varphi(f)$ of f is defined as

$$\mathcal{K}_\varphi(f)(x):=\left(\frac{1-|\varphi(x)|^2}{1-|x|^2}\right)^{(n-2)/2} f(\varphi(x)).$$

The factor multiplying $f\circ\varphi$ stems from the Kelvin transform; see subsection 2.3. If f is harmonic on $\mathbb{B}$, then so is $\mathcal{K}_\varphi(f)$. This is verified later, but in the literature there are also different proofs. See, for example, [2, Corollary 2.3], [9, §2.3], [19, Proposition 3.1] or [23, Theorem 2].

If the dimension is $n=2$, the first factor disappears and we just have $\mathcal{K}_\varphi(f)=f\circ\varphi$. Because of this there are differences between the cases $n=2$ and $n\ge 3$. When $n=2$, the *harmonic* case is very similar to the *holomorphic* case studied in [3,5,25]; nevertheless there are details that require attention. We deal with that in a different work [22] and throughout this paper we consider the case $n\ge 3$.

Remark 1.2. We show in subsection 2.3 that $\mathcal{K}_\varphi: h(\mathbb{B})\to h(\mathbb{B})$ is invertible with $\mathcal{K}_\varphi^{-1}=\mathcal{K}_{\varphi^{-1}}$.

We now define a Kelvin-Möbius-invariant harmonic function space following [25] and [31]. Let $(E,\|\cdot\|_E)$ be a Banach space of harmonic functions on $\mathbb{B}$. A non-zero continuous linear functional L on $h(\mathbb{B})$ is called *decent on E* if L is also continuous on E with respect to the norm $\|\cdot\|_E$.

We denote by **1** the constant function whose value is 1.

Definition 1.3. Let $n\ge 3$. A Banach space $(E,\|\cdot\|_E)$ of harmonic functions on $\mathbb{B}$ is called *Kelvin-Möbius invariant* if the following properties hold:

(i) E contains **1**.
(ii) There exists a decent linear functional on E.
(iii) For every $f\in E$ and $\varphi\in\mathcal{M}(\mathbb{B})$, the Kelvin-Möbius transform $\mathcal{K}_\varphi(f)$ belongs to E and

$$\|\mathcal{K}_\varphi(f)\|_E\le C\|f\|_E, \tag{1}$$

for some constant C independent of f and φ.

Since E is linear we can write the first condition as (i) E contains constant functions.

If the constant in (1) is $C = 1$, then by Remark 1.2, $\|\mathcal{K}_\varphi(f)\|_E = \|f\|_E$ for every $f \in E$ and $\varphi \in \mathcal{M}(\mathbb{B})$ and in this case we call E *strictly* Kelvin-Möbius invariant as done in [5, Definition 1]. In the general case we only have $\|\mathcal{K}_\varphi(f)\|_E \sim \|f\|_E$, where we use the notation $A \sim B$ to mean that A/B is bounded from above and below by some constants that are independent of the parameters involved. If A/B is just bounded above, we write $A \lesssim B$.

Remark 1.4. If E is Kelvin-Möbius invariant, then we can define a new, equivalent norm on E with which E becomes strictly Kelvin-Möbius invariant (see [5, p. 111]). Thus weakening *strictly* Kelvin-Möbius invariance and using Kelvin-Möbius invariance with "$\sim$" is not very important.

Remark 1.5. We note that in the holomorphic case the invariant space is taken as a complete *semi-normed* space. In the harmonic case the same is true when $n = 2$ but when $n \geq 3$ we do not need semi-norms because of the extra multiplying factor in $\mathcal{K}_\varphi$, and E in Definition 1.3 is a Banach space.

The harmonic function spaces we are mainly interested in this work are Bergman-Besov spaces b^p_q and weighted Bloch spaces b^∞_s with $q, s \in \mathbb{R}$ in both. Let ν be the normalized Lebesgue measure on $\mathbb{B}$ and for $q \in \mathbb{R}$, let

$$d\nu_q(x) := c_q(1 - |x|^2)^q \, d\nu(x).$$

The measure ν_q is finite only when $q > -1$ and in this case we pick the constant c_q so that $\nu_q(\mathbb{B}) = 1$. When $q \leq -1$, we just set $c_q = 1$. For $0 < p < \infty$, we denote the Lebesgue classes with respect to ν_q by L^p_q.

For $q > -1$, the well-known harmonic Bergman space b^p_q is defined as $b^p_q := h(\mathbb{B}) \cap L^p_q$ with norm $\|f\|_{b^p_q} := \|f\|_{L^p_q}$. The next definition extends this class to all $q \in \mathbb{R}$. To denote partial derivatives we use multi-indices and write

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

where the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition 1.6. Let $0 < p < \infty$ and $q \in \mathbb{R}$. Pick a nonnegative integer N such that

$$q + pN > -1. \tag{2}$$

The *harmonic Bergman-Besov space* b^p_q consists of all $f \in h(\mathbb{B})$ such that

$$(1 - |x|^2)^N \partial^\alpha f \in L^p_q$$

for every multi-index α with $|\alpha| = N$. A norm (quasinorm when $0 < p < 1$) on b^p_q is

$$\|f\|^p_{b^p_q} := \sum_{|\alpha|<N} |\partial^\alpha f(0)|^p + \sum_{|\alpha|=N} \int_{\mathbb{B}} \left|(1 - |x|^2)^N \partial^\alpha f(x)\right|^p \, d\nu_q(x). \tag{3}$$

When $0 < p < 1$, while we still use the notation $\|\cdot\|_{b^p_q}$, b^p_q is not a normed space, but it is a complete metric space with respect to the metric $d(f, g) = \|f - g\|^p_{b^p_q}$.

The space b^p_q does not depend on the choice of N. Different choices of N satisfying (2) give rise to equivalent norms and in the notation $\|\cdot\|_{b^p_q}$ we do not indicate the dependence on N. The partial derivatives in the above definition can be replaced with radial derivatives or various other suitable differential operators.

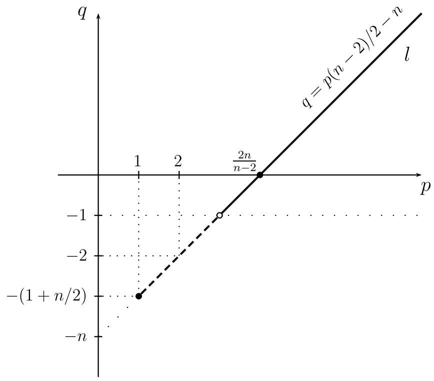

Fig. 1. Equation of the ray l is $q = p(n-2)/2 - n$, $p \geq 1$. By Lemma 1.7, if (p, q) is on the solid part of the ray l, then b^p_q is Kelvin-Möbius invariant.

The region $q > -1$ is the Bergman region in which we take $N = 0$, and $q \leq -1$ is the proper Besov region. When $p = 2$, b^2_q is a Hilbert space endowed with the inner product

$$\langle f, g\rangle_{b^2_q} := \sum_{|\alpha|<N} \partial^\alpha f(0)\overline{\partial^\alpha g(0)} + \sum_{|\alpha|=N} \int_{\mathbb{B}} \partial^\alpha f(x)\overline{\partial^\alpha g(x)}(1-|x|^2)^{2N}\, d\nu_q(x), \tag{4}$$

with N satisfying (2). In the special case $q = -n$, the spaces b^p_{-n} are studied in [20,29]. For the whole family b^p_q with $q \in \mathbb{R}$, see [17] (when $p \geq 1$) and [13] (when $0 < p < 1$).

We first determine which harmonic Bergman-Besov spaces are Kelvin-Möbius invariant. In the Bergman region $q > -1$, this is very easy.

Lemma 1.7. *Let $n \geq 3$. Let $1 \leq p < \infty$ and $q > -1$. Then b^p_q is Kelvin-Möbius invariant if and only if $q = p(n-2)/2 - n$ and in this case b^p_q is strictly Kelvin-Möbius invariant.*

It is clear that $\mathbf{1} \in b^p_q$, and it is well known that point-evaluation functionals are bounded on Bergman spaces. To check condition (iii) of Definition 1.3, let $f \in b^p_q$ and $\varphi \in \mathcal{M}(\mathbb{B})$. Then

$$\|\mathcal{K}_\varphi(f)\|^p_{b^p_q} = c_q \int_{\mathbb{B}} |f(\varphi(x))|^p \left(\frac{1-|\varphi(x)|^2}{1-|x|^2}\right)^{p(n-2)/2} (1-|x|^2)^q\, d\nu(x).$$

Using $p(n-2)/2 = q + n$, changing variables as $\tilde{x} = \varphi(x)$, and using (12) below, we immediately obtain $\|\mathcal{K}_\varphi(f)\|_{b^p_q} = \|f\|_{b^p_q}$. We show the only-if part of Lemma 1.7 in Section 4 within the proof of Theorem 1.8.

Lemma 1.7 tells that if (p, q) is on the solid part of the ray l in Fig. 1, then the space b^p_q is Kelvin-Möbius invariant. This suggests that the Banach spaces corresponding to the dashed part of the ray l may also be Kelvin-Möbius invariant. This is true and it is our first theorem.

Theorem 1.8. *Let $n \geq 3$. Let $p \geq 1$ and $q \in \mathbb{R}$. Then b^p_q is a Kelvin-Möbius invariant space if and only if $q = p(n-2)/2 - n$.*

Of course the interesting part of Theorem 1.8 is the case $q \leq -1$. We prove this part in Section 4 using complex interpolation.

By Remark 1.4, when $q = p(n-2)/2 - n$, the space b^p_q is *strictly* Kelvin-Möbius invariant when endowed with a suitable equivalent norm.

Remark 1.9. Theorem 1.8 is true also when $n = 2$. In this case $q = p(n-2)/2 - n = -2$ is less than -1 and there is no Kelvin-Möbius-invariant harmonic *Bergman* space. When $n = 2$, as in the holomorphic case, all Kelvin-Möbius-invariant spaces belong to the proper Besov region and lie on the horizontal line $q = -2$.

Observe that when $p = 2$, the corresponding q in Theorem 1.8 is $q = -2$ and the Hilbert space $(b^2_{-2}, \langle\cdot,\cdot\rangle_{b^2_{-2}})$ is Kelvin-Möbius invariant. The space b^2_{-2} is *strictly* Kelvin-Möbius invariant when endowed with a suitable inner product $\langle\cdot,\cdot\rangle_I$ which we now describe.

For $m \geq 0$, let $\mathcal{H}_m(\mathbb{S})$ be the vector space of spherical harmonics of degree m and δ_m be its dimension (see subsection 3.1 for more details). Let $\{Y_m^j : j = 1, \dots, \delta_m\}$ be an orthonormal basis of $\mathcal{H}_m(\mathbb{S})$. If f is harmonic on $\mathbb{B}$, then f has the expansion

$$f(x) = \sum_{m=0}^{\infty} \sum_{j=1}^{\delta_m} f_m^j Y_m^j(x) \qquad (x \in \mathbb{B}),$$

where f_m^j's are complex numbers. The above series converges absolutely and uniformly on compact subsets of $\mathbb{B}$. By [16, Theorem 3.8] or [17, Theorem 5.1], the space b^2_{-2} can also be described as

$$b^2_{-2} = \left\{ f = \sum_{m=0}^{\infty} \sum_{j=1}^{\delta_m} f_m^j Y_m^j \in h(\mathbb{B}) : \|f\|_I^2 = \sum_{m=0}^{\infty} \sum_{j=1}^{\delta_m} \frac{m + n/2 - 1}{n/2 - 1} |f_m^j|^2 < \infty \right\}.$$

The above norm is equivalent to the norm(s) given in Definition 1.6 and is induced by the inner product

$$\langle f, g\rangle_I := \sum_{m=0}^{\infty} \sum_{j=1}^{\delta_m} \frac{m + n/2 - 1}{n/2 - 1} f_m^j \overline{g_m^j} \qquad (f, g \in b^2_{-2}). \tag{5}$$

When endowed with the above inner product $\langle\cdot,\cdot\rangle_I$, the space b^2_{-2} is *strictly* Kelvin-Möbius-invariant Hilbert space and it is the only strictly Kelvin-Möbius-invariant Hilbert space. This is our second theorem.

Theorem 1.10. (i) *The Hilbert space $(b^2_{-2}, \langle\cdot,\cdot\rangle_I)$ is strictly Kelvin-Möbius invariant.*
(ii) *If $(H, \langle\cdot,\cdot\rangle_H)$ is a strictly Kelvin-Möbius-invariant Hilbert space, then $H = b^2_{-2}$ and $\langle\cdot,\cdot\rangle_H = C\langle\cdot,\cdot\rangle_I$ for some $C > 0$.*

In Proposition 6.1, we give an integral description of the inner product $\langle\cdot,\cdot\rangle_I$ similar to (4), but in terms of radial derivatives.

When (p, q) is on the ray $q = p(n-2)/2 - n$, $p \geq 1$, the Kelvin-Möbius-invariant spaces b^p_q increase as p increases. This is a consequence of [15, Theorem 1.2]. If $p = 1$, the corresponding q is $-(1 + n/2)$, and this suggests that $b^1_{-(1+n/2)}$ might be the *smallest* Kelvin-Möbius-invariant space. This is true and it is our next theorem.

Theorem 1.11. *Let $n \geq 3$. The space $b^1_{-(1+n/2)}$ is the smallest Kelvin-Möbius-invariant space. More precisely, if E is Kelvin-Möbius invariant, then $b^1_{-(1+n/2)} \subset E$ and there exists a $C > 0$ such that $\|f\|_E \leq C\|f\|_{b^1_{-(1+n/2)}}$ for every $f \in b^1_{-(1+n/2)}$.*

We next determine the *largest* Kelvin-Möbius-invariant space. For this we need to consider weighted harmonic Bloch spaces.

Definition 1.12. Let $s \in \mathbb{R}$. Pick a non-negative integer N so that $s + N > 0$. The *weighted harmonic Bloch space* b^∞_s consists of all $f \in h(\mathbb{B})$ such that for every multi-index α with $|\alpha| = N$,

$$\sup_{x\in\mathbb{B}} (1-|x|^2)^{s+N}\,|\partial^\alpha f(x)| < \infty.$$

A norm on b^∞_s is

$$\|f\|_{b^\infty_s} := \sum_{|\alpha|<N} |\partial^\alpha f(0)| + \sum_{|\alpha|=N} \sup_{x\in\mathbb{B}} (1-|x|^2)^{s+N}\,|\partial^\alpha f(x)|\,.$$

As before, the space b^∞_s does not depend on the choice of N as long as $s+N>0$ and different choices of N give rise to equivalent norms (see [14] for more details). When $n\ge 3$ we are interested in the space $b^\infty_{(n-2)/2}$. Choosing $N=0$ shows

$$b^\infty_{(n-2)/2} = \Big\{ f\in h(\mathbb{B}) : \|f\|_{b^\infty_{(n-2)/2}} = \sup_{x\in\mathbb{B}} (1-|x|^2)^{(n-2)/2}|f(x)| < \infty \Big\}. \tag{6}$$

It is immediate that the space $b^\infty_{(n-2)/2}$ is strictly Kelvin-Möbius invariant. Clearly $\mathbf{1}\in b^\infty_{(n-2)/2}$ and for a decent linear functional, we can take any point-evaluation functional. That $\|\mathcal{K}_\varphi(f)\|_{b^\infty_{(n-2)/2}} = \|f\|_{b^\infty_{(n-2)/2}}$ is obvious.

By [15, Theorem 1.3], the increasing family of Kelvin-Möbius-invariant spaces b^p_q, $p\ge 1$, $q=p(n-2)/2-n$ are all included in the weighted Bloch space $b^\infty_{(n-2))/2}$. This suggests that $b^\infty_{(n-2)/2}$ might be the largest Kelvin-Möbius-invariant space. This is true and it is our next theorem.

Theorem 1.13. *Let $n\ge 3$. The space $b^\infty_{(n-2)/2}$ is the largest Kelvin-Möbius-invariant space. More precisely, if E is Kelvin-Möbius invariant, then $E\subset b^\infty_{(n-2)/2}$ and there exists a $C>0$ such that $\|f\|_{b^\infty_{(n-2)/2}} \le C\|f\|_E$ for every $f\in E$.*

For completeness we also look at the spaces b^p_q, $0<p<1$. Because these are not normed spaces, they cannot be Kelvin-Möbius invariant in the sense of Definition 1.3. Nevertheless if $q=p(n-2)/2-n$, the properties (i)-(iii) in Definition 1.3 do hold even when $0<p<1$, where we understand $\|\cdot\|$ as a quasinorm.

Theorem 1.14. *Let $n\ge 3$. Let $0<p<1$ and $q\in\mathbb{R}$. Then the properties* (i)-(iii) *in Definition 1.3 hold for b^p_q if and only if $q=p(n-2)/2-n$.*

We note that the spaces b^p_q, $0<p<1$, $q=p(n-2)/2-n$ are smaller than $b^1_{-(1+n/2)}$ by [15, Theorem 1.2]. This, however, does not contradict Theorem 1.11 since these spaces are not Banach spaces. An analogous result concerning Möbius invariant *holomorphic* function spaces on the unit ball of $\mathbb{C}^n$ is obtained in [33].

Another well-studied space of harmonic functions is the class of harmonic Hardy spaces. For each $n\ge 3$, one Hardy space can be shown to be Kelvin-Möbius invariant by using [19, Theorem 1.2]. We discuss this in Remark 4.4.

Lastly, we consider subspaces of $h(\mathbb{B})$ and show that there is no non-trivial closed subspace of $h(\mathbb{B})$ that is invariant under taking Kelvin-Möbius transform.

Theorem 1.15. *Let $n\ge 3$ and $A\subset h(\mathbb{B})$ be a closed subspace. If $\mathcal{K}_\varphi(f)\in A$ for every $f\in A$ and $\varphi\in\mathcal{M}(\mathbb{B})$, then $A=\{0\}$ or $A=h(\mathbb{B})$.*

Corollary 1.16. *Let $n\ge 3$ and $f\in h(\mathbb{B})$ be non-zero. Then* $\operatorname{span}\{\mathcal{K}_\varphi(f):\varphi\in\mathcal{M}(\mathbb{B})\}$ *is dense in $h(\mathbb{B})$.*

The paper is organized as follows. In Section 2 we justify Definition 1.1 and present some elementary properties of the Kelvin-Möbius transform. In Section 3 we review reproducing kernels and atomic decompositions of harmonic Bergman-Besov spaces b^p_q. We consider the Kelvin-Möbius invariance of the b^p_q spaces and prove Theorems 1.8 and 1.14 in Section 4. In Section 5 we first show that point-evaluation functionals are bounded on Kelvin-Möbius-invariant spaces and as a consequence determine the minimal and maximal invariant spaces. We prove Theorem 1.15 also in this section. Section 6 is devoted to the Hilbert-space case. We repeat that in this paper we take $n \geq 3$.

2. Kelvin-Möbius transform

In this section we first recall some known facts about Möbius transformations. For more detail about these transformations see [1,7,30]. We then justify the definition of Kelvin-Möbius transform and prove some of its elementary properties.

2.1. Orthogonal transformations

We denote the group of all orthogonal transformations of $\mathbb{R}^n$ by $O(n)$. Fixing an orthonormal basis of $\mathbb{R}^n$ we can represent each element of $O(n)$ with an orthogonal matrix of size $n \times n$. Identifying matrices of size $n \times n$ with elements of $\mathbb{R}^{n^2}$ induces a natural topology on such matrices which makes $O(n)$ a compact topological group. The elements of $O(n)$ whose corresponding matrices have determinant 1 form a subgroup of $O(n)$ denoted by $SO(n)$. We denote the normalized Haar measure on $SO(n)$ by μ_0, where normalized means $\mu_0(SO(n)) = 1$.

For a proof of the lemma below see [11, Theorem 3.1] or [26, Lemma 1.4.7 (3)].

Lemma 2.1. *Let f be continuous on $\mathbb{S}$ and $\eta \in \mathbb{S}$. Then*

$$\int_{SO(n)} f(U(\eta))\, d\mu_0(U) = \int_{\mathbb{S}} f(\zeta)\, d\sigma(\zeta).$$

2.2. Möbius transformations

A Möbius transformation of $\hat{\mathbb{R}}^n := \mathbb{R}^n \cup \{\infty\}$ is a finite composition of inversions in spheres and reflections in planes. We denote the group of all Möbius transformations that map the unit ball $\mathbb{B}$ to itself by $\mathcal{M}(\mathbb{B})$.

For $a \in \mathbb{B}$, the involutive Möbius transformation φ_a that exchanges a and 0 is defined by

$$\varphi_a(x) := \frac{(1-|a|^2)(a-x) + |x-a|^2 a}{[x,a]^2},$$

where

$$[x,a] := \sqrt{1 - 2x \cdot a + |x|^2|a|^2}.$$

The map φ_a can be decomposed into simple maps. Let J be the inversion with respect to the unit sphere $\mathbb{S}$,

$$J(x) := x^* := \frac{x}{|x|^2}.$$

Note that when $a \neq 0$,

$$[x,a] = |a|\,|x - a^*|. \tag{7}$$

More generally, for $c \in \mathbb{R}^n$ and $r > 0$, let $\mathbb{S}(c, r)$ be the sphere with center c and radius r, and let $J_{c,r}$ denote the inversion with respect to $\mathbb{S}(c, r)$,

$$J_{c,r}(x) := c + r^2 J(x - c).$$

For $0 \neq a \in \mathbb{B}$, φ_a can be decomposed as

$$\varphi_a = -P_a \circ J_{a^*,\rho},$$

where

$$\rho := \frac{\sqrt{1 - |a|^2}}{|a|} \tag{8}$$

and

$$P_a(x) := x - 2\frac{x \cdot a}{|a|^2}a$$

is the reflection about the plane passing through the origin and perpendicular to the vector a. For details see [1, Section 2.6] which uses $T_a = -\varphi_a$. The sphere $\mathbb{S}(a^*, \rho)$ is orthogonal to $\mathbb{S}$ and thus $J_{a^*,\rho}(\mathbb{B}) = \mathbb{B}$.

Let $\varphi \in \mathcal{M}(\mathbb{B})$ be arbitrary and $a = \varphi^{-1}(0)$. If $a = 0$, then φ is an orthogonal transformation. If $a \neq 0$, then by [30, Theorem 2.1.2] there exists an orthogonal transformation $U \in O(n)$ such that $\varphi = U \circ \varphi_a$ and φ can be decomposed as

$$\varphi = U \circ (-P_a) \circ J_{a^*,\rho}. \tag{9}$$

Thus, since P_a is also orthogonal, we can write φ as composition of an orthogonal transformation and an inversion.

We list some identities involving φ. The most useful identity is

$$1 - |\varphi(x)|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{[x, a]^2} \qquad (a = \varphi^{-1}(0)). \tag{10}$$

The ratio $[x, y]^2/((1 - |x|^2)(1 - |y|^2))$ is Möbius invariant, that is, for every $x, y \in \mathbb{B}$ and $\varphi \in \mathcal{M}(\mathbb{B})$,

$$\frac{[\varphi(x), \varphi(y)]^2}{(1 - |\varphi(x)|^2)(1 - |\varphi(y)|^2)} = \frac{[x, y]^2}{(1 - |x|^2)(1 - |y|^2)}. \tag{11}$$

The derivative $\varphi'(x)$ of $\varphi = (\varphi_1, \ldots, \varphi_n) : \mathbb{B} \to \mathbb{B}$, is the $n \times n$ matrix

$$\varphi'(x) = \left[\frac{\partial \varphi_i}{\partial x_j}\right]_{i,j=1}^{n}.$$

The absolute value of the Jacobian determinant of φ is ([30, Theorem 3.3.1])

$$|\det(\varphi'(x))| = \left(\frac{1 - |\varphi(x)|^2}{1 - |x|^2}\right)^n. \tag{12}$$

2.3. Kelvin-Möbius transform

The Kelvin transform of f with respect to the unit sphere $\mathbb{S}$ is defined by

$$K_{\mathbb{S}}(f)(x) := \frac{1}{|x|^{n-2}} f(J(x)).$$

More generally, *Kelvin transform* of f with respect to the sphere $\mathbb{S}(c,r)$ is defined by (see [18, p. 39])

$$K_{\mathbb{S}(c,r)}(f)(x) := \frac{r^{n-2}}{|x-c|^{n-2}} f(J_{c,r}(x)).$$

If f is harmonic on a domain $\Omega \subset \mathbb{R}^n$, then $K_{\mathbb{S}(c,r)}(f)$ is harmonic on $J_{c,r}(\Omega)$.

Let $\varphi \in \mathcal{M}(\mathbb{B})$, $\varphi^{-1}(0) = a \neq 0$ and φ have the decomposition (9). By (7) and (8),

$$K_{\mathbb{S}(a^*,\rho)}(f)(x) = \frac{(1-|a|^2)^{(n-2)/2}}{[x,a]^{n-2}} f(J_{a^*,\rho}(x)).$$

Because $J_{a^*,\rho}(\mathbb{B}) = \mathbb{B}$, if f is harmonic on $\mathbb{B}$, then $K_{\mathbb{S}(a^*,\rho)}(f)$ is harmonic on $\mathbb{B}$. Replacing f with $f \circ U \circ (-P_a)$ we deduce that if $f \in h(\mathbb{B})$, then so is

$$\frac{(1-|a|^2)^{(n-2)/2}}{[x,a]^{n-2}} f(\varphi(x)) = \left(\frac{1-|\varphi(x)|^2}{1-|x|^2}\right)^{(n-2)/2} f(\varphi(x)),$$

where we also use the formula (10). The function above is the *Kelvin-Möbius transform* of f.

If $\varphi = U \in O(n)$, then we just have

$$\mathcal{K}_U(f) = f \circ U. \tag{13}$$

For future reference let us also record that if $f = \mathbf{1}$, then

$$\mathcal{K}_{\varphi_a}(\mathbf{1})(x) = \frac{(1-|a|^2)^{(n-2)/2}}{[x,a]^{n-2}}. \tag{14}$$

Some basic properties of the transform $\mathcal{K}_\varphi$ are listed in the following lemma.

Lemma 2.2. *Let $\varphi, \psi \in \mathcal{M}(\mathbb{B})$. The Kelvin-Möbius transform $\mathcal{K}_\varphi : h(\mathbb{B}) \to h(\mathbb{B})$ satisfies the following properties:*

(i) *$\mathcal{K}_\varphi$ is linear.*
(ii) *$\mathcal{K}_\psi \circ \mathcal{K}_\varphi = \mathcal{K}_{\varphi\circ\psi}$.*
(iii) *$\mathcal{K}_\varphi$ is one-to-one and onto and $\mathcal{K}_\varphi^{-1} = \mathcal{K}_{\varphi^{-1}}$.*
(iv) *$\mathcal{K}_{\varphi_a}^{-1} = \mathcal{K}_{\varphi_a}$.*
(v) *$\mathcal{K}_\varphi$ is continuous.*

Proof. Part (i) is clear. Part (ii) is pure computation. Part (iii) follows from part (ii) and part (iv) is true because φ_a is an involution. To see part (v) suppose $f_m \to f$ in $h(\mathbb{B})$, that is, f_m converges to f uniformly on compact subsets of $\mathbb{B}$. We have

$$\mathcal{K}_\varphi(f_m)(x) = \left(\frac{1-|\varphi(x)|^2}{1-|x|^2}\right)^{(n-2)/2} f_m(\varphi(x))$$

and if x lies in the compact set $|x| \le r < 1$, then there exists $s < 1$ such that $|\varphi(x)| \le s$. The first factor on the right is bounded and second factor converges uniformly to $f(\varphi(x))$ since f_m converges uniformly to f on $|x| \le s$. Hence $\mathcal{K}_\varphi(f_m)$ converges uniformly to $\mathcal{K}_\varphi(f)$ on $|x| \le r$. □

3. Harmonic Bergman-Besov spaces

This section is for review purposes. We recall some known facts about zonal harmonics and harmonic Bergman-Besov spaces b^p_q that will be used in the sequel. For more detail about zonal harmonics see [6, Chapter 5], and about the spaces b^p_q see [17] and [13].

3.1. Spherical and zonal harmonics

Let $L^2(\mathbb{S})$ be the Hilbert space of square integrable functions on $\mathbb{S}$ with respect to the inner product $\langle f, g\rangle = \int_{\mathbb{S}} f\overline{g}\, d\sigma$, where σ is the normalized surface area measure on $\mathbb{S}$. Let $\mathcal{H}_m(\mathbb{R}^n)$ denote the complex vector space of all homogeneous harmonic polynomials of degree m in n real variables. Restriction of an element of $\mathcal{H}_m(\mathbb{R}^n)$ to $\mathbb{S}$ is called a (surface) spherical harmonic of degree m. The collection $\mathcal{H}_m(\mathbb{S})$ of all spherical harmonics of degree m is a finite-dimensional subspace of $L^2(\mathbb{S})$ with dimension δ_m.

For $m \ge 0$, let $\{Y^j_m,\ j = 1, \dots, \delta_m\}$ be an orthonormal basis of $\mathcal{H}_m(\mathbb{S})$. If $m \ne k$, then $\mathcal{H}_m(\mathbb{S}) \perp \mathcal{H}_k(\mathbb{S})$ in $L^2(\mathbb{S})$, so

$$\int_{\mathbb{S}} Y^j_m(\xi)\overline{Y^i_k(\xi)}\, d\sigma(\xi) = 0 \tag{15}$$

unless $m = k$ and $j = i$. For $\eta \in \mathbb{S}$, the point-evaluation functional $f \mapsto f(\eta)$ is bounded on $\mathcal{H}_m(\mathbb{S})$ and therefore there exists a unique $Z_m(\cdot, \eta) \in \mathcal{H}_m(\mathbb{S})$ such that for all $f \in \mathcal{H}_m(\mathbb{S})$

$$f(\eta) = \int_{\mathbb{S}} f(\xi)\overline{Z_m(\xi, \eta)}\, d\sigma(\xi).$$

The zonal harmonic $Z_m(\xi, \eta)$ is real-valued, it is symmetric in its variables, and in terms of Y^j_m it equals

$$Z_m(\xi, \eta) = \sum_{j=1}^{\delta_m} Y^j_m(\xi)\overline{Y^j_m(\eta)} \qquad (\xi, \eta \in \mathbb{S}).$$

The above formula extends to $\mathbb{R}^n \times \mathbb{R}^n$ by homogeneity

$$Z_m(x, y) = \sum_{j=1}^{\delta_m} Y^j_m(x)\overline{Y^j_m(y)} \qquad (x, y \in \mathbb{R}^n), \tag{16}$$

where $Y^j_m(x) = |x|^m Y^j_m(\xi)$ for $x = |x|\xi$.

When $n \ge 3$, as a function of x, $1/[x, a]^{n-2}$ is harmonic on $\overline{\mathbb{B}}$ by (7). Its homogeneous expansion is given in the following lemma.

Lemma 3.1. *Let $n \ge 3$ and $a \in \mathbb{B}$. Then*

$$\frac{1}{[x, a]^{n-2}} = \sum_{m=0}^{\infty} \frac{n/2 - 1}{m + n/2 - 1} Z_m(x, a),$$

where the series converges uniformly for $x \in \overline{\mathbb{B}}$.

Proof. Lemma is clear when $a = 0$, so we assume $a \neq 0$. For $d > 0$, the Gegenbauer polynomial G_m^d of degree m is defined by the generating function

$$\frac{1}{(1-2rt+t^2)^d} = \sum_{m=0}^{\infty} G_m^d(r)\, t^m.$$

Writing $x = |x|\xi$, $a = |a|\eta$, we see that

$$\frac{1}{[x,a]^{n-2}} = \frac{1}{(1-2\xi\cdot\eta\,|x||a| + |x|^2|a|^2)^{n/2-1}} = \sum_{m=0}^{\infty} G_m^{n/2-1}(\xi\cdot\eta)|x|^m|a|^m.$$

It is known that (see, for example, [17, Equation (14.8)])

$$G_m^{n/2-1}(\xi\cdot\eta) = \frac{n/2-1}{m+n/2-1} Z_m(\xi,\eta).$$

Thus

$$\frac{1}{[x,a]^{n-2}} = \sum_{m=0}^{\infty} \frac{n/2-1}{m+n/2-1} Z_m(\xi,\eta)|x|^m|a|^m = \sum_{m=0}^{\infty} \frac{n/2-1}{m+n/2-1} Z_m(x,a),$$

where the last equality follows from the homogeneity of Z_m. For fixed $a \in \mathbb{B}$, the series converges uniformly for $x \in \overline{\mathbb{B}}$ since $|Z_m(\xi,\eta)| \lesssim m^{n-2}$ (see [6, Proposition 5.27 (e) and Exercise 10, p. 107]). □

3.2. Reproducing kernels of harmonic Bergman-Besov spaces

For all $q \in \mathbb{R}$, the space b_q^2 is a reproducing kernel Hilbert space. We denote the reproducing kernel by $R_q(x,\cdot)$. When $q > -1$, the natural inner product on b_q^2 is $\langle f,g\rangle_{b_q^2} := \int_{\mathbb{B}} f\overline{g}\, d\nu_q$ and with respect to this inner product (see [12, p. 164])

$$R_q(x,y) = \sum_{m=0}^{\infty} \frac{(n/2+q+1)_m}{(n/2)_m} Z_m(x,y) \qquad (q > -1),$$

where the Pochhammer symbol $(a)_b$ is defined by

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ.

When $q \leq -1$, it is necessary to consider derivatives of the functions in the inner product and there are various choices. Different choices of the inner product would give rise to different reproducing kernels and vice versa. We follow the approach of [16] and extend the reproducing kernels to whole $q \in \mathbb{R}$ in the following way. Define

$$\gamma_m(q) := \begin{cases} \dfrac{(n/2+q+1)_m}{(n/2)_m}, & \text{if } q > -(1+n/2); \\[2ex] \dfrac{(m!)^2}{(1-(n/2+q))_m (n/2)_m}, & \text{if } q \leq -(1+n/2); \end{cases} \tag{17}$$

and

$$R_q(x,y) := \sum_{m=0}^{\infty} \gamma_m(q) Z_m(x,y).$$

For an (integral) inner product that makes this $R_q(x,y)$ the reproducing kernel of b^2_q ($q \in \mathbb{R}$) see [17, Theorem 5.2]. Every $R_q(x,y)$ is symmetric in its variables since every $Z_m(x,y)$ is.

Using the reproducing kernels above we can define radial differential operators D^t_s of order t for every $t, s \in \mathbb{R}$. If $f \in h(\mathbb{B})$ with homogeneous expansion $f = \sum_{m=0}^{\infty} f_m$, then we define

$$D^t_s f := \sum_{m=0}^{\infty} \frac{\gamma_m(s+t)}{\gamma_m(s)} f_m.$$

The operators D^t_s are compatible with reproducing kernels in the sense that for every $t, s \in \mathbb{R}$

$$D^t_s R_s(x,y) = R_{s+t}(x,y), \tag{18}$$

where the operator acts on the first variable. Because of this in the study of the properties of harmonic Bergman-Besov spaces it is more convenient to use the operators D^t_s rather than the partial derivatives. Similar to Definition 1.6 the spaces b^p_q can be described in terms of the operators D^t_s. For $0 < p < \infty$ and $q \in \mathbb{R}$, pick $t \in \mathbb{R}$ such that $q + pt > -1$ and any $s \in \mathbb{R}$. Then $f \in b^p_q$ if and only if

$${}_{(t,s)}\|f\|^p_{b^p_q} := \int_{\mathbb{B}} \left|(1-|x|^2)^t D^t_s f(x)\right|^p \, d\nu_q(x) < \infty \tag{19}$$

and the norm (quasinorm when $0 < p < 1$) ${}_{(t,s)}\|\cdot\|^p_{b^p_q}$ is equivalent to the norm $\|\cdot\|_{b^p_q}$ in (3); see [17, Theorems 1.1 and 1.2] and [13, Theorem 1.1]. We need the D^t_s in the proof of the only-if part of Theorems 1.8 and 1.14.

As suggested by Theorem 1.10, for our purposes the most important kernel is R_{-2}. In this case, by (17) and Lemma 3.1, the following closed formula holds.

Lemma 3.2. *Let $n \geq 3$. Then*

$$R_{-2}(x,y) = \frac{1}{[x,y]^{n-2}} \qquad (x, y \in \mathbb{B}).$$

Combining the above lemma with (14) we see that

$$\mathcal{K}_{\varphi_a}(\mathbf{1})(x) = \frac{(1-|a|^2)^{(n-2)/2}}{[x,a]^{n-2}} = (1-|a|^2)^{(n-2)/2} R_{-2}(x,a). \tag{20}$$

For a proof of the following estimate of the weighted integrals of powers of $R_q(x,y)$, see [17, Theorem 1.5].

Lemma 3.3. *Let $q \in \mathbb{R}$, $a > 0$, and $b > -1$. Set $c = a\,(n+q) - (n+b)$. Then*

$$\int_{\mathbb{B}} |R_q(x,y)|^a (1-|y|^2)^b \, d\nu(y) \sim \begin{cases} \dfrac{1}{(1-|x|^2)^c}, & \text{if } c > 0; \\ 1 + \log \dfrac{1}{1-|x|^2}, & \text{if } c = 0; \\ 1, & \text{if } c < 0. \end{cases}$$

3.3. Atomic decomposition of harmonic Bergman-Besov spaces

Every function in the space b^p_q can be written as an infinite sum of reproducing kernels (atoms). For harmonic Bergman spaces ($q > -1$) this is proved in [10] and is extended to all $q \in \mathbb{R}$ in [17, Theorem 10.1] when $p \geq 1$ and in [13, Theorem 1.4] when $0 < p < 1$. To describe this atomic decomposition we need some definitions.

For $a, b \in \mathbb{B}$, the *pseudo-hyperbolic metric* $d_{\mathcal{H}}(a, b)$ is defined by $d_{\mathcal{H}}(a, b) := |\varphi_a(b)|$. It is well known that $d_{\mathcal{H}}$ is Möbius invariant,

$$d_{\mathcal{H}}(\varphi(a), \varphi(b)) = d_{\mathcal{H}}(a, b) \qquad (\varphi \in \mathcal{M}(\mathbb{B}), a, b \in \mathbb{B}). \tag{21}$$

For $a \in \mathbb{B}$ and $0 < r < 1$, let $D(a, r) := \{x \in \mathbb{B} : d_{\mathcal{H}}(x, a) < r\}$ be the pseudo-hyperbolic ball with center a and radius r. A sequence (a_m) in $\mathbb{B}$ is called *r-separated* if the balls $D(a_m, r)$ are pairwise disjoint. The sequence (a_m) is called an *r-lattice* if $\mathbb{B} = \cup_{m=1}^{\infty} D(a_m, r)$ and (a_m) is $r/2$-separated.

Theorem 3.4. *Let $0 < p < \infty$ and $q \in \mathbb{R}$. Choose s such that*

$$\begin{aligned} q + 1 &< p(s + 1), \quad \textit{if } p \geq 1; \\ q + n &< p(s + n), \quad \textit{if } 0 < p < 1. \end{aligned}$$

There exists an $r_0 > 0$ such that if (a_m) is an r-lattice with $r < r_0$, then the following hold:

(i) *For every $(c_m) \in \ell^p$, the function*

$$f(x) = \sum_{m=1}^{\infty} c_m (1 - |a_m|^2)^{s+n-(q+n)/p}\, R_s(x, a_m) \tag{22}$$

is in b^p_q and $\|f\|_{b^p_q} \lesssim \|(c_m)\|_{\ell^p}$. The series in (22) converges to f absolutely and uniformly on compact subsets of $\mathbb{B}$ and also in $\|\cdot\|_{b^p_q}$.

(ii) *For every $f \in b^p_q$, there exists $(c_m) \in \ell^p$ with $\|(c_m)\|_{\ell^p} \lesssim \|f\|_{b^p_q}$ such that the representation (22) holds.*

4. Kelvin-Möbius-invariant harmonic Bergman-Besov spaces

In this section we consider harmonic Bergman-Besov spaces b^p_q for the whole range $0 < p < \infty$, $q \in \mathbb{R}$ and prove Theorems 1.8 and 1.14. That is, we show that the properties (i)-(iii) in Definition 1.3 hold for b^p_q if and only if $q = p(n-2)/2 - n$. Note that clearly $\mathbf{1} \in b^p_q$ for every p and q and point-evaluation functionals are bounded on all b^p_q by [17, Theorem 13.1] and [13, Theorem 5.1]. Therefore all we need to check is condition (iii) of Definition 1.3.

We first prove the if parts of Theorems 1.8 and 1.14 and defer the only-if parts to the end of the section. We begin with the case $0 < p \leq 1$ which we handle by using atomic decomposition.

Proposition 4.1. *Let $n \geq 3$, $0 < p \leq 1$ and $q = p(n-2)/2 - n$. Then there exists a constant $C > 0$ such that $\|\mathcal{K}_\varphi(f)\|_{b^p_q} \leq C\|f\|_{b^p_q}$ for every $f \in b^p_q$ and $\varphi \in \mathcal{M}(\mathbb{B})$.*

Proof. We apply Theorem 3.4 with $s = -2$ which is possible since $n \geq 3$. Let r_0 be as asserted in that theorem and pick an r-lattice (a_m) with $r < r_0$. Let $f \in b^p_q$ and $\varphi \in \mathcal{M}(\mathbb{B})$ be arbitrary. By Theorem 3.4 (ii), there exists $(c_m) \in \ell^p$ with $\|(c_m)\|_{\ell^p} \lesssim \|f\|_{b^p_q}$ such that

$$f(x) = \sum_{m=1}^{\infty} c_m (1 - |a_m|^2)^{(n-2)/2} R_{-2}(x, a_m) = \sum_{m=1}^{\infty} c_m \mathcal{K}_{\varphi_{a_m}}(\mathbf{1})(x),$$

where the second equality follows from (20). Apply $\mathcal{K}_\varphi$ to f and pass it through the sum to each term of the series. This is possible by Lemma 2.2 (v) and the uniform convergence of the series on compact subsets of $\mathbb{B}$. Applying also Lemma 2.2 (ii) we obtain

$$\mathcal{K}_\varphi(f) = \sum_{m=1}^{\infty} c_m \mathcal{K}_\varphi \circ \mathcal{K}_{\varphi_{a_m}}(\mathbf{1}) = \sum_{m=1}^{\infty} c_m \mathcal{K}_{\varphi_{a_m} \circ \varphi}(\mathbf{1}).$$

Let $b_m = \varphi^{-1}(a_m)$. By (21), (b_m) is an r-lattice too. Because $(\varphi_{a_m} \circ \varphi)(b_m) = 0$, there exist $U_m \in O(n)$ such that $\varphi_{a_m} \circ \varphi = U_m \circ \varphi_{b_m}$ and

$$\mathcal{K}_\varphi(f) = \sum_{m=1}^{\infty} c_m \mathcal{K}_{\varphi_{a_m} \circ \varphi}(\mathbf{1}) = \sum_{m=1}^{\infty} c_m \mathcal{K}_{U_m \circ \varphi_{b_m}}(\mathbf{1}) = \sum_{m=1}^{\infty} c_m \mathcal{K}_{\varphi_{b_m}}(\mathbf{1}),$$

where the last equality holds because $\mathcal{K}_{U_m \circ \varphi_{b_m}}(\mathbf{1}) = \mathcal{K}_{\varphi_{b_m}} \circ \mathcal{K}_{U_m}(\mathbf{1})$ and $\mathcal{K}_{U_m}(\mathbf{1}) = \mathbf{1}$ by (13). Using (20) again we obtain

$$\mathcal{K}_\varphi(f)(x) = \sum_{m=1}^{\infty} c_m (1 - |b_m|^2)^{(n-2)/2} R_{-2}(x, b_m).$$

Finally, Theorem 3.4 (i) implies $\|\mathcal{K}_\varphi(f)\|_{b^p_q} \lesssim \|(c_m)\|_{\ell^p}$ and we conclude that $\|\mathcal{K}_\varphi(f)\|_{b^p_q} \lesssim \|f\|_{b^p_q}$. □

Proposition 4.1 shows that property (iii) in Definition 1.3 holds for b^p_q when $0 < p \le 1$, $q = p(n-2)/2 - n$ and this proves the if part of Theorem 1.14. We separate the Banach space case $p = 1$.

Corollary 4.2. *The Bergman-Besov space $b^1_{-(1+n/2)}$ is Kelvin-Möbius invariant.*

We next prove the if part of Theorem 1.8. In view of Lemma 1.7 we only need to deal with the $q \le -1$ case, that is, we need to establish that the spaces corresponding to the dashed part of the ray l in Fig. 1 are Kelvin-Möbius invariant.

Note that on the ray l in Fig. 1, when $q = 0$, the corresponding p is $2n/(n-2)$ and by Lemma 1.7, the unweighted Bergman space $b^{2n/(n-2)}_0$ is Kelvin-Möbius invariant. Next, observe that the coordinates of the left end point of the ray l are $p = 1$, $q = -(1+n/2)$, and the corresponding space $b^1_{-(1+n/2)}$ is Kelvin-Möbius invariant by Corollary 4.2. To finish the proof it suffices to show that the spaces corresponding to the line segment joining the points $(1, -(1+n/2))$ and $(2n/(n-2), 0)$ in Fig. 1 are Kelvin-Möbius invariant. We do this by using complex interpolation.

The complex interpolation space between two Bergman-Besov spaces is determined in [17, Theorem 13.5] which we repeat below.

Theorem 4.3. *Let $1 \le p_0 < p_1 < \infty$ and $q_0, q_1 \in \mathbb{R}$. If*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \tag{23}$$

for some $0 < \theta < 1$ and

$$\frac{q}{p} = \frac{(1-\theta)q_0}{p_0} + \frac{\theta q_1}{p_1}, \tag{24}$$

then $[b^{p_0}_{q_0}, b^{p_1}_{q_1}]_\theta$*, the complex interpolation space between* $b^{p_0}_{q_0}$ *and* $b^{p_1}_{q_1}$*, is* b^p_q*.*

Proof of the if part of Theorem 1.8. With $p_0 = 1$, $q_0 = -(1+n/2)$, $p_1 = 2n/(n-2)$, $q_1 = 0$, and $p_0 < p < p_1$, if θ satisfies (23) and q satisfies (24), then the complex interpolation space b^p_q given by Theorem 4.3 lies on the ray l in Fig. 1.

Now consider the linear transformation $K_\varphi(f)$ acting on f. Corollary 4.2 says that K_φ is bounded on $b^1_{-(1+n/2)}$, and Lemma 1.7 says that K_φ is bounded on $b^{2n/(n-2)}_0$, both uniformly for $\varphi \in \mathcal{M}(\mathbb{B})$. On the other hand, [8, Theorem 4.1.2] says that the complex interpolation method is an interpolation functor, which means that K_φ is also bounded on b^p_q with $1 < p < 2n/(n-2)$ and $q = p(n-2)/2 - n$ uniformly for $\varphi \in \mathcal{M}(\mathbb{B})$. Consequently the b^p_q on the dashed part of the ray are all Kelvin-Möbius invariant. □

We now deal with the only-if parts of Theorems 1.8 and 1.14 and show that if $q \neq p(n-2)/2 - n$, then property (iii) in Definition 1.3 does not hold. This includes the Bergman ($q > -1$) case asserted in Lemma 1.7 too.

Proof of the only-if parts of Theorems 1.8 and 1.14. Consider the function $f = \mathbf{1}$. We have $\|\mathbf{1}\|_{b^p_q} = 1$ and by (20)

$$\mathcal{K}_{\varphi_a}(\mathbf{1})(x) = (1-|a|^2)^{(n-2)/2} R_{-2}(x,a),$$

for $a \in \mathbb{B}$. Pick $t \in \mathbb{R}$ such that $q + pt > -1$. Applying the operator D^t_{-2} and using (18) we obtain

$$D^t_{-2}\mathcal{K}_{\varphi_a}(\mathbf{1}) = (1-|a|^2)^{(n-2)/2} D^t_{-2} R_{-2}(x,a) = (1-|a|^2)^{(n-2)/2} R_{t-2}(x,a).$$

Thus by (19)

$$\|\mathcal{K}_{\varphi_a}(\mathbf{1})\|^p_{b^p_q} \sim (1-|a|^2)^{p(n-2)/2} \int_{\mathbb{B}} |R_{t-2}(x,a)|^p \, (1-|x|^2)^{q+pt} \, d\nu(x).$$

We estimate the above integral using Lemma 3.3. We check three distinct cases depending on the sign of $c = p(n-2) - q - n$.

If $p(n-2) - q - n > 0$, then

$$\|\mathcal{K}_{\varphi_a}(\mathbf{1})\|^p_{b^p_q} \sim \frac{1}{(1-|a|^2)^{p(n-2)/2-q-n}},$$

and right-hand side tends to ∞ or 0 as $|a| \to 1^-$ unless $q = p(n-2)/2 - n$.

If $p(n-2)-q-n = 0$, then $\|\mathcal{K}_{\varphi_a}(\mathbf{1})\|^p_{b^p_q} \to 0$ as $|a| \to 1^-$ since $(1-|a|^2)^{p(n-2)/2}$ dominates $\log(1/(1-|a|^2))$.

If $p(n-2) - q - n < 0$, then $\|\mathcal{K}_{\varphi_a}(f)\|^p_{b^p_q} \to 0$ as $|a| \to 1^-$.

Thus $\|\mathcal{K}_{\varphi_a}(\mathbf{1})\|_{b^p_q} \sim \|\mathbf{1}\|_{b^p_q} = 1$ as $|a| \to 1^-$ only if $q = p(n-2)/2 - n$. □

Remark 4.4. We finish this section by looking at another class of harmonic function spaces, *harmonic Hardy spaces*. For $1 \le p < \infty$, the space h^p consists of all $f \in h(\mathbb{B})$ such that

$$\|f\|^p_{h^p} := \sup_{0 \le r < 1} \int_{\mathbb{S}} |f(r\xi)|^p \, d\sigma(\xi) < \infty.$$

For $\varphi \in \mathcal{M}(\mathbb{B})$, the norm of $\mathcal{K}_\varphi : h^p \to h^p$ is computed in [19, Theorem 1.2]. It follows from this theorem that when $n \ge 3$, h^p is Kelvin-Möbius invariant if and only if $p = 2(n-1)/(n-2)$. We note that in the

Bergman-Besov case if we take $q = -1$ in Theorem 1.8, we find the same p. In view of Theorems 1.11 and 1.13, the inclusions $b^1_{-(1+n/2)} \subset h^{2(n-1)/(n-2)} \subset b^\infty_{(n-2)/2}$ follow from this invariance. We also note that although there is no Möbius-invariant *holomorphic* Hardy space on the unit ball of $\mathbb{C}^n$, there is one Kelvin-Möbius-invariant *harmonic* Hardy space.

5. Minimal and maximal spaces

We show in this section that of all the Kelvin-Möbius-invariant spaces, the Bergman-Besov space $b^1_{-(1+n/2)}$ is minimal and the weighted Bloch space $b^\infty_{(n-2)/2}$ is maximal. A major part of the proof is to verify that point-evaluation functionals are bounded on Kelvin-Möbius-invariant spaces. We begin with some lemmas. Recall that we always take $n \geq 3$.

Lemma 5.1. $\operatorname{span}\{1/[x,a]^{n-2} : a \in \mathbb{B}\}$ *is dense in* $h(\mathbb{B})$.

Proof. It is well known that harmonic polynomials are dense in $h(\mathbb{B})$. Let g be a harmonic polynomial. It is clear that g belongs to every b^p_q, in particular it belongs to $b^1_{-(1+n/2)}$. We apply Theorem 3.4 to $b^1_{-(1+n/2)}$ with $s = -2$ which shows that there exist suitable $a_m \in \mathbb{B}$ and $(c_m) \in \ell^1$ such that

$$g(x) = \sum_{m=1}^{\infty} c_m (1 - |a_m|^2)^{(n-2)/2} R_{-2}(x, a_m) = \sum_{m=1}^{\infty} c_m \frac{(1 - |a_m|^2)^{(n-2)/2}}{[x, a_m]^{n-2}},$$

where in the second equality we use Lemma 3.2. Since the above series converges uniformly on compact subsets of $\mathbb{B}$, in $h(\mathbb{B})$, we can approximate g with elements of $\operatorname{span}\{1/[x,a]^{n-2} : a \in \mathbb{B}\}$. □

Lemma 5.2. *Let E be a Kelvin-Möbius-invariant space. Then there exists a decent linear functional L on E with $L(\mathbf{1}) \neq 0$.*

Proof. As E is Kelvin-Möbius invariant, there is a continuous linear functional $L \neq 0$ on $h(\mathbb{B})$ which is also continuous on $(E, \|\cdot\|_E)$. For $a \in \mathbb{B}$, define L_a by $L_a(f) = L(\mathcal{K}_{\varphi_a}(f))$. By Lemma 2.2 (v), L_a is continuous on $h(\mathbb{B})$ and L_a is also continuous on $(E, \|\cdot\|_E)$ since

$$|L_a(f)| = |L(\mathcal{K}_{\varphi_a}(f))| \leq \|L\| \, \|\mathcal{K}_{\varphi_a}(f)\|_E \lesssim \|L\| \, \|f\|_E,$$

by Kelvin-Möbius invariance of E, where $\|L\|$ is the norm of L on $(E, \|\cdot\|_E)$. If $L_a(\mathbf{1}) = 0$ for every $a \in \mathbb{B}$, then by (14) we would have $L(1/[x,a]^{n-2}) = 0$ for every $a \in \mathbb{B}$. But by Lemma 5.1 this would imply $L = 0$. Thus there exists $a \in \mathbb{B}$ such that $L_a(\mathbf{1}) \neq 0$ and L_a is the asserted decent linear functional. □

Theorem 5.3. *Let E be a Kelvin-Möbius-invariant space. Then the point evaluation functional $f \mapsto f(0)$ is bounded on E.*

Proof. Let L be a decent linear functional on E with $L(\mathbf{1}) \neq 0$ which exists by Lemma 5.2. Define a new linear functional $\mathcal{L}$ on $h(\mathbb{B})$ by

$$\mathcal{L}(f) = \int_{SO(n)} L(f \circ U)\, d\mu_0(U) \qquad (f \in h(\mathbb{B})), \tag{25}$$

where μ_0 is as in subsection 2.1. The above integral is well defined since the map $U \mapsto f \circ U$ is continuous from $SO(n)$ to $h(\mathbb{B})$ and $SO(n)$ is compact.

We first verify that $\mathcal{L}$ is continuous on $h(\mathbb{B})$. A linear functional Λ on $h(\mathbb{B})$ is continuous if and only if there exists a compact set $K \subset \mathbb{B}$ and a constant $C > 0$ such that $|\Lambda(f)| \leq C \sup\{|f(x)| : x \in K\}$, for every $f \in h(\mathbb{B})$ [25, p. 1]. Hence there exists a $K_r = \{x \in \mathbb{B} : |x| \leq r < 1\}$ and a $C > 0$ such that

$$|L(f)| \leq C \sup\{|f(x)| : x \in K_r\} \qquad (f \in h(\mathbb{B})).$$

The same inequality also holds when $|L(f)|$ is replaced by $|L(f \circ U)|$ since K_r is $O(n)$-invariant. Thus

$$|\mathcal{L}(f)| \leq \int\limits_{SO(n)} |L(f \circ U)| \, d\mu_0(U) \leq C \sup\{|f(x)| : x \in K_r\},$$

because μ_0 is normalized. This shows that $\mathcal{L}$ is continuous on $h(\mathbb{B})$.

It is clear that $\mathcal{L}(\mathbf{1}) = L(\mathbf{1}) \neq 0$. We proceed to show that $\mathcal{L}(Y_m^j) = 0$ for every $j = 1, 2, \ldots, \delta_m$, $m = 1, 2, \ldots$, where Y_m^j is as in subsection 3.1. For this, for fixed $f \in h(\mathbb{B})$, we consider the vector-valued integral

$$\int\limits_{SO(n)} f \circ U \, d\mu_0(U) \tag{26}$$

as defined in [27, Definition 3.26]. Noting that the map $U \mapsto f \circ U$ is continuous from $SO(n)$ to $h(\mathbb{B})$, by [27, Theorem 3.27] and the remark preceding it, the integral in (26) exists in the sense that there exists a unique $g \in h(\mathbb{B})$ such that

$$\Lambda(g) = \int\limits_{SO(n)} \Lambda(f \circ U) \, d\mu_0(U)$$

for every continuous linear functional Λ on $h(\mathbb{B})$. The value of the integral in (26) is then defined to be equal to g.

Since L is continuous on $h(\mathbb{B})$, taking $\Lambda = L$ we obtain

$$\mathcal{L}(Y_m^j) = \int\limits_{SO(n)} L(Y_m^j \circ U) \, d\mu_0(U) = L\left(\int\limits_{SO(n)} Y_m^j \circ U \, d\mu_0(U) \right) = L(V_m^j), \tag{27}$$

where $V_m^j = \int_{SO(n)} Y_m^j \circ U \, d\mu_0(U)$. To find V_m^j note that for every $x \in \mathbb{B}$, the point-evaluation functional Λ_x given by $\Lambda_x(f) = f(x)$ is continuous on $h(\mathbb{B})$. Thus taking $\Lambda = \Lambda_x$ and writing $\xi = x/|x|$ for $x \neq 0$, we see that

$$\begin{aligned} V_m^j(x) = \Lambda_x(V_m^j) = \Lambda_x \left(\int\limits_{SO(n)} Y_m^j \circ U \, d\mu_0(U) \right) &= \int\limits_{SO(n)} \Lambda_x \left(Y_m^j \circ U \right) \, d\mu_0(U) \\ &= \int\limits_{SO(n)} Y_m^j(U(x)) \, d\mu_0(U). \end{aligned}$$

The last integral can be computed to be

$$\int\limits_{SO(n)} Y_m^j(U(x)) \, d\mu_0(U) = |x|^m \int\limits_{SO(n)} Y_m^j(U(\xi)) \, d\mu_0(U) = |x|^m \int\limits_{\mathbb{S}} Y_m^j(\zeta) \, d\sigma(\zeta) = |x|^m Y_m^j(0) = 0,$$

where in the first equality we use the homogeneity of Y_m^j, in the second equality we use Lemma 2.1, in the third equality we use the mean-value property, and in the last that $m \geq 1$. Hence $V_m^j(x) = 0$ for $x \neq 0$ and by continuity $V_m^j \equiv 0$. By (27) we conclude that $\mathcal{L}(Y_m^j) = 0$ for $m \geq 1$.

We can now find $\mathcal{L}(f)$ for any $f \in h(\mathbb{B})$ easily. If

$$f(x) = f(0) + \sum_{m=1}^{\infty} \sum_{j=1}^{\delta_m} f_m^j Y_m^j(x) \qquad (f_m^j \in \mathbb{C},\ x \in \mathbb{B})$$

is the expansion of f, then, since the above series converges in $h(\mathbb{B})$, we have

$$\mathcal{L}(f) = \mathcal{L}(f(0)\mathbf{1}) + \sum_{m=1}^{\infty} \sum_{j=1}^{\delta_m} f_m^j \mathcal{L}(Y_m^j) = f(0)L(\mathbf{1}). \tag{28}$$

We now show that $\mathcal{L}$ is decent on E, that is, $\mathcal{L}$ is continuous also on $(E, \|\cdot\|_E)$. To see this, first note that since L is decent, it is continuous on $(E, \|\cdot\|_E)$. Next, by the Kelvin-Möbius invariance of E and (13), we have $\|f \circ U\|_E = \|\mathcal{K}_U(f)\|_E \sim \|f\|_E$ for every $f \in E$ and $U \in SO(n)$. Therefore

$$|\mathcal{L}(f)| \leq \int_{SO(n)} |L(f \circ U)|\, d\mu_0(U) \leq \|L\| \int_{SO(n)} \|f \circ U\|_E\, d\mu_0(U) \lesssim \|f\|_E. \tag{29}$$

To finish the proof, let $f \in E$ be arbitrary. Then by (28) and (29),

$$|f(0)L(\mathbf{1})| = |\mathcal{L}(f)| \lesssim \|f\|_E,$$

and since $L(\mathbf{1}) \neq 0$, we conclude that $|f(0)| \lesssim \|f\|_E$. □

Corollary 5.4. *Let E be a Kelvin-Möbius-invariant space. Then for every $a \in \mathbb{B}$, the point-evaluation functional $\Lambda_a : E \to \mathbb{C}$, $\Lambda_a(f) = f(a)$ is bounded on E.*

Proof. Since $\mathcal{K}_{\varphi_a}(f)(0) = (1 - |a|^2)^{(n-2)/2} f(a)$, using that $\|\mathcal{K}_{\varphi_a}(f)\|_E \sim \|f\|_E$ and Theorem 5.3, we obtain

$$(1 - |a|^2)^{(n-2)/2} |f(a)| = |\mathcal{K}_{\varphi_a}(f)(0)| \lesssim \|\mathcal{K}_{\varphi_a}(f)\|_E \lesssim \|f\|_E. \quad \square \tag{30}$$

The maximality of $b^\infty_{(n-2)/2}$ among Kelvin-Möbius-invariant spaces is now immediate.

Proof of Theorem 1.13. If E is Kelvin-Möbius invariant and $f \in E$, then (30) and (6) shows $f \in b^\infty_{(n-2)/2}$ and $\|f\|_{b^\infty_{(n-2)/2}} \lesssim \|f\|_E$. □

We next show the minimality of $b^1_{-(1+n/2)}$.

Proof of Theorem 1.11. Suppose E is Kelvin-Möbius invariant. Pick $f \in b^1_{-(1+n/2)}$. Applying Theorem 3.4 with $p = 1$, $q = -(1 + n/2)$ and $s = -2$ shows that there exist $a_m \in \mathbb{B}$ and a sequence $(c_m) \in \ell^1$ with $\|(c_m)\|_{\ell^1} \lesssim \|f\|_{b^1_{-(1+n/2)}}$ such that

$$f(x) = \sum_{m=1}^{\infty} c_m (1 - |a_m|^2)^{(n-2)/2} R_{-2}(x, a_m) = \sum_{m=1}^{\infty} c_m \mathcal{K}_{\varphi_{a_m}}(\mathbf{1})(x), \tag{31}$$

where the second equality follows from (20). The above series converges to f absolutely and uniformly on compact subsets of $\mathbb{B}$. We claim that it converges to f also in $(E, \|\cdot\|_E)$. To verify this, first note that since $\mathbf{1} \in E$, for each m, $\mathcal{K}_{\varphi_{a_m}}(\mathbf{1}) \in E$ and $\|\mathcal{K}_{\varphi_{a_m}}(\mathbf{1})\|_E \sim \|\mathbf{1}\|_E$. Because $(c_m) \in \ell^1$,

$$\sum_{m=1}^{\infty} \left\| c_m \mathcal{K}_{\varphi_{a_m}}(\mathbf{1}) \right\|_E \lesssim \|\mathbf{1}\|_E \sum_{m=1}^{\infty} |c_m| < \infty,$$

and the series $\sum_{m=1}^{\infty} c_m \mathcal{K}_{\varphi_{a_m}}(\mathbf{1})$ converges in the Banach space $(E, \|\cdot\|_E)$ to some $g \in E$. Since point-evaluation functionals are bounded on E by Corollary 5.4, we must have

$$g(x) = \sum_{m=1}^{\infty} c_m \mathcal{K}_{\varphi_{a_m}}(\mathbf{1})(x) \qquad (x \in \mathbb{B}).$$

Comparing with (31) we conclude that $f = g \in E$ and

$$\|f\|_E = \left\| \sum_{m=1}^{\infty} c_m \mathcal{K}_{\varphi_{a_m}}(\mathbf{1}) \right\|_E \leq \sum_{m=1}^{\infty} |c_m| \left\| \mathcal{K}_{\varphi_{a_m}}(\mathbf{1}) \right\|_E \lesssim \|\mathbf{1}\|_E \|(c_m)\|_{\ell^1} \lesssim \|\mathbf{1}\|_E \|f\|_{b^1_{-(1+n/2)}}.$$

This completes the proof. □

The following corollary follows from the fact that every harmonic polynomial belongs to $b^1_{-(1+n/2)}$.

Corollary 5.5. *A Kelvin-Möbius-invariant space contains all harmonic polynomials.*

We finish this section by proving Theorem 1.15 and its corollary. The proof utilizes same ideas we used before.

Proof of Theorem 1.15. First observe that if $\mathbf{1} \in A$, then $A = h(\mathbb{B})$. This is because if $\mathbf{1} \in A$, then by the invariance of A under the Kelvin-Möbius transform and (14), $1/[x, a]^{n-2} \in A$ for every $a \in \mathbb{B}$. Since A is closed, $A = h(\mathbb{B})$ by Lemma 5.1.

Suppose that $A \neq h(\mathbb{B})$. Then $\mathbf{1} \notin A$ and since $h(\mathbb{B})$ is locally convex, by [27, Theorem 3.5], there exists a continuous linear functional L on $h(\mathbb{B})$ with $L(\mathbf{1}) = 1$ and $L(f) = 0$ for every $f \in A$. Define $\mathcal{L}$ as in (25). Then, as shown in the proof of Theorem 5.3, $\mathcal{L}$ is continuous on $h(\mathbb{B})$ and for every $f \in h(\mathbb{B})$,

$$\mathcal{L}(f) = f(0)L(\mathbf{1}) = f(0) \tag{32}$$

by (28).

Now let $f \in A$. Since $\mathcal{K}_U(f) = f \circ U \in A$ for every $U \in SO(n)$ and L vanishes on A, we have $\mathcal{L}(f) = 0$ by the definition (25). Thus, by (32), $f(0) = 0$ for every $f \in A$. Next, since $\mathcal{K}_{\varphi_a}(f) \in A$ for every $a \in \mathbb{B}$, we have $\mathcal{K}_{\varphi_a}(f)(0) = 0$, and because $\mathcal{K}_{\varphi_a}(f)(0) = (1 - |a|^2)^{(n-2)/2} f(a)$, we conclude that $f(a) = 0$ for every $a \in \mathbb{B}$. Thus f vanishes on $\mathbb{B}$ and $A = \{0\}$. □

Proof of Corollary 1.16. Let A be the closure of $\operatorname{span}\{\mathcal{K}_\varphi(f) : \varphi \in \mathcal{M}(\mathbb{B})\}$. Then A is a closed subspace of $h(\mathbb{B})$ and $A \neq \{0\}$. By Lemma 2.2 (ii) and (v), A is invariant under the Kelvin-Möbius transform and thus by Theorem 1.15, $A = h(\mathbb{B})$. □

6. The unique Hilbert space

We first show that b^2_{-2}, endowed with the inner product $\langle \cdot, \cdot \rangle_I$ given in (5), is strictly Kelvin-Möbius invariant.

Proof of Theorem 1.10 (i). By Theorem 1.8, $(b^2_{-2}, \langle \cdot, \cdot \rangle_{b^2_{-2}})$ is Kelvin-Möbius invariant and since $\|\cdot\|_I$ is equivalent to $\|\cdot\|_{b^2_{-2}}$, it follows that for all $\varphi \in \mathcal{M}(\mathbb{B})$ and $f \in b^2_{-2}$

$$\|\mathcal{K}_\varphi(f)\|_I \sim \|f\|_I. \tag{33}$$

We need to show that exact equality holds in (33). For this we show

$$\langle \mathcal{K}_\varphi(f), \mathcal{K}_\varphi(g)\rangle_I = \langle f, g\rangle_I, \qquad \text{for every } \varphi \in \mathcal{M}(\mathbb{B}) \text{ and } f, g \in b^2_{-2}. \tag{34}$$

By Corollary 5.4, point-evaluation functionals are bounded on $(b^2_{-2}, \langle\cdot,\cdot\rangle_I)$. We first determine the corresponding reproducing kernels. For $a \in \mathbb{B}$, let

$$R_a(x) := \frac{1}{[x,a]^{n-2}} = \sum_{m=0}^{\infty}\sum_{j=1}^{\delta_m} \frac{n/2-1}{m+n/2-1}\overline{Y_m^j(a)}Y_m^j(x),$$

where second equality follows from Lemma 3.1 and (16). As R_a is harmonic on $\overline{\mathbb{B}}$, it belongs to b^2_{-2}, and by (5), $\langle f, R_a\rangle_I = f(a)$ for all $f \in b^2_{-2}$. Thus $R_a(x)$ is the reproducing kernel of b^2_{-2} with respect to the inner product $\langle\cdot,\cdot\rangle_I$.

We next compute $\mathcal{K}_\varphi(R_a)$. Because

$$\mathcal{K}_\varphi(R_a)(x) = \left(\frac{1-|\varphi(x)|^2}{1-|x|^2}\right)^{(n-2)/2} \frac{1}{[\varphi(x),a]^{n-2}},$$

using (11) with y replaced by $\varphi^{-1}(a)$ we obtain

$$\begin{aligned}\mathcal{K}_\varphi(R_a)(x) &= \left(\frac{1-|\varphi^{-1}(a)|^2}{1-|a|^2}\right)^{(n-2)/2} \frac{1}{[x,\varphi^{-1}(a)]^{n-2}}\\ &= \left(\frac{1-|\varphi^{-1}(a)|^2}{1-|a|^2}\right)^{(n-2)/2} R_{\varphi^{-1}(a)}(x).\end{aligned} \tag{35}$$

We verify (34) first when $g = R_a$. Let $a \in \mathbb{B}$, $\varphi \in \mathcal{M}(\mathbb{B})$ and $f \in b^2_{-2}$. Then

$$\begin{aligned}\langle \mathcal{K}_\varphi(f), \mathcal{K}_\varphi(R_a)\rangle_I &= \left(\frac{1-|\varphi^{-1}(a)|^2}{1-|a|^2}\right)^{(n-2)/2} \langle \mathcal{K}_\varphi(f), R_{\varphi^{-1}(a)}\rangle_I\\ &= \left(\frac{1-|\varphi^{-1}(a)|^2}{1-|a|^2}\right)^{(n-2)/2} \mathcal{K}_\varphi(f)(\varphi^{-1}(a)) = f(a) = \langle f, R_a\rangle_I,\end{aligned}$$

where the first equality follows from (35), the second equality from the reproducing property, and the third equality from the definition of $\mathcal{K}_\varphi$. Thus (34) holds when $g \in \operatorname{span}\{R_a\}_{a\in\mathbb{B}}$ by the linearity of $\mathcal{K}_\varphi$.

Finally, let $f, g \in b^2_{-2}$ be arbitrary. By the general properties of reproducing kernels, $\operatorname{span}\{R_a\}_{a\in\mathbb{B}}$ is dense in $(b^2_2, \langle\cdot,\cdot\rangle_I)$. So there exists a sequence (g_m) with $g_m \in \operatorname{span}\{R_a\}_{a\in\mathbb{B}}$ such that

$$g_m \to g \quad \text{in } (b^2_2, \langle\cdot,\cdot\rangle_I).$$

By (33) it also holds that

$$\mathcal{K}_\varphi(g_m) \to \mathcal{K}_\varphi(g) \quad \text{in } (b^2_2, \langle\cdot,\cdot\rangle_I).$$

Hence

$$\langle \mathcal{K}_\varphi(f), \mathcal{K}_\varphi(g)\rangle_I = \lim_{m\to\infty} \langle \mathcal{K}_\varphi(f), \mathcal{K}_\varphi(g_m)\rangle_I = \lim_{m\to\infty} \langle f, g_m\rangle_I = \langle f, g\rangle_I.$$

This shows (34) and the proof is complete.

Note that (33) and therefore Theorem 1.8 plays an essential role here. That is, we first verified the Kelvin-Möbius invariance of $(b^2_{-2}, \langle\cdot,\cdot\rangle_I)$ and then using this we showed its strictness. □

We next show that $(b^2_{-2}, \langle\cdot,\cdot\rangle_I)$ is essentially the only strictly Kelvin-Möbius-invariant Hilbert space.

Proof of Theorem 1.10 (ii). Let $(H, \langle\cdot,\cdot\rangle_H)$ be a strictly Kelvin-Möbius-invariant Hilbert space. That is, $\|\mathcal{K}_\varphi(f)\|_H = \|f\|_H$ for every $\varphi \in \mathcal{M}(\mathbb{B})$ and $f \in H$. By polarization we also have

$$\langle \mathcal{K}_\varphi(f), \mathcal{K}_\varphi(g)\rangle_H = \langle f, g\rangle_H \qquad (\varphi \in \mathcal{M}(\mathbb{B}),\ f, g \in H). \tag{36}$$

By Corollary 5.4, the point-evaluation functional $f \mapsto f(a)$ is bounded on H for every $a \in \mathbb{B}$. Therefore for each $a \in \mathbb{B}$, there exists $S_a \in H$ such that for all $f \in H$,

$$f(a) = \langle f, S_a\rangle_H.$$

We show that S_a is a positive scalar multiple of the function $1/[x,a]^{n-2}$. For this let us first compute $\mathcal{K}_\varphi(S_a)$. By (36) and Lemma 2.2 (iii),

$$\begin{aligned}\langle f, \mathcal{K}_\varphi(S_a)\rangle_H &= \langle \mathcal{K}_{\varphi^{-1}}(f), S_a\rangle_H = \mathcal{K}_{\varphi^{-1}}(f)(a) \\ &= \left(\frac{1-|\varphi^{-1}(a)|^2}{1-|a|^2}\right)^{(n-2)/2} f(\varphi^{-1}(a)) = \left(\frac{1-|\varphi^{-1}(a)|^2}{1-|a|^2}\right)^{(n-2)/2} \langle f, S_{\varphi^{-1}(a)}\rangle_H.\end{aligned}$$

Since this is true for all $f \in H$, we conclude that

$$\mathcal{K}_\varphi(S_a) = \left(\frac{1-|\varphi^{-1}(a)|^2}{1-|a|^2}\right)^{(n-2)/2} S_{\varphi^{-1}(a)}, \tag{37}$$

which is the same as (35) with S_a in place of R_a.

We first determine S_0. Taking $\varphi = U \in O(n)$ and $a = 0$ in (37) and using (13) we obtain

$$S_0 \circ U = \mathcal{K}_U(S_0) = S_0.$$

This implies that S_0 is constant on the spheres $|x| = r$, $0 < r < 1$. Because S_0 is also harmonic on $\mathbb{B}$, by the mean value property, S_0 is constant on $\mathbb{B}$ and $S_0 = \lambda\mathbf{1}$, for some $\lambda \in \mathbb{C}$. Since

$$1 = \langle \mathbf{1}, S_0\rangle_H = \langle \mathbf{1}, \lambda\mathbf{1}\rangle_H = \overline{\lambda}\|\mathbf{1}\|_H^2,$$

we see that $\lambda = 1/\|\mathbf{1}\|_H^2$ is positive.

We now find S_a. For any $a \in \mathbb{B}$, taking $\varphi = \varphi_a$ in (37) shows

$$\mathcal{K}_{\varphi_a}(S_a) = \frac{1}{(1-|a|^2)^{(n-2)/2}} S_0.$$

Applying $\mathcal{K}_{\varphi_a}^{-1} = \mathcal{K}_{\varphi_a^{-1}} = \mathcal{K}_{\varphi_a}$ to both sides and using that $S_0 = \lambda\mathbf{1}$, we obtain

$$S_a(x) = \frac{1}{(1-|a|^2)^{(n-2)/2}} \mathcal{K}_{\varphi_a}(S_0)(x) = \frac{\lambda}{(1-|a|^2)^{(n-2)/2}} \mathcal{K}_{\varphi_a}(\mathbf{1})(x) = \frac{\lambda}{[x,a]^{n-2}},$$

where in the last equality we use (14). It follows that the reproducing kernel of $(H, \langle\cdot,\cdot\rangle_H)$ is $\lambda/[x,a]^{n-2}$ and therefore the reproducing kernel of $(H, \lambda\langle\cdot,\cdot\rangle_H)$ is $1/[x,a]^{n-2}$. On the other hand, by the proof of part (i), $1/[x,a]^{n-2}$ is the reproducing kernel of $(b^2_{-2}, \langle\cdot,\cdot\rangle_I)$. By the uniqueness of reproducing kernel Hilbert spaces ([28, Theorem 1.4]), we conclude that $H = b^2_{-2}$ and $\lambda\langle\cdot,\cdot\rangle_H = \langle\cdot,\cdot\rangle_I$. □

We finish this section by giving an integral description of the invariant inner product $\langle\cdot,\cdot\rangle_I$. This is similar to [24, Definition 5.4], but since in our case the invariant Hilbert space corresponds to $q = -2$, first-order derivatives are sufficient. On the other hand, in the holomorphic case the Möbius-invariant Hilbert space is $B^2_{-(n+1)}$ and as n gets larger higher order derivatives are needed in the integral description of the inner product.

Let $f \in h(\mathbb{B})$ and $f = \sum_{m=0}^{\infty} f_m$ be its homogeneous expansion. The *radial derivative* $\mathcal{R}f$ of f is defined by

$$\mathcal{R}f := \sum_{m=1}^{\infty} m\, f_m.$$

For $n \geq 3$, let $\tilde{\mathcal{R}} := (n/2-1)^{-1}\mathcal{R} + I$, where I is the identity. It is clear that

$$\tilde{\mathcal{R}}f = \sum_{m=0}^{\infty} \frac{m + n/2 - 1}{n/2 - 1} f_m.$$

Theorem 6.1. *Let $n \geq 3$. For $f, g \in b^2_{-2}$,*

$$\begin{aligned}\langle f, g\rangle_I &= \frac{n/2-1}{n/2} \int_{\mathbb{B}} \frac{1-|x|^2}{|x|} \tilde{\mathcal{R}}f(x) \frac{1-|x|^2}{|x|} \overline{\tilde{\mathcal{R}}g(x)}\, d\nu_{-2}(x) \\ &= \frac{n/2-1}{n/2} \int_{\mathbb{B}} \frac{1}{|x|^2} \tilde{\mathcal{R}}f(x) \overline{\tilde{\mathcal{R}}g(x)}\, d\nu(x).\end{aligned}$$

Note that the factor $1/|x|^2$ causes no integrability problem since $n \geq 3$.

Proof. Since $f \in b^2_{-2}$, the first-order derivative $\mathcal{R}f$ is in b^2_0 by [17, Theorem 1.2]. Also since $b^2_{-2} \subset b^2_0$, f is in b^2_0. Thus $\tilde{\mathcal{R}}f$ is in $b^2_0 \subset L^2(d\nu)$. Similarly $\tilde{\mathcal{R}}g$ is in $L^2(d\nu)$ and we can pass to polar coordinates to write

$$\int_{\mathbb{B}} \frac{1}{|x|^2} \tilde{\mathcal{R}}f(x)\overline{\tilde{\mathcal{R}}g(x)}\, d\nu(x) = \int_0^1 n r^{n-1} \frac{1}{r^2} \int_{\mathbb{S}} \tilde{\mathcal{R}}f(r\xi)\overline{\tilde{\mathcal{R}}g(r\xi)}\, d\sigma(\xi)\, dr.$$

Let $f = \sum_{m=0}^{\infty}\sum_{j=1}^{\delta_m} f_m^j Y_m^j$ and $g = \sum_{m=0}^{\infty}\sum_{j=1}^{\delta_m} g_m^j Y_m^j$, where Y_m^j is as in subsection 3.1. Then

$$\tilde{\mathcal{R}}f(x) = \sum_{m=0}^{\infty}\sum_{j=1}^{\delta_m} \frac{m+n/2-1}{n/2-1} f_m^j Y_m^j(x), \quad \tilde{\mathcal{R}}g(x) = \sum_{m=0}^{\infty}\sum_{j=1}^{\delta_m} \frac{m+n/2-1}{n/2-1} g_m^j Y_m^j(x),$$

where the series converge uniformly on $r\mathbb{S}$. Using also the orthogonality (15), and the homogeneity of Y_m^j, we deduce that

$$\begin{aligned}\int_{\mathbb{B}} \frac{1}{|x|^2} \tilde{\mathcal{R}}f(x)\overline{\tilde{\mathcal{R}}g(x)}\, d\nu(x) &= \sum_{m=0}^{\infty}\sum_{j=1}^{\delta_m} \left(\frac{m+n/2-1}{n/2-1}\right)^2 f_m^j \overline{g_m^j} \int_0^1 n\, r^{2m+n-3}\, dr \\ &= \frac{n/2}{n/2-1} \langle f, g\rangle_I. \quad \square\end{aligned}$$

The integral inner product given in Proposition 6.1 is in terms of radial derivatives which are well known. If we want to use the differential operators D^t_s, then we have other choices. Note that the reproducing kernel of $(b^2_{-2}, \langle\cdot,\cdot\rangle_I)$ is $R_{-2}(x,y)$. So [17, Theorem 5.2] provides us with an integral inner product such that R_{-2} is the reproducing kernel. For example, if we follow the proof of [17, Theorem 5.2] and choose $t = 1$, $s_1 = -1$ and $s_2 = -2$, then we see that

$$\langle f, g\rangle_I = \int_{\mathbb{B}} D^1_{-1} f(x)\overline{D^1_{-2} g(x)}\, d\nu(x).$$

To verify the above formula, just integrate in polar coordinates and use the homogeneous expansions of the functions. The integral on the right does not look like (conjugate) symmetric in f and g, but it actually is.

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