Gleason’s problem and homogeneous interpolation in Hardy and Dirichlet-type spaces of the ball

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Abstract

We solve Gleason’s problem in the reproducing kernel Hilbert spaces with reproducing kernels \(1/(1 - \sum_1^N z_j w_j)^r\) for real \(r > 0\) and their counterparts for \(r \leq 0\), and study the homogeneous interpolation problem in these spaces.

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1. Introduction

A complete Nevanlinna–Pick kernel is a function \(K(z, w)\) that is positive in a set \(\Omega\) and such that \(1/K(z, w)\) has one positive square in \(\Omega\). A typical example is given by

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\[ K_1(z, w) = \frac{1}{1 - \langle z, w \rangle}, \]  

where \( z = (z_1, z_2, \ldots, z_N) \) and \( w = (w_1, w_2, \ldots, w_N) \) are in the unit ball

\[ \mathbb{B}_N = \left\{ z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N \mid |z|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 < 1 \right\} \]

of \( \mathbb{C}^N \) and \( \langle z, w \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2 + \cdots + z_N \bar{w}_N \). We will denote by \( \mathcal{H}_1 \) the reproducing kernel Hilbert space with reproducing kernel (1).

Complete Nevanlinna–Pick kernels originate with the work of Agler (see Quiggin’s paper [9]) and play a central role in the extension of the Nevanlinna–Pick interpolation problem from the case of the disk to more general settings such as the ball. In [8] McCullough and Trent extended Beurling’s theorem to reproducing kernel Hilbert spaces with complete Pick kernels. The theorem they proved reads as follows for \( \mathcal{H}_1^n \), the Hilbert space whose elements are \( n \)-vectors and with reproducing kernel (1).

**Theorem 1.1.** Let \( \mathcal{M} \) be a closed subspace of \( \mathcal{H}_1^n \) invariant under the operators of multiplication by \( z_j \) (\( j = 1, \ldots, N \)). Then there is an \( (n \times m) \)-valued function \( \Phi \) (with \( m \) possibly infinite) that takes contractive values in \( \mathbb{B}_N \) and whose radial limits satisfy \( \Phi(z)\Phi(z)^* = I_n \) on the sphere \( S = \partial \mathbb{B}_N \) such that \( \mathcal{M} = \Phi \mathcal{H}_1^m \).

Motivated by a problem in interpolation of analytic functions, we proved in [2] and [3] a particular case of Theorem 1.1 for a class of spaces \( \mathcal{M} \) of finite codimension. The last hypothesis allowed us to get explicit formulas and to have an estimate on the size of \( \Phi \).

In this paper we prove the counterpart of our results of [2] for all reproducing kernel Hilbert spaces \( \mathcal{H}_r \) with reproducing kernels

\[ K_r(z, w) = \frac{1}{(1 - \langle z, w \rangle)^r} \]

for \( r \) real and positive and

\[ K_0(z, w) = \log \frac{1}{1 - \langle z, w \rangle}. \]

The principal value of the logarithm is used in all kernels. These spaces are not, except for \( 0 \leq r \leq 1 \), complete Nevanlinna–Pick kernels. However, the spaces include in particular the Dirichlet space \( (r = 0) \), the Hardy space \( (r = N) \), and the weighted Bergman spaces \( (r > N) \) of the ball. They are also called Dirichlet-type spaces. Denoting the inner product and norm of \( \mathcal{H}_r \) by \( \langle \cdot, \cdot \rangle_{\mathcal{H}_r} \) and \( \| \cdot \|_{\mathcal{H}_r} \), the reproducing property of the kernels is that

\[ \langle f(z), K_r(z, w) \rangle_{\mathcal{H}_r} = f(w) \]

for any \( f \in \mathcal{H}_r \) and any \( w \in \mathbb{B}_N \).
Our computations require frequent use of the gamma function $\Gamma$ with domain all real numbers except for nonpositive integers. The gamma function satisfies $\Gamma(x + 1) = x\Gamma(x)$ for such real numbers $x$ and $n! = \Gamma(n + 1)$ for nonnegative integers $n$. Stirling’s formula for the asymptotic behavior of the gamma function and its counterpart for the factorial are

$$\Gamma(x) \sim \frac{x^x}{e^x \sqrt{x}} \quad (x \to \infty) \quad \text{and} \quad m! \sim \frac{m^m \sqrt{m}}{e^m} \quad (m \to \infty),$$

where $A \sim B$ means that $|A/B|$ is bounded above and below by two positive constants. From this it follows that

$$\frac{\Gamma(a + b)}{\Gamma(a + c)} \sim a^{b-c} \quad (a \to \infty)$$

for fixed $b$ and $c$.

We also use the multi-index notation in which $\alpha = (\alpha_1, \ldots, \alpha_N)$ is an $N$-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$, and $0^0 = 1$.

Recall that a $\mathbb{C}^{n \times m}$-valued function $\Phi$ defined in $\mathbb{B}_N$ is called a Schur multiplier from $\mathcal{H}_m^{n \times 1}$ into $\mathcal{H}_r^{n \times 1}$ if the operator of multiplication by $\Phi$ is a contraction from $\mathcal{H}_m^{n \times 1}$ into $\mathcal{H}_r^{n \times 1}$, or, equivalently, if the kernel

$$\frac{I_n - \Phi(z)\Phi(w)^*}{(1 - \langle z, w \rangle)^r}$$

is positive in $\mathbb{B}_N$. The function $\Phi$ is in particular holomorphic in $\mathbb{B}_N$ and takes contractive values there. There are functions that are holomorphic and contractive in $\mathbb{B}_N$ and are not Schur multipliers. Examples are presented in [2] for $r = 1$. Another family of examples is presented in Proposition 2.5 below.

The theorem below is the main result of this paper.

**Theorem 1.2.** Let $a_1, \ldots, a_m \in \mathbb{B}_N$ and $c_1, \ldots, c_m \in \mathbb{C}^{n \times 1}$. The set $\mathcal{M}$ of functions $f$ in $\mathcal{H}_r^{n \times 1}$ such that

$$c_j^* f(a_j) = 0 \quad (j = 1, \ldots, m)$$

is of the form $\Phi \mathcal{M}_r^{(n+m'(N-1)) \times 1}$, where $m' \leq m$ independently of $r$, and $\Phi$ is a $\mathbb{C}^{n \times (n+m'(N-1))}$-valued holomorphic function in the ball taking coisometric radial limits almost everywhere on the sphere $\mathbb{S}$. The operator of multiplication by $\Phi$ is bounded in all the spaces $\mathcal{H}_r$ and is a contraction for $r \geq 1$.

For related results on the Dirichlet and Bergman spaces when $N = 1$, we refer to [1].

A recent publication [12] shows how the results of this work can be extended to the case $r < 0$ in the spirit of Remark 3.1. See the note added in proof at the end of this paper.
2. The spaces $\mathcal{H}_r$

We gain much insight on the structure of the spaces $\mathcal{H}_r$ by considering the expansion

$$
\frac{1}{(1 - \langle z, w \rangle)^r} = \sum_{j=0}^{\infty} \frac{\Gamma(r + j)}{\Gamma(r) j!} \langle z, w \rangle^j = \sum_{\alpha} \frac{\Gamma(r + |\alpha|)}{\Gamma(r) \alpha!} z^\alpha \bar{w}^\alpha,
$$

that is valid in the set that $\Gamma(r)$ makes sense, which includes $r > 0$. Expansion (5) gives

$$
(z^\alpha, z^\beta)_{\mathcal{H}_r} = 0 \quad (\alpha \neq \beta), \quad \|z^\alpha\|_{\mathcal{H}_r}^2 = \frac{\Gamma(r) \alpha!}{\Gamma(r + |\alpha|)},
$$

and that $\mathcal{H}_r$ is the space of holomorphic functions $f(z) = \sum_{\alpha} f_\alpha z^\alpha$ in the ball for which

$$
\|f\|^2_{\mathcal{H}_r} = \sum_{\alpha} |f_\alpha|^2 \frac{\Gamma(r) \alpha!}{\Gamma(r + |\alpha|)} < \infty.
$$

When $r = 0$, we have the expansion

$$
\log \frac{1}{1 - \langle z, w \rangle} = \sum_{j=1}^{\infty} \frac{\langle z, w \rangle^j}{j} = \sum_{|\alpha| > 0} \frac{|\alpha|!}{|\alpha| \alpha!} z^\alpha \bar{w}^\alpha,
$$

which yields the norms

$$
\|z^\alpha\|^2_{\mathcal{H}_0} = \frac{|\alpha| \alpha!}{|\alpha|!} \quad \text{and} \quad \|f\|^2_{\mathcal{H}_0} = \sum_{|\alpha| > 0} |f_\alpha|^2 \frac{|\alpha| \alpha!}{|\alpha|!}.
$$

In other words, the monomials \{z^\alpha\} form a complete orthogonal set in each $\mathcal{H}_r$ (for $|\alpha| > 0$ when $r = 0$). By the reproducing property of $K_r$ applied on itself, we have $\|K_r(\cdot, w)\|_{\mathcal{H}_r}^2 = K_r(w, w)$ in each $\mathcal{H}_r$.

We denote by $k_{r,j}$ for any $r \geq 0$ the coefficients in the expansion of $K_r(z, w)$ in powers of $\langle z, w \rangle$ given in (5) and (8). So

$$
k_{r,j} = \frac{\Gamma(r + j)}{\Gamma(r) j!} \quad (r > 0, \ j = 0, 1, \ldots) \quad \text{and} \quad k_{0,j} = \frac{1}{j} \quad (j = 1, 2, \ldots).
$$

The two expressions in (10) are quite similar: by (3), if $r > 0$, then $k_{r,j} \sim \frac{1}{j^{1-r}}$ for large $j$, which tends to $1/j$ as $r \to 0$. All $k_{r,j}$ are positive.

We next give integral formulas for the inner products of the spaces $\mathcal{H}_r$. Those corresponding to $r = 0$ and $r = 1$ are in [7] and [4].
**Proposition 2.1.** Consider the pairing

\[ L_s(f, g) = \frac{(s - N) \cdots (s - 2)(s - 1)}{N!} \int_{B_N} f(z) g(z) (1 - |z|^2)^{s-N-1} \, d\nu(z), \]

where \( \nu \) is the volume measure on \( B_N \) with total mass 1. Then \( (f, g)_{\mathcal{H}_r} = \lim_{s \to r} L_s(f, g) \) for \( r > 0 \). Also

\[ (f, g)_{\mathcal{H}_0} = \lim_{s \to 0} \frac{(s - N) \cdots (s - 1)}{N!} \int_{B_N} f(z) g(z) (1 - |z|^2)^{s-N-1} \, d\nu(z). \]

**Proof.** We prove only the formula for \( r > 0 \). It suffices to obtain the result on the monomials since they form a complete orthogonal set in each \( \mathcal{H}_r \). By Lemma 1 of [4], \( L_s(z^\alpha, z^\beta) = 0 \) if \( \alpha \neq \beta \). By the same lemma, for \( s > N \) we have

\[ L_s(z^\alpha, z^\alpha) = \frac{\alpha!}{s(s+1) \cdots (s-1+|\alpha|)}. \]

The function \( s \mapsto L_s \) continues analytically at least to the region \( \{ \text{Re} \, s > 0 \} \) and

\[ \| z^\alpha \|_{r}^2 \overset{\text{def}}{=} \lim_{s \to r} L_s(z^\alpha, z^\alpha) = \frac{\Gamma(r) \alpha!}{\Gamma(r + |\alpha|)} = \| z^\alpha \|_{\mathcal{H}_r}. \]

The Hilbert space completion of the polynomials in the norm \( \| \cdot \|_r \) coincides with \( \mathcal{H}_r \). \( \square \)

**Remark 2.2.** When \( r = N + 1 \) and \( r = N \), the formulas of Proposition 2.1 actually reduce to the classical forms of the inner products of the Bergman and Hardy spaces of the ball. We consider the measures

\[ d\mu_s(\rho) = \frac{(s - N) \cdots (s - 2)(s - 1)}{N!} 2N \rho^{2N+1} (1 - \rho^2)^{s-N-1} d\rho \]

on the interval \([0, 1]\). Integration in polar coordinates (§1.4.3 of [10]) shows that

\[ \| f \|_r^2 = \lim_{s \to r} \int_0^1 d\mu_s(\rho) \int_{\mathbb{S}} \left| f(\rho \zeta) \right|^2 d\sigma(\zeta), \]

where \( \sigma \) is the area measure on \( \mathbb{S} \) with total mass 1. The measure \( \mu_s \) is finite if and only if \( s > N \), and then a computation with the beta function gives \( \mu_s([0, 1]) = 1 \). The distribution function of \( \mu_s \) is defined as \( F_s(x) = \mu_s([0, x]) \) and given by

\[ F_s(x) = \frac{(s - N) \cdots (s - 2)(s - 1)}{N!} 2N \int_0^x \rho^{2N+1} (1 - \rho^2)^{s-N-1} d\rho \]

\[ = \sum_{j=0}^{N-1} \frac{(-1)^j (s - N) \cdots (s - 2)(s - 1)}{(N - 1 - j)! (s - N + j)!} [1 - (1 - x^2)^{s-N+j}] \]
for $0 \leq x < 1$; also $F_s(1) = 1$ as stated above. By Theorem 2.3 of [5], for all $x \in [0, 1],$

$$F_s^{N+1}(x) = \lim_{s \to N+1} F_s(x) = x^{2N}$$

is the distribution function of the weak* limit $\mu_s^{N+1}$ of $\mu_s$ as $s \to N + 1$. In other words, $d\mu_s^{N+1} = dF_s^{N+1} d\rho = 2N \rho^{2N-1} d\rho$ for $\rho \in [0, 1]$, and

$$\|f\|_{H_s}^2 = 2N \int_0^1 \rho^{2N-1} \int_\mathbb{S} |f(\rho \zeta)|^2 d\sigma(\zeta) = \int_{\mathbb{B}_N} |f(z)|^2 d\nu(z)$$

$$= \|f\|_{H_{N+1}}^2.$$

Similarly, for $0 \leq x < 1,$

$$F_s^N(x) = \lim_{s \to N} F_s(x) = 0$$

is the distribution function of the weak* limit $\mu_s^N$ of $\mu_s$ as $s \to N$. But $F_s^N(1) = 1$. That means $\mu_s^N$ is the unit point mass at $\rho = 1$, and

$$\|f\|_{H_r}^2 = \lim_{\rho \to 1^-} \int_\mathbb{S} |f(\rho \zeta)|^2 d\sigma(\zeta) = \sup_{0 \leq \rho < 1} \int_\mathbb{S} |f(\rho \zeta)|^2 d\sigma(\zeta) = \|f\|_{H_r}^2$$

since $\int_\mathbb{S} |f(\rho \zeta)|^2 d\sigma(\zeta)$ is an increasing function of $\rho$.

**Remark 2.3.** The spaces $H_r$ are ordered by inclusion and the inclusion maps between some are contractions. If $0 < r < s$, since

$$\frac{\Gamma(s)\alpha!}{\Gamma(s + |\alpha|)} = \frac{(\alpha - 1 + r) \cdots (r + 1)r}{(\alpha - 1 + s) \cdots (s + 1)s} < 1$$

for any $\alpha$, we have $\|f\|_{H_s} < \|f\|_{H_r}$ by (7), and thus $H_r \subset H_s$ contractively. When $0 = r < s$, then

$$\frac{\Gamma(s)\alpha!}{\Gamma(s + |\alpha|)} = \frac{(\alpha - 1 + r) \cdots (r + 1)r}{(\alpha - 1 + s) \cdots (s + 1)s} \cdot \frac{2}{s+2} \cdot \frac{1}{s+1} \cdot \frac{1}{s} < 1$$

(11)

This ratio is $< 1$ for $s \geq 1$, and thus $H_0 \subset H_s$ contractively. For $0 < s < 1$, this ratio is bounded above but not necessarily by 1. Thus $H_0 \subset H_s$, but we cannot say whether the inclusion is a contraction or not. However, all the inclusions are proper as we show now. The case $r = 1$ and $s = N$ is in [2].

**Example 2.4.** For $0 \leq r < s$, let

$$f(z_1, \ldots, z_N) = \sum_{j=1}^{\infty} j^{(r+s-4)/4} z_1^j.$$
Then by (3) as \( j \to \infty \)
\[
\| f \|_{H_r}^2 \sim \sum_{j=1}^{\infty} j^{(r+s-4)/2} j^{1-r} = \sum_{j=1}^{\infty} \frac{1}{j^{1-(s-r)/2}} = \infty,
\]
whereas
\[
\| f \|_{H_s}^2 \sim \sum_{j=1}^{\infty} j^{(s-r-4)/2} j^{1-s} = \sum_{j=1}^{\infty} \frac{1}{j^{1+(s-r)/2}} < \infty,
\]
giving us a function \( f \) in \( \mathcal{H}_s \) that is not in \( \mathcal{H}_r \).

**Proposition 2.5.** When \( N > 1 \), a nonconstant inner function is not a Schur multiplier of \( \mathcal{H}_r \) for \( 0 \leq r < N \). In contrast, every function in the unit ball of \( H^\infty(\mathbb{B}_N) \), hence every inner function, is a Schur multiplier for the Hardy space \( \mathcal{H}_N \).

**Proof.** Let \( 0 \leq r < N \) and \( f(z) = \sum_{\alpha} f_\alpha z^\alpha \) be an inner function in \( \mathbb{B}_N \). Since \( \| 1 \|_{\mathcal{H}_r} = 1 \), the norm of the operator of multiplication by \( f \) is at least \( \| f \|_{\mathcal{H}_r} \). But the inner functions are precisely those functions \( f \in \mathcal{H}_N \) that have \( \| f \|_{\mathcal{H}_N} = 1 \) [11, p. ix]. With \( s = N \geq 1 \), the ratios in (11) and (12) are both less than 1 for the values of \( r \) considered. Hence \( 1 = \| f \|_{\mathcal{H}_N} < \| f \|_{\mathcal{H}_r} \) for the same \( r \), proving the first claim.

For \( r = N \), \( \mathcal{H}_N \) is the Hardy space of the ball and the square of its norm has the traditional form of the integral of the square of its boundary values. So if \( f \) is a bounded holomorphic function with \( \| f \|_{H^\infty(\mathbb{B}_N)} \leq 1 \) and \( g \in \mathcal{H}_N \), then \( \| fg \|_{\mathcal{H}_N} \leq \| g \|_{\mathcal{H}_N} \). Thus the operator of multiplication by \( f \) on \( \mathcal{H}_N \) has norm \( \leq 1 \). \( \square \)

We now identify the spaces that result from restricting the spaces \( \mathcal{H}_r \) to slices of the ball. A *slice* is the intersection of a complex one-dimensional subspace of \( \mathbb{C}^N \) (a complex line) with the ball \( \mathbb{B}_N \) and is completely determined by a point \( \zeta \) on \( \mathbb{S} \). Given a function \( f \) in the ball and \( \zeta \in \mathbb{S} \), the *slice function* \( f_\zeta \) is defined as \( f_\zeta(\lambda) = f(\lambda \zeta) \) for \( \lambda \) in the disc \( \mathbb{B}_1 \). If \( f \) is holomorphic in the ball, then each \( f_\zeta \) is holomorphic in the disc. We work out the restrictions from the point of view of weighted Hardy spaces.

A *weighted Hardy space* \( \mathcal{H} \) in the ball is a Hilbert space of holomorphic functions in which the monomials \( \{ z^\alpha \} \) form a complete orthogonal set with
\[
\frac{\| z^{\alpha_1} \|_{\mathcal{H}_N}}{\| z^{\alpha_1} \|_{\mathcal{H}_N}} = \frac{\| z^{\alpha_2} \|_{\mathcal{H}_N}}{\| z^{\alpha_2} \|_{\mathcal{H}_N}},
\]
whenever \( |\alpha_1| = |\alpha_2| \); see [6, p. 23]. A glance at (6) and (9) shows that each of the spaces \( \mathcal{H}_r \) including \( \mathcal{H}_0 \) is a weighted Hardy space in the ball.
We define our extension and restriction operators $E$ and $R$ only with respect to the slice obtained from $\zeta = (1, 0, \ldots, 0)$ since any slice can be transformed to any other slice by a unitary rotation, and such a rotation carries holomorphic functions to holomorphic functions. For $f$ holomorphic in the disc $D = B_1$ and $F$ holomorphic in the ball $B_N$, we set $E(f)(z_1, z_2, \ldots, z_N) = f(z_1)$ and $R(F)(\lambda) = F(\lambda, 0, \ldots , 0)$. So $R(F)(\lambda) = F(1,0,\ldots,0)(\lambda)$.

**Proposition 2.6** (Proposition 2.21 of [6]). The map $E$ is an isometry from $H_r(D)$ into $H_r(B_N)$, and $R$ is a norm-decreasing map from $H_r(B_N)$ onto $H_r(D)$, so that $RE = I$, the identity operator.

Therefore the restriction of the Dirichlet space of the ball to a slice is the Dirichlet space of the disc, the restriction of the space $H_1$ of the ball to a slice is the Hardy space of the disc, the restriction of the space $H_2$ of the ball to a slice is the Bergman space of the disc, the restriction of the Hardy space of the ball to a slice is the weighted Bergman space $A^2_{N-2}$ of the disc, the restriction of the Bergman space of the ball to a slice is the weighted Bergman space $A^2_{N-1}$ of the disc, and similar results hold for intermediate spaces and weighted Bergman spaces of the ball too.

There is also the converse problem, which is the question of whether a function is in $H_r(B_N)$ for $N > 1$ if all its slice functions are in $H_r(D)$. This question is answered in the negative in Example 7.2.11 of [10] for the case $r = N = 2$. That example can be modified to answer the same question in the negative for all $N > 1$ and $r \geq 0$.

**Example 2.7.** For $N > 1$ and $j = 1, 2, \ldots$, let $F_j(z) = N^{Nj}(z_1 \cdots z_N)^{2j}$. So $F_j(z) = N^{Nj}z^\alpha$ with $\alpha = (2j, \ldots, 2j)$ and $|\alpha| = 2Nj$. Then $F_{j\zeta}(\lambda) = F_j(\zeta)\lambda^{2Nj}$, and $|F_j(\zeta)| \leq 1$ for any $\zeta \in \mathbb{S}$ by Proposition 7.2.8(i) of [10]. This last fact along with (6), (9), and (3) give

$$\|F_{j\zeta}\|_{H_r(D)} \sim \frac{1}{j^{r-1}}$$

for all $r \geq 0$. On the other hand, (6), (9), and (2) together imply

$$\|F_j\|_{H_r(B_N)} \sim \frac{1}{j^{r-(N+1)/2}}$$

for all $r \geq 0$. Now pick $c_j \in \mathbb{C}$ so that

$$\sum_{k=1}^{\infty} \frac{|c_j|^2}{j^{r-1}} < \infty \quad \text{but} \quad \sum_{j=1}^{\infty} \frac{|c_j|^2}{j^{r-(N+1)/2}} = \infty;$$

for example, pick $c_j = j^{r/2-1-\varepsilon}$ with $0 < \varepsilon \leq (N - 1)/4$. Then $F(z) = \sum_{j=1}^{\infty} c_j F_j(z)$ is not in $H_r(B_N)$ although all its slice functions $F_{j\zeta}$ are in $H_r(D)$. 


3. Gleason’s problem and resolvent operators

In this section we solve Gleason’s problem in the spaces $\mathcal{H}_r$. Recall that Gleason’s problem consists of finding functions $g_1, \ldots, g_N$ in $\mathcal{H}_r$ satisfying

$$f(z) - f(a) = \sum_{k=1}^{N} (z_k - a_k)g_k(z,a)$$

for all $z \in \mathbb{B}_N$ for given $f$ in $\mathcal{H}_r$ and $a \in \mathbb{B}_N$. Said differently, it is the problem of determining if the coordinate functions shifted to $a$ generate the maximal ideal consisting of all $f \in \mathcal{H}_r$ with $f(a) = 0$; see [10, §6.6.1]. The case $r = 1$ was considered in [2], where $g_k$ turned out to be the $k$th backward shift (resolvent) operator in $\mathcal{H}_1$. Here the situation is reversed, and we define the backward shift operators by solving Gleason’s problem first. We obtain our results in two stages, first for the kernel functions $K_r(\cdot, w)$, and then for arbitrary functions in $\mathcal{H}_r$. The cases $r > 0$ and $r = 0$ are treated the same way. So let first $f(z) = K_r(z, w)$ for some arbitrary but fixed $w \in \mathbb{B}_N$. Following §6.6.2 of [10], we see that the functions

$$g_k(z,a) = \frac{\overline{w}_k}{\langle z - a, w \rangle} (K_r(z,w) - K_r(a,w))$$

(13)
do the job provided they lie in $\mathcal{H}_r$. To prove they do, we use (5) and (6). Then

$$g_k(z,a) = \frac{\overline{w}_k}{\langle z - a, w \rangle} \sum_{j=1}^{\infty} k_{r,j} (\langle z, w \rangle^j - \langle a, w \rangle^j)$$

$$= \frac{\overline{w}_k (\langle z, w \rangle - \langle a, w \rangle)}{\langle z - a, w \rangle} \sum_{j=1}^{\infty} k_{r,j} \sum_{l=0}^{j-1} \langle a, w \rangle^l \langle z, w \rangle^{j-l-1}$$

$$= \overline{w}_k \sum_{l=0}^{\infty} \langle a, w \rangle^l \sum_{j=l+1}^{\infty} k_{r,j} \langle z, w \rangle^{j-l-1}$$

$$= \overline{w}_k \sum_{l=0}^{\infty} \langle a, w \rangle^l \sum_{j=0}^{\infty} k_{r,j+l+1} \langle z, w \rangle^j,$$

which converges absolutely. Call the inner sum $K_{r+1}^l(z,w)$. Then

$$K_{r+1}^l(z,w) = \frac{1}{\langle z, w \rangle^{l+1}} \sum_{j=l+1}^{\infty} k_{r,j} \langle z, w \rangle^j$$

$$= \frac{1}{\langle z, w \rangle^{l+1}} \left( K_r(z,w) - \sum_{j=0}^{l} k_{r,j} \langle z, w \rangle^j \right)$$

$$= k_{r,l+1} + k_{r,l+2} \langle z, w \rangle + k_{r,l+3} \langle z, w \rangle^2 + \cdots,$$
where $k_{r,j}$ are defined by (10). So by (7), $\|K_{r+1}^{l+1}(\cdot, w)\|_{\mathcal{H}_r} \leq \|K(\cdot, w)\|_{\mathcal{H}_r}$ for any $w$. Hence the linear operator $A_l$ defined on the span of the functions $K_r(\cdot, w)$ that takes $K_r(z, w)$ to $K_{r+1}^{l+1}(z, w)$ is bounded with $\|A_l\| \leq 1$, and $K_{r+1}^{l+1}(z, w)$ belongs to $\mathcal{H}_r$. Then also

$$\|g_k\|_{\mathcal{H}_r} \leq \sum_{l=0}^{\infty} |a|^l \|K(\cdot, w)\|_{\mathcal{H}_r} \leq K_r(w, w) \sum_{l=0}^{\infty} |a|^l = \frac{K_r(w, w)}{1 - |a|} < \infty.$$ 

Thus the $g_k$ belong to $\mathcal{H}_r$ and the solution to Gleason’s problem on the span of the functions $K_r(\cdot, w)$ is complete.

Another consequence we derive from a computation like the last one is that the operator $T_k^r(a)^* = \sum_{l=0}^{\infty} (a, w)^l A_l$ defined on the span of the functions $K_r(\cdot, w)$ is bounded with

$$\|T_k^r(a)^*\| \leq \frac{1}{1 - |a|},$$

which does not depend on $w$. This fact allows us to extend the operator $T_k^r(a)^*$ as a bounded operator with the bound in (14) to all of $\mathcal{H}_r$ since it contains the span of the functions $K_r(\cdot, w)$ densely. We conclude that Gleason’s problem is solvable for $f \in \mathcal{H}_r$ with $g_k = T_k^r(a)^*(f)$. We call the operator $T_k^r(a)^*$ the backward shift (resolvent) operator on $\mathcal{H}_r$ in analogy with the results in [2].

The adjoint $T_k^r(a)$ of each $T_k^r(a)^*$ is also bounded on $\mathcal{H}_r$, and we call it the forward shift operator on $\mathcal{H}_r$. The computation

$$T_k^r(a)(K_r(z, w))(v) = (T_k^r(a)(K_r(z, w)), K_r(z, v))_{\mathcal{H}_r} = (K_r(z, w), \frac{\overline{v}_k}{(z - a, v)} (K_r(z, v) - K_r(a, v)))_{\mathcal{H}_r} = \frac{\overline{v}_k}{(w - a, v)} (K_r(w, v) - K_r(a, v))$$

which uses the reproducing property of $K_r(z, w)$ yields

$$T_k^r(a)(K_r(z, w)) = \frac{zk}{\langle z, w - a \rangle} (K_r(z, w) - K_r(z, a)).$$

In terms of series, this is
\[ T_k^r(a)(K_r(z, w)) = z_k \sum_{l=0}^{\infty} \langle z, a \rangle^l \sum_{j=l+1}^{\infty} k_{r,j} \langle z, w \rangle^{j-1-l} \]

\[ = z_k \sum_{l=0}^{\infty} \langle z, a \rangle^l K_r^{l+1}(z, w). \]

Setting \( w = 0 \) when \( r > 0 \) gives

\[ T_k^r(a)^*(1) = 0 \quad \text{and} \quad T_k^r(a)(1) = \frac{z_k}{\langle z, a \rangle} (K_r(z, a) - 1). \]

**Remark 3.1.** A careful examination of the above shows that our procedure solves Gleason’s problem and defines shift operators in any reproducing kernel Hilbert space on the ball whose reproducing kernel is a holomorphic function of \( \langle z, w \rangle \).

It is possible to write down an explicit formula for the action of \( T_k^r(a) \) on an arbitrary \( f \in \mathcal{H}_r \). By the reproducing property of \( K_r \) and using (6), (9), and (10), we obtain

\[ T_k^r(a)(f)(w) = \left( T_k^r(a)(f(z)), K_r(z, w) \right)_{\mathcal{H}_r} \]

\[ = \left( \sum_{\alpha} f_\alpha z^\alpha, \sum_{l=0}^{\infty} \langle a, w \rangle^l \sum_{j=0}^{\infty} k_{r,j,l+1} \langle z, w \rangle^j \right)_{\mathcal{H}_r} \]

\[ = \left( \sum_{\alpha} f_\alpha z^\alpha, \sum_{l=0}^{\infty} \langle a, w \rangle^l \sum_{j=0}^{\infty} k_{r,j,l+1} \langle z, w \rangle^j \right)_{\mathcal{H}_r} \]

\[ = w_k \sum_{l=0}^{\infty} \langle w, a \rangle^l \sum_{\alpha} k_{r,|\alpha|+l+1} \frac{|\alpha|!}{\alpha!} z^\alpha \|z\|_r^2 \|w\|_r^2 f_\alpha w_\alpha \]

\[ = w_k \sum_{l=0}^{\infty} \langle w, a \rangle^l \sum_{\alpha} k_{r,|\alpha|+l+1} \frac{1}{k_{r,|\alpha|}} f_\alpha w_\alpha, \]

which yields

\[ T_k^r(a)(f(z)) = z_k \sum_{l=0}^{\infty} \langle z, a \rangle^l \sum_{j=0}^{\infty} \frac{k_{r,j+l+1}}{k_{r,j}} \sum_{|\alpha| = j} f_\alpha z^\alpha. \]

An application of (3) gives \( k_{r,j+l+1}/k_{r,j} \sim 1 \) as \( j \to \infty \) for each \( l \), and this implies that \( T_k^r(a)(f) \) is in \( \mathcal{H}_r \) since \( f \) is. In particular, when \( f(z) = z^\alpha \), we have

\[ T_k^r(a)(z^\alpha) = \frac{z_k z^\alpha}{k_{r,|\alpha|}} \sum_{l=0}^{\infty} k_{r,|\alpha|+l+1} \langle z, a \rangle^l = \frac{z_k z^\alpha}{k_{r,|\alpha|}} K_r^{|\alpha|+1}(z, a). \]
A similar computation shows that $T_k^0(a)(1) = 0$ since $\|1\|_{\mathcal{H}_0} = 0$. From this it follows by the orthogonality of $\{z^\alpha\}$ in $\mathcal{H}_0$ that $T_k^0(a)^*(1) = 0$ too.

The corresponding results for $T_k^r(a)^*$ are more complicated to write because of the presence of $z$ in too many places in $T_k^r(a)(K_r(z, w))$. Denoting by $1_k$ the multi-index whose coordinates are all 0 except for a 1 in $k$th position, we have

$$T_k^r(a)^*(f(z)) = \sum_\alpha \sum_{\beta + \gamma + 1_k = \alpha} \frac{\gamma!|\beta|!}{\alpha!} \frac{\alpha!}{\gamma!\beta!} f_\alpha a^\gamma w^\beta.$$

A case with a simple answer is $T_k^r(a)^*(z_l) = \delta_{kl}$, where $\delta$ is Kronecker’s delta. This follows from the obvious identity $z_l - a_l = \sum_{k=1}^N (z_k - a_k)\delta_{kl}$.

Let us investigate further the case that $r$ is a positive integer. Equation (13) then takes the simpler form

$$T_k^r(a)^*(K_r(z, w)) = \frac{\overline{w}_k}{\langle z - a, w \rangle} \frac{\langle z - a, w \rangle^r - (1 - \langle z, w \rangle)^r}{\langle z, w \rangle - \langle a, w \rangle} \frac{\langle z, w \rangle - \langle a, w \rangle}{\langle z, w \rangle^r(1 - \langle a, w \rangle)^r} \times \sum_{l=0}^{r-1} (1 - \langle a, w \rangle)^{r+1-l} (1 - \langle z, w \rangle)^l = \frac{1}{\overline{w}_k} \sum_{l=1}^r (1 - \langle a, w \rangle)^{r+1-l} (1 - \langle z, w \rangle)^l.$$

Similarly

$$T_k^r(a)(K_r(z, w)) = z_k \sum_{l=1}^r \frac{1}{(1 - \langle z, a \rangle)^{r+1-l} (1 - \langle z, w \rangle)^l}$$

and

$$T_k^r(a)(z_\alpha) = \left(\frac{r - 1 + |\alpha|}{r - 1}\right)^{-1} z_k^\alpha \sum_{l=1}^r \frac{(l - 1 + |\alpha|)}{l - 1} \frac{1}{(1 - \langle z, a \rangle)^{r+1-l}}.$$

The results for $T_k^1(a)$ and $T_k^1(a)^*$ are identical to those given in [2].

4. Homogeneous interpolation in the spaces $\mathcal{H}_r$

In [2] we proved the following theorem for $r = 1$.

**Theorem 4.1.** Let $a \in \mathbb{B}_N$ and $c \in \mathbb{C}^{n \times 1}$. Then

$$\mathcal{N} \overset{\text{def}}{=} \{ f \in \mathcal{H}^{n \times 1}_r \mid c^* f(a) = 0 \} = B_a \mathcal{H}^{(n+N-1) \times 1}_r,$$
where the equality is as sets and $B_a$ is the $C_{n \times (n+N-1)}$-valued function given by

$$B_a(z) = U \begin{pmatrix} b_a(z) & 0_{(n-1)\times N} \\ 0_{(n-1)\times N} & I_{n-1} \end{pmatrix}.$$ 

In the above expression, $U$ is any unitary matrix whose first column is $c/|c|$, and $b_a$ is the $1 \times N$ Blaschke factor

$$b_a(z) = \frac{(1 - |a|^2)^{1/2}}{1 - \langle z, a \rangle}(z - a)(I_N - a^*a)^{-1/2}.$$ (15)

**Proof.** We first prove that the operator of multiplication by $z_j$ is bounded in the spaces $H^r (r > 0)$ and in the Dirichlet space. Equivalently, we have to show that there is a positive constant $p$ such that the function $(p - \langle z, w \rangle)K_r(z, w)$ is positive in the ball. For $r > 0$ and $p$ to be determined in the sequel, we have

$$K_r(z, w) = \frac{p - \langle z, w \rangle}{1 - \langle z, w \rangle} = \left( \sum_{j=0}^{\infty} \frac{\Gamma(r+j)}{\Gamma(r)j!} \langle z, w \rangle^j \right)$$

$$= p + \sum_{j=1}^{\infty} \left( p \frac{\Gamma(r+j)}{\Gamma(r)j!} - \frac{\Gamma(r+j-1)}{\Gamma(r)(j-1)!} \right) \langle z, w \rangle^j.$$ (16)

Hence the function (16) is positive for $p$ satisfying

$$p \frac{\Gamma(r+j)}{\Gamma(r)j!} - \frac{\Gamma(r+j-1)}{\Gamma(r)(j-1)!} \geq 0 \quad (j = 1, 2, \ldots),$$

which is equivalent to finding a $p \geq j/(r+j-1)$ for all $j = 1, 2, \ldots$. So we can pick $p = 1/r$ for the given value of $r$. When $r = 0$, we use expansion (8) and see that it suffices to take $p = 1$.

Next we prove the claim of the theorem for $n = 1$ and $c = 1$. By Section 3, $f \in \mathcal{N}$ if and only if $f(z) = (z-a)g(z)$ with $g \in H^N_r$. Thus

$$f(z) = (z-a)g(z) = \frac{1}{1 - \langle z, a \rangle}(z - a)(1 - \langle z, a \rangle)g(z) = b_a(z)h(z),$$

where, in view of the preceding step, the vector-valued function

$$h(z) = \frac{1}{(1 - |a|^2)^{1/2}}(1 - \langle z, a \rangle)(I_N - a^*a)^{1/2}g(z)$$

has its entries in $H^r_r$.

When $n > 1$, by the previous case we have

$$c^*f(z) = b_a(z)g(z)$$

for some $g \in H^N_{r \times 1}$. Thus
\[ f(z) = \frac{cc^*}{c^*c} f(z) + \left( I_n - \frac{cc^*}{c^*c} \right) f(z) = \frac{cb_a(z)g(z)}{c^*c} + \left( I_n - \frac{cc^*}{c^*c} \right) f(z) = \left( \frac{c}{c^*c} \right) \left( \frac{b_a(z)g(z)}{c^*c} \right) \] 

To conclude, we first note that \( c^*f(z) \) does not depend on 
\[ \left( I_n - \frac{cc^*}{c^*c} \right) f(z) \] 
and thus this last expression is arbitrary. Also \( \text{rank}(I_n - \frac{cc^*}{c^*c}) = n - 1 \), and the only nonzero eigenvalue of this matrix is 1 since \( I_n - \frac{cc^*}{c^*c} \) is a projection. So there is a unitary matrix \( U_1 \) such that 
\[ I_n - \frac{cc^*}{c^*c} = U_1 (\text{diag}(1,1,\ldots,1,0)) U_1^*. \] 

Let \( V_1 \) denote the \( n \times (n-1) \) matrix that consists of the first \( n-1 \) columns of \( U_1 \). We can write 
\[ f(z) = \left( \frac{c}{c^*c} V_1 \right) \left( \begin{array}{c} b_a \\ 0_{(n-1)\times N} \\ 0_{1 \times (n-1)} \\ I_{n-1} \\ g_3(z) \end{array} \right), \]

where \( g_3 \) is the \( \mathbb{C}^{n-1} \)-valued function whose components are the first \( n-1 \) components of \( U_1^*g_2(z) \). The matrix 
\[ U = \left( \frac{c}{c^*c} V_1 \right) \]
is unitary since the ranges of \( c/(c^*c) \) and \( V_1 \) are orthogonal, and this concludes the proof. \( \square \)

**Proof of Theorem 1.2.** This is obtained from Theorem 4.1 recursively. A function \( f \in \mathcal{H}_r^{n \times 1} \) satisfies the interpolation conditions (4) if and only if it can be written as 
\[ f(z) = B_{a_1,c_1}(z)g(z), \]
where \( g \in \mathcal{H}_r^{(n+N-1) \times 1} \) satisfies the interpolation conditions 
\[ c_j^*B_{a_1,c_1}(a_j)g(a_j) = d_j^*g(a_j) = 0 \quad (j = 2,\ldots,m), \]
where \( d_j^* = c_j^*B_{a_1,c_1}(a_j) \) for \( j = 2,\ldots,m \). We thus find the set of all \( g \in \mathcal{H}_r^{(n+N-1) \times 1} \) for which \( d_2^*g(a_2) = 0 \) first and then iterate on \( g \). \( \square \)

**5. The case of integer \( r \)**

We now specialize to the case that \( r \) is a positive integer. The analysis is then very close to that of [2].
Theorem 5.1. Let $r$ be a positive integer. Then

$$N \overset{\text{def}}{=} \{ f \in \mathcal{H}_r^n \times 1 \mid c^* f(a) = 0 \} = \mathcal{B}_a \mathcal{H}_r^{(n+N-1) \times 1},$$

where equality and $\mathcal{B}_a$ are as in Theorem 4.1, and the norm in $N$ is equivalent to the norm

$$\| \mathcal{B}_a f \| = \| (I - \pi) f \|_{\mathcal{H}_r^{(n+N-1) \times 1}}$$

with $\pi$ denoting the orthogonal projection on the set of functions $f \in \mathcal{H}_r^{(n+N-1) \times 1}$ satisfying $\mathcal{B}_a(z)f(z) \equiv 0$.

Proof. We set $N^\perp = \mathcal{M}$ to be the one-dimensional subspace of $\mathcal{H}_r^n \times 1$ spanned by the function $c(1 - \langle z, a \rangle)^{-r}$. Then $\mathcal{M}$ is a reproducing kernel Hilbert space whose reproducing kernel is

$$K_{\mathcal{M}}(z, w) = \left( \frac{I_n - \mathcal{B}_a(z) \mathcal{B}_a(w)}{1 - \langle z, a \rangle} \right)^r.$$

The arguments are as in [2] and repeated for completeness. We first consider the case $n = 1$ and $c = 1$. Then the reproducing kernel of $\mathcal{M}$ is

$$K_{\mathcal{M}}(z, w) = \frac{(1 - \langle a, a \rangle)^r}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)} = \left( \frac{1 - \mathcal{B}_a(z) \mathcal{B}_a(w)}{1 - \langle z, w \rangle} \right)^r,$$

where we have used the formula (see [10, Theorem 2.2.2] and [2] for a different proof)

$$\frac{1 - \mathcal{B}_a(z) \mathcal{B}_a(w)^*}{1 - \langle z, w \rangle} = \frac{1 - \langle a, a \rangle}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)} (z, w \in \mathcal{B}_N)$$

to get the second identity. Next the choice of $c_0 = (1 \ 0 \ \ldots \ 0)^*$ leads to

$$K_{\mathcal{M}}(z, w) = \left( \frac{(1 - \langle a, a \rangle)^r}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)} \right)^r \mathbb{1}.$$
which is of the asserted form with
\[ B_a(z) = \begin{pmatrix} b_a(z) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix}. \]

Finally, let \( c \neq 0_{n \times 1} \) and \( U \) be a unitary matrix such that \( Uc_0 = c/|c| \). Then \( K_M(z, w) \) is equal to
\[
\frac{cc^*}{c^*c} \left( 1 - \frac{b_a(z)b_a(w)^*}{1 - \langle z, w \rangle} \right)^r = Uc_0c_0^* \frac{1 - b_a(z)b_a(w)^*}{1 - \langle z, w \rangle} U^* \\
= U \left( \frac{I_n - \begin{pmatrix} b_a(z) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix}}{1 - \langle z, w \rangle} \begin{pmatrix} b_a(w) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix}^* \right)^r \\
= \left( \frac{I_n - U \begin{pmatrix} b_a(z) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix} \begin{pmatrix} b_a(w) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix}^*}{1 - \langle z, w \rangle} \right)^r,
\]
since \( U \) is unitary.

Now we prove the claim of the theorem for \( n = 1 \) and \( c = 1 \). We write
\[
\frac{1 - (1 - b_a(z)b_a(w))^r}{(1 - \langle z, w \rangle)^r} = b_a(z)C_r(z, w)b_a(w)^*,
\]
where \( C_r(z, w) \) is the \( C^{N \times N} \)-valued positive function defined by
\[
C_r(z, w) = \sum_{j=0}^{r-1} \left( \frac{1 - b_a(z)b_a(w)}{1 - \langle z, w \rangle} \right)^j \frac{I_N}{(1 - \langle z, w \rangle)^{r-j}}.
\]
Each of the functions
\[
\left( \frac{1 - b_a(z)b_a(w)}{1 - \langle z, w \rangle} \right)^j \frac{I_N}{(1 - \langle z, w \rangle)^{r-j}} \quad (j = 0, 1, \ldots, r - 1)
\]
is positive in the ball and hence
\[
\frac{I_N}{(1 - \langle z, w \rangle)^r} \leq C_r(z, w).
\]
Since
\[
\frac{I_N}{(1 - \langle z, w \rangle)^r} - \left( \frac{1 - b_a(z)b_a(w)^*}{1 - \langle z, w \rangle} \right)^j \frac{I_N}{(1 - \langle z, w \rangle)^{r-j}} \\
= \frac{1 - (1 - b_a(z)b_a(w)^*)^j}{(1 - \langle z, w \rangle)^r} I_N \\
= \frac{b_a(z)b_a(w)^*}{1 - \langle z, w \rangle} (\ldots)^k I_N \geq 0,
\]
we have

\[
\frac{I_N}{(1 - \langle z, w \rangle)^r} \leq C_r(z, w) \leq m \frac{I_N}{(1 - \langle z, w \rangle)^r}.
\]

Therefore, as vector spaces,

\[
\mathcal{H}(C_r) = \mathcal{H}\left( \frac{I_N}{(1 - \langle z, w \rangle)^r} \right),
\]

their norms are equivalent (this last fact is also a consequence of Banach’s theorem), and \(M_{\perp} = b_a \mathcal{H}^{N \times 1}_r\). The last claim on the norm is easily verified. Next for \(n > 1\) we first take

\[
c = c_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

The reproducing kernel of \(M_{\perp}\) is then

\[
K_{M_{\perp}}(z, w) = \frac{I_n}{(1 - \langle z, a \rangle)^r}
\]

\[
- \frac{(1 - \langle a, a \rangle)^r}{(1 - \langle z, a \rangle)^r (1 - \langle a, w \rangle)^r} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 - (1-b_a(z)b_a(w))r \\ (1 - \langle z, w \rangle)^r \end{pmatrix} \begin{pmatrix} 0_{(n-1) \times 1} \\ I_{n-1} \end{pmatrix}.
\]

Thus, with the current choice of \(c = c_0\) and taking into account the results for \(n = 1\), we have

\[
M_{\perp} = \left\{ F(z) = \begin{pmatrix} b_a(z) f(z) \\ g(z) \end{pmatrix} \bigg| f \in \mathcal{H}^{N \times 1}_r, \ g \in \mathcal{H}^{(n-1) \times 1}_r \right\}
\]

\[
= B_{a} \mathcal{H}^{(\alpha+N-1) \times 1}_r.
\]

The case of arbitrary \(c\) is easily adapted. \(\square\)

**Remark 5.2.** The case where \(r = N\) is of special interest since it concerns the Hardy space of the ball. Assume that \(B\) is \(\mathbb{C}^{1 \times \ell}\)-valued. Then for any \(f \in \mathcal{H}^{\ell \times \ell}_N\) we have

\[
\|Bf\|_{\mathcal{H}^{\ell \times \ell}_N} = \int_{\mathbb{S}} \text{Tr} f(z)^* B(z)^* B(z) f(z) \, d\sigma(z)
\]

\[
= \int_{\mathbb{S}} \text{Tr} B(z) f(z) f(z)^* B(z)^* \, d\sigma(z),
\]

which is equal to 1 for functions \(f\) taking unitary values on \(\mathbb{S}\).
Note added in proof

The reproducing kernel Hilbert spaces $\mathcal{H}_r$ for $r \leq 0$ with reproducing kernels

$$K_r(z, w) = 1 + \sum_{j=1}^{\infty} \frac{\Gamma(1-r)(j-1)!}{\Gamma(1-r+j)} \langle z, w \rangle^j \quad (r \leq 0)$$

are studied in [12]. These spaces naturally extend the spaces $\mathcal{H}_r$ ($r \geq 0$) of this paper to all real $r$, and the two families have many properties in common. In fact, when $r = 0$, (17) reduces to the $K_0(z, w)$ given earlier. The $k_{r,j}$ corresponding to (17) satisfies $k_{r,j} \sim 1/j^{1-r}$ for $r < 0$ and large $j$ exactly as it does for $r \leq 0$, although

$$\|z^\alpha\|^2_{\mathcal{H}_r} = \frac{\Gamma(1-r+|\alpha|)|\alpha|!}{\Gamma(1-r)(|\alpha|!)^2} \quad (r \leq 0)$$

is ostensively different.

All the new kernels are complete Nevanlinna–Pick kernels by Lemma 7.38 of [13]. Thus the spaces $\mathcal{H}_r$ have the complete Nevanlinna–Pick property if and only if $r \leq 1$. By contrast, a reworking of Example 5.17 of [13] shows that none of the spaces $\mathcal{H}_r$ for $r > 1$ has the weaker two-point scalar Pick property.

Continuing, $\mathcal{H}_r \subset \mathcal{H}_s$ if $-\infty < r < s < \infty$. In addition to Remark 2.3, the inclusion is a contraction also when $r < 0$ if $s \geq 1$ or $s \leq 0$. The first statement of Proposition 2.5 and Proposition 2.6 hold for $\mathcal{H}_r$ when $r < 0$ too, as well as Examples 2.4 and 2.7.

Remark 3.1 already shows that the work of Section 3 carries over to $\mathcal{H}_r$ for $r < 0$. Finally, repeating the first part of the proof of Theorem 4.1 yields that the operator of multiplication by $z_j$ is a contraction (choosing $p = 1$ works) on $\mathcal{H}_r$ when $r < 0$. Thus the homogeneous interpolation described in Theorem 4.1 is valid in $\mathcal{H}_r$ for any real $r$.

References


