Russian Journal of Mathematical Physics, Vol. 12, No. 3, 2005, pp. 363–368.

Bohr Radii of Elliptic Regions

H. T. Kaptanoğlu* and N. Sadık**

**Mathematics Department, Middle East Technical University, Ankara 06531, Turkey*
http://www.math.metu.edu.tr/∼kaptan/
E-mail: kaptan@math.metu.edu.tr
***Mathematics Department, Istanbul University, Vezneciler, Istanbul 34459, Turkey*
http://www.istanbul.edu.tr/fen/matematik/nazsad.htm
E-mail: sadnaz@istanbul.edu.tr

Received April 14, 2004

Abstract. We use Faber series to define the Bohr radius for a simply connected planar domain bounded by an analytic Jordan curve. We estimate the value of the Bohr radius for elliptic domains of small eccentricity and show that these domains do not exhibit Bohr phenomenon when the eccentricity is large. We obtain the classical Bohr radius as the eccentricity tends to 0.

1. INTRODUCTION

A classical theorem of Bohr states that, if a holomorphic function f with power series of the form $f(z) = \sum a_n z^n$ maps the open unit disc $\mathbb{D}$ to itself, then $\sum |a_n z^n| < 1$ for all $|z| < 1/3$. The value $1/3$ is the best possible and called the *Bohr radius* of $\mathbb{D}$. The interest in the Bohr phenomena was revived in the nineties due to the discovery of generalizations to domains in $\mathbb{C}^n$ and to more abstract settings; see [1–3, 6, 8, 12]. But the only domain on which the exact value of the Bohr radius is known is $\mathbb{D}$. Exact values for harmonic functions were found in [4, 10].

In this paper, we return to $\mathbb{C}$ and study Bohr phenomena on a planar domain E. There are two problems to be resolved. Power series must be replaced by another series that converge globally on E. Certain subsets of E must be associated to discs in such a way that the concept of radius can make sense. Under some conditions on E, the space of holomorphic functions $H(E)$ on E can be equipped with a basis $\{F_n\}$ of polynomials, the so-called Faber polynomials of E, which means that, if $f \in H(E)$, then $f = \sum a_n F_n$ on the entire set E. Faber polynomials are defined by using a mapping of the complement of E onto the complement of a disc. Properties of Faber polynomials enable us to define the corresponding Bohr radius for E in a natural way in Section 2.

In Section 3, we concentrate on elliptic regions E. The reasons are that the computation of the Bohr radius must be made in a specific basis, and an ellipse is a region whose Faber polynomials are known (and sufficiently simple); they are essentially Chebyshev polynomials. We find lower and upper bounds for the Bohr radius of E if the eccentricity of the ellipse is less than about 0.37 and 0.41, respectively. These bounds are roots of a certain quadratic polynomials. Moreover, we show that an ellipse fails to have the Bohr property in question if the eccentricity is more than about 0.41. For precise statements, see Theorems 7 and 8 below, which are our main results. However, as the eccentricity reduces to 0, the set E becomes the disk $\mathbb{D}$, and both the bounds for the Bohr radius of E increase and converge to the classical Bohr radius equal to $1/3$.

2. FABER POLYNOMIALS AND FABER SERIES

Let us first survey the basics of Faber theory to define the related Bohr radius for a sufficiently general domain. Much of this section can be found in our standard references.

Let E be a nonempty bounded simply connected domain in $\mathbb{C}$. The closure $\overline{E}$ is compact, and the complement $\overline{E}^c$ of this closure in the extended plane $\mathbb{C} \cup \{\infty\}$ is a simply connected set containing ∞.

The work of the second author is supported by the Research Fund of İstanbul University, project number UDP-259/29032004.

By the Riemann mapping theorem, there is a unique conformal (univalent) map Φ of $\overline{E}^c$ onto $\overline{\mathbb{D}}^c$ such that

$$\Phi(\infty) = \infty \qquad \text{and} \qquad \Phi'(\infty) = \lim_{z\to\infty} \Phi(z)/z = 1/d > 0.$$

Here d stands for the *logarithmic capacity* (or *transfinite diameter*) of $\overline{E}$ and can be defined by $d = \exp(\lim_{z\to\infty}(\log|z| - \mathcal{G}(z,\infty)))$, where $\mathcal{G}$ is the Green function for $\overline{E}^c$ with singularity at ∞; see [9], pp. 114–119.

The function Φ and its nonnegative integer powers Φ^n have Laurent series at ∞ of the form $\Phi(z) = (1/d)z + \gamma_0 + \gamma_{-1}/z + \gamma_{-2}/z^2 + \cdots$ and

$$\Phi^n(z) = (1/d^n)z^n + \gamma_{n,n-1}z^{n-1} + \cdots + \gamma_{n,0} + \gamma_{n,-1}/z + \gamma_{n,-2}/z^2 + \cdots .$$

The principal part of the Laurent expansion of $\Phi^n(z)$ at ∞ is called the nth *Faber polynomial* $P_n(z)$ of $\overline{E}$ (or of Φ). Hence,

$$P_0(z) = 1, \qquad P_1(z) = z/d + \gamma_0, \qquad P_n(z) = z^n/d^n + \gamma_{n,n-1}z^{n-1} + \cdots + \gamma_{n,1}z + \gamma_{n,0},$$

and P_n is a polynomial of degree n.

Suppose now that the boundary ∂E of E is an analytic Jordan curve, which is the situation originally treated by Faber. This occurs, for example, if E is bounded by an ellipse, which is our main case of interest. Let $\Psi = \Phi^{-1} : \overline{\mathbb{D}}^c \to \overline{E}^c$. Then Ψ continues holomorphically across $\partial\mathbb{D}$, and there is a least radius $R < 1$ such that Ψ is univalent for $|w| > R$; see [7]. For $r > R$, let $C_r = \{\, z = \Psi(w) \,:\, |w| = r \,\}$ be a level set of Φ. This is a Jordan curve. Let E_r be the interior of C_r. Specifically, $C_1 = \partial E$ and $E_1 = E$. Then the above extension result concerning Ψ can be restated as follows: Φ is holomorphic and univalent on $\overline{E_R}^c$.

The following analogs of Abel theorems and Cauchy estimates are due to Faber and can be found in [13, Chap. III; 11, Sec. 3.14]. If ∂E is not analytic, then R is taken as 1.

Theorem 1. *Suppose that the series* $\sum a_n P_n$ *satisfies the condition* $\limsup \sqrt[n]{|a_n|} = 1/r < 1/R$. *Then the series converges absolutely on* E_r *and uniformly on its compact subsets, and thus the sum is in* $H(E_r)$.

Theorem 2. *If* $r > R$ *and* $f \in H(E_r)$, *then* f *can be represented on* E_r *uniquely as the sum of a series of the form* $f(z) = \sum a_n P_n(z)$, *which converges absolutely on* E_r *and uniformly on its compact subsets.*

Thus, the Faber polynomials $\{P_n\}$ form a natural basis for the space $H(E)$ if ∂E is analytic. This basis satisfies the necessary condition $P_0 \equiv 1$ in Proposition 3.1 of [3] for the domain to have the Bohr property.

The series in Theorems 1 and 2 are called *Faber series*. Let $F_n(w) = P_n(\Psi(w))$ for $|w| > R$. These functions are often easier to work with than $P_n(z)$. For $f \in H(E_r)$, set $F(w) = f(\Psi(w))$. Then we also have $F(w) = \sum a_n F_n(w)$, at least for $R < |w| < r$.

Corollary 3. *Under the conditions of Theorem* 2, $|a_n| \leqslant (\max\{\, |f(z)| \,:\, z \in C_r \,\})/r^n$.

Thus, for Faber series, the curves C_r play the role played by circles for power series.

Definition 4. Let E be a bounded simply connected domain in $\mathbb{C}$ whose boundary is an analytic Jordan curve. We say that E has the *Bohr property* if there is an r with $R < r < 1$ such that, if $f\colon E \to \mathbb{D}$ is holomorphic and has the Faber expansion $f(z) = \sum a_n P_n(z)$, then

$$\sum |a_n| \sup_{z\in E_r} |P_n| < 1.$$

We define the *Bohr radius* B of E as the least upper bound of the numbers r.

Equivalently, B is the greatest lower bound of the numbers r such that

$$\sum |a_n| \sup_{R<|w|<r} |F_n(w)| < 1.$$

We can also say that the basis $\{P_n\}$ has the Bohr property, and B is the Bohr radius of $\{P_n\}$.

Example 5. Let us study a test case. Let $E = D(c,d)$ be the disc of radius d centered at c. Then $\Phi\colon \overline{D(c,d)}^c \to \overline{\mathbb{D}}^c$ is given by $\Phi(z) = (z-c)/d$ and this shows that the capacity of a disc is its radius. The inverse of Φ is $\Psi(w) = dw+c$. Both Φ and Ψ are holomorphic and univalent on $\mathbb{C}$. Then $P_n(z) = \Phi^n(z) = (z-c)^n/d^n$ and $F_n(w) = w^n$. For any $r > 0$, we have $E_r = \{\, z \in E \,:\, |z-c| < rd \,\}$ and $\sup\{\, |P_n(z)| \,:\, z \in E_r \,\} = r^n$. If $f(z) = \sum a_n P_n(z)$ takes E to $\mathbb{D}$, then $\sum |a_n| \sup_{E_r} |P_n| = \sum |a_n| r^n$, and this is the series used in the classical Bohr theorem. Therefore, $E_{1/3} = D(c, d/3)$ is the largest set which is the Φ-inverse image of a disc on which $\sum |a_n| \sup_{E_r} |P_n| < 1$.

We need the following lemma, which is a Carathéodory inequality for a function with singularity at 0.

Lemma 6. *If a function*

$$G(w) = b_0 + \sum_{n=1}^{\infty} \big(b_n w^n + c_n/w^n\big)$$

is holomorphic on $R < |w| < 1$ for some $R \geqslant 0$ and has positive real part, then $\big||b_n| - |c_n|\big| \leqslant 2\operatorname{Re} b_0$ for $n = 1, 2, \dots$

Proof. By considering radii reducing to 1, we may assume that G is holomorphic in a neighborhood of $\partial\mathbb{D}$. We take $n = 1, 2, \dots$ and compute

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\, G(e^{i\theta})\, d\theta = b_n \qquad \text{and} \qquad \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\, \overline{G(e^{i\theta})}\, d\theta = \overline{c_n}.$$

Adding yields

$$\big||b_n| - |c_n|\big| \leqslant |b_n + \overline{c_n}| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |e^{-in\theta}|\, |G(e^{i\theta}) + \overline{G(e^{i\theta})}|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} 2\operatorname{Re} G(e^{i\theta})\, d\theta = 2\operatorname{Re} b_0.$$

If G is holomorphic in $\mathbb{D}$, then all $c_n = 0$, and the result reduces to the Carathéodory inequality. $\square$

3. ELLIPTIC REGIONS

In the rest of the paper, E is the domain bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a > b > 0$ and $c = \sqrt{a^2 - b^2}$ is the focal distance. The Zhukovski map $\Psi(w) = \frac{1}{2}\big((a+b)w + (a-b)/w\big)$ takes the unit circle onto ∂E, and hence $\overline{\mathbb{D}}^c$ onto $\overline{E}^c$. We obtain the well-known fact:

$$d = \lim_{w\to\infty} \Psi(w)/w = (a+b)/2.$$

The derivative $\Psi'(w)$ is equal to 0 at $w = \pm R$, where

$$R = \sqrt{(a-b)/(a+b)} = c/(2d) < 1. \tag{1}$$

Thus, although Ψ is holomorphic for all $|w| > 0$, it is univalent for $|w| > R$ only. The circle $|w| = R$ is mapped onto the line segment $[-c, c] \subset E$ in a two-to-one way. The inverse of Ψ is $\Phi(z) = (z + \sqrt{z^2 - c^2})/(a+b)$, and it is holomorphic and univalent on $E \setminus [-c, c]$.

The *eccentricity* $\varepsilon < 1$ of an ellipse is the ratio $c/a = \sqrt{1 - (b/a)^2}$. Solving with respect to the ratio b/a in (1), we obtain ε in terms of R; inverting, we obtain R in terms of ε. Thus,

$$\varepsilon = (2R)/(1+R^2) \qquad \text{and} \qquad R = \big(1 - \sqrt{1-\varepsilon^2}\,\big)/\varepsilon. \tag{2}$$

Let $n = 1, 2, \dots$ A computation shows that $R^{2n}/\Phi^n(z)$ has no principal part at ∞. Then $\Phi^n(z)$ and $\Phi^n(z) + R^{2n}/\Phi^n(z)$ have the same principal part at ∞. But

$$\Phi^n(z) + R^{2n}/\Phi^n(z) = (a+b)^{-n}\big[\big(z + \sqrt{z^2 - c^2}\,\big)^n + \big(z - \sqrt{z^2 - c^2}\,\big)^n\big] \tag{3}$$

is a polynomial of degree n (since the odd powers of the radical cancel out and its even powers are polynomials). Thus, this sum is the nth Faber polynomial $P_n(z)$ of $\overline{E}$. An explicit expression for $P_n(z)$ is given in [7] in the form

$$P_n(z) = 2(a+b)^{-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} z^{n-2k} \left(z^2 - c^2\right)^k.$$

On the other hand, $F_n(w) = P_n(\Psi(w)) = \Phi^n(\Psi(w)) + R^{2n}/\Phi^n(\Psi(w)) = w^n + R^{2n}/w^n$. Of course, $P_0(z) = F_0(w) = 1$. Substituting $z = c\cos t$ into (3), we obtain $P_n(z) = 2R^n \cos(n \cos^{-1}(z/c))$. Thus, the Faber polynomials of E are suitably normalized Chebyshev polynomials.

Let $R < r < 1$. We have

$$\max_{z\in C_r} |P_n(z)| = \max_{|w|=r} |F_n(w)| = \max_{\theta} \left| r^n e^{in\theta} + \frac{R^{2n}}{r^n e^{in\theta}} \right| = r^n + \frac{R^{2n}}{r^n} > 2R^n = \max_{|w|=R} |F_n(w)|.$$

Thus,

$$\sup_{z\in E_r} |P_n(z)| = \sup_{R<|w|<r} |F_n(w)| = \max_{|w|=r} |F_n(w)| = r^n + R^{2n}/r^n. \tag{4}$$

The polynomials P_n have the property that $P_n(0) = 0$ if n is odd, and

$$P_n(0) = 2(-1)^{n/2} R^n = 2(-1)^{n/2} \big((a-b)/(a+b)\big)^{n/2} \neq 0 \qquad (2 \leqslant n \in 2\mathbb{Z}).$$

This property of Chebyshev polynomials persists also at other points the interval $[-c, c] \subset E_r$. Thus, the natural basis $\{P_n\}$ for $H(E)$ does not vanish at some fixed point, and the sufficient condition of Theorem 3.3 in [3] for the Bohr property fails. Nevertheless, it has the above Bohr property, at least for some values of R, as the following theorem shows.

Theorem 7. *Let E be a domain bounded by an ellipse of eccentricity ε. There is a constant $\varepsilon_1 \approx 0.373814$ such that, if $\varepsilon < \varepsilon_1$, then a lower bound for the Bohr radius of E is the larger root of the polynomial $q(r) = (3-R^2)r^2 - (1+4R^2-R^4)r + R^2(3-R^2)$, where R and ε are related by* (2).

Proof. Let $f(z) = \sum a_n P_n(z)$ be the Faber expansion of a nonconstant holomorphic function $f\colon E \to \mathbb{D}$. Applying the substitution $z = \Psi(w)$, we obtain a nonconstant holomorphic function $F\colon \mathbb{D} \setminus \overline{D(0,R)} \to \mathbb{D}$ with an expansion of the form

$$F(w) = a_0 + \sum_{n=1}^{\infty} a_n \big(w^n + R^{2n}/w^n\big) \qquad (|w| = r > R).$$

By Corollary 3, $|a_0| < 1$ and $|a_n| \leqslant 1$ for $n = 1, 2, \ldots$

Choose ϕ such that $e^{i\phi} a_0 = |a_0|$ and set $G(w) = 1 - e^{i\phi} F(w)$. Then G is holomorphic if F is, and the real part of G is positive. Applying Lemma 6 to G for $b_0 = 1 - e^{i\phi} a_0$, $b_n = a_n$, and $c_n = a_n R^{2n}$, we obtain $|a_n| \leqslant (2(1-|a_0|))/(1-R^{2n}) \leqslant (2(1-|a_0|))/(1-R^2)$. Then using (4) gives

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n| \sup_{z\in E_r} |P_n(z)| &\leqslant |a_0| + \frac{2(1-|a_0|)}{1-R^2} \sum_{n=1}^{\infty} \left(r^n + \frac{R^{2n}}{r^n} \right) = |a_0| + \frac{2(1-|a_0|)}{1-R^2} \left(\frac{r}{1-r} + \frac{R^2/r}{1-R^2/r} \right) \\ &= |a_0| + (1-|a_0|) \left[\frac{2}{1-R^2} \quad \frac{r^2 - 2R^2 r + R^2}{-r^2 + (1+R^2)r - R^2} \right].\end{aligned}$$

The last expression is less than 1 if and only if the product of the two bracketed fractions is less than 1, which holds if and only if

$$\frac{(3-R^2)r^2 - (1+4R^2-R^4)r + R^2(3-R^2)}{2(1-r)(r-R^2)} = \frac{q(r)}{2(1-r)(r-R^2)} < 0.$$

The denominator and the coefficient at r^2 in $q(r)$ are positive, and hence the fraction takes negative values if and only if $q(r)$ has two real roots, which occurs if and only if its discriminant $\delta(R) = (1-R^2)(1-27R^2+11R^4-R^6)$ is positive. The polynomial $1-27x+11x^2-x^3$ has two real roots that are greater than 1 (and of no interest for us) and the smallest real root between 0 and 1. This polynomial is positive for x between 0 and the smallest root. Let r_0 be the square root of the smallest root; we compute $r_0 \approx 0.193937$. Then $\delta(r)$ is positive, and $q(r)$ takes negative values if $R < r_0$. The corresponding condition on the eccentricity is $\varepsilon < \varepsilon_1 \approx 0.373814$. If $r_1 < r_2$ are the roots of $q(r)$, then $q(r) < 0$ for $r_1 < r < r_2$. After some tedious work simplifying polynomial expressions, we can show that $0 < r_1 < R < r_2 < 1$. Since $r > R$, we have $q(r) < 0$, which implies that $\sum |a_n| \sup_{E_r} |P_n| < 1$ if and only if $R < r < r_2$. Thus, the Bohr radius of E satisfies the condition $B \geqslant r_2$. $\square$

Theorem 8. *Let E be a domain bounded by an ellipse of eccentricity ε. There is a constant $\varepsilon_2 \approx 0.408804$ with the following property. If $\varepsilon < \varepsilon_2$, then an upper bound for the Bohr radius of E is the larger root of the quadratic polynomial $Q(r) = (3+5R^2)r^2 - (1+8R^2+7R^4)r + R^2(3+5R^2)$, where R and ε are related by (2). If $\varepsilon \geqslant \varepsilon_2$, then E fails to have the Bohr property.*

Proof. To obtain the upper bound in the classical Bohr theorem, one can use the disc automorphism

$$h_t(z) = (t-z)/(1-tz) = t - ((1-t^2)/t)\sum_{n=1}^{\infty} t^n z^n,$$

where $0 < t < 1$ (see [5]). By analogy with this approach, we set

$$G_t(w) = t - \frac{1-t^2}{t}\sum_{n=1}^{\infty} t^n\left(w^n + \frac{R^{2n}}{w^n}\right) \qquad (R < |w| < 1)$$

for the same values of t. The corresponding function on E is

$$g_t(z) = G_t(\Phi(z)) = t - \frac{1-t^2}{t}\sum_{n=1}^{\infty} t^n\left(\Phi^n(z) + \frac{R^{2n}}{\Phi^n(z)}\right) = t - \frac{1-t^2}{t}\sum_{n=1}^{\infty} t^n P_n(z).$$

Theorem 1 shows that g_t converges and is holomorphic on $E_{1/t} \supset E$. This can also be seen from the fact that G_t converges and is holomorphic for $tR^2 < |w| < 1/t$. This set contains the annulus $R \leqslant |w| < 1$, even if $t = 1$. Summing the series, we obtain

$$G_t(w) = \frac{w^2 - t(1+2R^2-t^2R^2)w + R^2}{tw^2 - (1+t^2R^2)w + tR^2} = \frac{1}{t} - \frac{(1-t^2)(1/t - tR^2)}{t(1/t-w)(1-tR^2/w)}.$$

Now it is clear that

$$\|g_t\| = \|G_t\| = \sup_{R<|w|<1} |G_t(w)| = G_t(-1) = \frac{1+R^2+tR^2(1-t)}{1+tR^2}.$$

We set $F_t(w) = G_t(w)/\|G_t\|$ and $f_t(z) = g_t(z)/\|g_t\| = \sum a_n P_n(z)$.

By virtue of (4),

$$\sum_{n=0}^{\infty} |a_n| \sup_{z\in E_r} |P_n(z)| = \frac{2t - G_t(r)}{\|G_t\|} = \frac{1}{\|G_t\|}\left[2t - \frac{r^2 - t(1+2R^2-t^2R^2)r + R^2}{-tr^2 + (1+t^2R^2)r - tR^2}\right]$$
$$= \frac{1}{\|G_t\|}\,\frac{(1-2t^2)r^2 + t(1-2R^2+3t^2R^2)r + R^2(1-2t^2)}{(1-tr)(r-tR^2)}.$$

The last expression is greater than or equal to 1 if and only if the final fraction is greater than or equal to $\|G_t\|$, which holds if and only if $((1-t)Q_t(r))/((1-tr)(r-tR^2)(1+tR^2)) \geqslant 0$, where $Q_t(r)$ is the quadratic polynomial

$$[(1+2t)+tR^2(2+3t)]r^2-[1+R^2(1+4t+3t^2)+t^2R^4(3+4t)]r+R^2[(1+2t)+tR^2(2+3t)]$$

obtained after a rather long computation. All factors multiplying Q_t in the above inequality are positive. We cancel them and then pass to the limit as $t \to 1^-$. Consequently, $\sum |a_n| \sup_{E_r} |P_n| \geqslant 1$ if and only if $Q(r) = (3+5R^2)r^2 - (1+8R^2+7R^4)r + R^2(3+5R^2) \geqslant 0$.

The discriminant of Q is $\Delta(R) = (1-R^2)(1-19R^2-61R^4-49R^6)$. The polynomial $1-19x-61x^2-49x^3$ has one positive root between 0 and 1 and two complex roots, and hence is nonnegative for x between 0 and the positive root. Let R_0 be the square root of the positive root; we compute $R_0 \approx 0.213740$. The corresponding eccentricity is $\varepsilon_2 \approx 0.408804$.

We first assume that $\varepsilon < \varepsilon_2$ or, equivalently, $R < R_0$. Then $\Delta(R) > 0$, and $Q(r)$ has two real roots. If $R_1 < R_2$ are these roots, then, after more tedious work, we can show that $0 < R_1 < R < R_2 < 1$. Then $Q(r) \geqslant 1$, which implies that $\sum |a_n| \sup_{E_r} |P_n| \geqslant 1$ if and only if $r \geqslant R_2$. Thus, the Bohr radius of E satisfies the condition $B \leqslant R_2$.

Next, assume that $\varepsilon \geqslant \varepsilon_2$ or, equivalently, $R \geqslant R_0$. Then $\Delta(R) \leqslant 0$, and $Q(r) \geqslant 0$ for any r. This implies that $\sum |a_n| \sup_{E_r} |P_n| \geqslant 1$ for all $r > R$. Thus, there is no Bohr phenomenon for $\{P_n\}$ in this case. $\square$

Remark 9. By (2), R is an increasing function of ε. Moreover, one can show that both r_2 and R_2 are decreasing functions of R (whenever they exist). We conclude that the upper and lower bounds for B increase as $\varepsilon \to 0^+$.

For example, $R = 0.19$ corresponds to $\varepsilon \approx 0.367$, where we have $r_2 \approx 0.225790$ and $R_2 \approx 0.278434$. Moreover, $R = 0.1$ corresponds to $\varepsilon \approx 0.198$, where $r_2 \approx 0.316163$ and $R_2 \approx 0.323407$. In each case, $r_2 \leqslant B \leqslant R_2$.

If $\varepsilon = 0$, then we have $b = a$, $c = 0$, and $R = 0$, and E becomes the disk of radius $d = a = b$. Both $q(r)$ and $Q(r)$ become the polynomial $3r^2 - r$, whose only relevant root is $r = r_2 = R_2 = 1/3$, which is the Bohr radius. Thus, this special case turns out to be the topic of Example 5.

REFERENCES

1. L. Aizenberg, "Multidimensional Analogues of Bohr's Theorem on Power Series," Proc. Amer. Math. Soc. **128**, 1147–1155 (2000).
2. L. Aizenberg, A. Aytuna, and P. Djakov, "An Abstract Approach to Bohr's Phenomenon," Proc. Amer. Math. Soc. **128**, 2611–2619 (2000).
3. L. Aizenberg, A. Aytuna, and P. Djakov, "Generalization of a Theorem of Bohr for Bases in Spaces of Holomorphic Functions of Several Complex Variables," J. Math. Anal. Appl. **258**, 429–447 (2001).
4. L. Aizenberg and N. Tarkhanov, "A Bohr Phenomenon for Elliptic Equations," Proc. London Math. Soc. **82**, 385–401 (2001).
5. H. P. Boas, "Majorant Series," J. Korean Math. Soc. **37**, 321–337 (2000).
6. H. P. Boas and D. Khavinson, "Bohr's Power Series Theorem in Several Variables," Proc. Amer. Math. Soc. **125**, 2975–2979 (1997).
7. J. H. Curtiss, "Faber Polynomials and the Faber Series," Amer. Math. Monthly **78**, 577–596 (1971).
8. A. Defant, D. Garcia, and M. Maestre, "Bohr's Power Series Theorem and Local Banach Space Theory," J. Reine Angew. Math. **557**, 173–197 (2003).
9. W. H. J. Fuchs, *Topics in the Theory of Functions of One Complex Variable*, Van Nostrand Math. Stud. **12** (Van Nostrand, Princeton, 1967).
10. H. T. Kaptanoğlu, "Bohr Phenomena for Laplace-Beltrami Operators," Preprint.
11. A. I. Markushevich, *Theory of Functions of a Complex Variable*, 2nd ed. (Chelsea, New York, 1977), Vol. III.
12. V. I. Paulsen, G. Popescu, and D. Singh, "On Bohr's Inequality," Proc. London Math. Soc. **85**, 493–512 (2002).
13. P. K. Suetin, *Series of Faber Polynomials* (Gordon & Breach, Amsterdam, 1998).