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Uncertainty principles in holomorphic function spaces on the unit ball

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Abstract. On all Bergman–Besov Hilbert spaces on the unit disk, we find self-adjoint weighted shift operators that are differential operators of half-order whose commutators are the identity, thereby obtaining uncertainty relations in these spaces. We also obtain joint average uncertainty relations for pairs of commuting tuples of operators on the same spaces defined on the unit ball. We further identify functions that yield equality in some uncertainty inequalities.

1 Introduction

The uncertainty principle of Heisenberg originates in quantum physics. The fact that quantum theory is based on operators on Hilbert spaces avails oneself of the consideration of uncertainty principles as inequalities involving Hilbert space operators.

Theorem 1.1. *Let L and M be self-adjoint operators on a Hilbert space H with inner product $\langle\cdot,\cdot\rangle_H$ and the associated norm $\|\cdot\|_H$. Then*

$$\|(L-\lambda I)u\|_H\|(M-\mu I)u\|_H \geq \frac{1}{2}\left|\langle(LM-ML)u,u\rangle_H\right| \tag{1}$$

for all $\lambda,\mu\in\mathbb{R}$ and all u that lie in the domain of both LM and ML. Equality holds if and only if $(L-\lambda I)u = i\gamma(M-\mu I)u$ for some $\gamma\in\mathbb{R}$.

Mathematically, as soon as self-adjoint operators L and M are found on a Hilbert space H whose *commutator* $[L,M] := LM - ML$ is a scalar multiple of the identity operator whence the right side of (1) simplifies (the case of conjugate observables), an explicit uncertainty principle appears. Theorem 1.1 and its easy proof can be found in [F, G], coincidentally on pages 27 and 28 in both.

A few remarks are in order. The equality $[L,M] = cI$ for some constant c cannot be satisfied with both L and M bounded.

Theorem 1.2. *If Z is a Banach algebra with unit e and $x,y\in Z$, then $xy-yx\neq e$.*

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Theorem 1.2 can be found in [R, Theorem 13.6] and its proof does not even use the completeness of Z. So, as warned in [FS, p. 211], the intersection of the domains of LM and ML is crucial even when $[L, M] = cI$ on the intersection.

Using the mathematical approach above, uncertainty principles with $[L, M] = cI$ are obtained in the Segal–Bargmann–Fischer–Fock space of entire functions weighted with the Gaussian in [CZ] or in its generalizations in [L]. Further, [CZ] poses the problem of finding uncertainty principles in Hardy and Bergman spaces. Some results are presented in [So1, So2] on certain, but not all, Bergman and Dirichlet spaces on the unit disk, but in these sources, one of the operators is always taken as the first-order derivative, resulting in $[L, M] \neq cI$. It is shown in [CD, Theorem 11] that there are no first-order self-adjoint differential operators on weighted Bergman spaces on the unit disk whose commutator is a nonzero multiple of the identity.

We find self-adjoint operators L and M with $LM - ML = cI$ on a large family of weighted symmetric (bosonic) Fock spaces of holomorphic functions on the unit disk as studied in [Ka] and obtain uncertainty relations from them. This family includes all Hilbert spaces among Bergman–Besov spaces, Dirichlet spaces, and the Hardy space H^2. The operators L and M are combinations of specific weighted shift operators, and these shifts are fractional differential operators of order 1/2 and also nothing but annihilation and creation operators. For contrast, uncertainty relations are obtained in [UT] in which the operators are annihilation and number operators. We also obtain joint average uncertainty relations for pairs of commuting tuples of operators in the same family of Fock spaces on the unit ball in $\mathbb{C}^n$ which includes the Drury-Arveson space. The formulation with tuples of operators seems new.

Our main results are Theorems 4.3 and 5.2.

2 Notation and preliminaries

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$ with respect to the usual Hermitian inner product $\langle z, w\rangle = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$, where we conjugate the second variable following the tradition in mathematics, and the associated norm $|z| = \sqrt{\langle z, z\rangle}$. When $n = 1$, the ball is the unit disk $\mathbb{D}$ in the complex plane.

In multi-index notation, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$, $0^0 = 1$, and $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$. We also let $\varepsilon_j := (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the jth position and the right side of $:=$ defines its left side.

The *Pochhammer symbol* $(p)_q$ is defined by

$$(p)_q = \frac{\Gamma(p+q)}{\Gamma(p)}$$

when p and $p+q$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. This is a shifted rising factorial since $(p)_k = p(p+1)\cdots(p+k-1)$ for positive integer k. In particular, $(1)_k = k!$ and $(p)_0 = 1$. Stirling formula gives

$$\text{(2)} \qquad \frac{\Gamma(r+p)}{\Gamma(r+q)} \sim r^{p-q}, \qquad \frac{(p)_r}{(q)_r} \sim r^{p-q}, \qquad \frac{(r)_p}{(r)_q} \sim r^{p-q} \qquad (\operatorname{Re} r \to \infty),$$

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where $P \sim Q$ means that $|P/Q|$ is bounded above and below by two strictly positive constants, that is, $P = \mathcal{O}(Q)$ and $Q = \mathcal{O}(P)$, for all P, Q of interest.

Definition 2.1. A function $K(z,w)$ is called the *reproducing kernel* of a Hilbert space H of functions defined on $\mathbb{B}$ if $K(z,\cdot) \in H$ for each $z \in \mathbb{B}$ and

$$u(z) = \langle u(\cdot), K(z,\cdot)\rangle_H \qquad (u \in H,\ z \in \mathbb{B}).$$

There is a one-to-one correspondence between reproducing kernel Hilbert spaces and positive definite kernels. We deal with Hilbert function spaces whose elements are holomorphic functions on the unit ball, the collection of all of which we denote by $H(\mathbb{B})$.

We use the term *operator* to mean a linear transformation whose domain $D(T)$ and range $R(T)$ are subspaces of a complex Hilbert space H with no requirement on boundedness.

If C is a densely defined operator on H, denoting its adjoint $A := C^*$, then by [R, Theorem 13.9], A is a closed operator. If A is also densely defined, then by [R, Theorem 13.12], $A^{**} = A$. Further, by [Kr, Theorem 10.2-1], $C \subset C^{**} = A^*$, that is, $C = A^*$ on the domain of C. If we let $L := C + C^*$ and $M := i(C - C^*)$, then L and M are self-adjoint and

$$[L, M] = 2i[A, C]. \tag{3}$$

Moreover, if $\lambda, \mu \in \mathbb{R}$, then $L - \lambda I$ and $M - \mu I$ are also self-adjoint and

$$[L - \lambda I, M - \mu I] = [L, M]. \tag{4}$$

Self-adjointness, (3), and (4) hold on the intersection of the domains of AC and CA, which is included in the intersection of the domains of C and A.

Two abstract uncertainty principles that follow from Theorem 1.1 and given in [F, pp. 27–28] are the following.

Corollary 2.2. *Let C and A be densely defined operators on a Hilbert space H with the properties that $A = C^*$ and $[A, C] = I$.*

(i) $\|(C + A - \lambda I)u\|_H \|(C - A - i\mu I)u\|_H \ge \|u\|_H^2$ *for all $\lambda, \mu \in \mathbb{R}$ and all u that lie in the domain of both CA and AC.*

(ii) $\|(C + A)u\|_H^2 + \|(C - A)u\|_H^2 \ge 2\|u\|_H^2$ *for all u that lie in the domain of both CA and AC.*

The passage from (i) to (ii) is via the elementary inequality $a^2 + b^2 \ge 2ab$, so we do not dwell on (ii) anymore. Inequality (1) can be generalized to hold also for complex λ and μ as explained in [CZ]. However, for equality, λ and μ must be real. This generalization works for any pairs of operators, and we do not dwell on this anymore either.

If $T = (T_1, \dots, T_n)$ and $S = (S_1, \dots, S_n)$ are tuples of operators on the same Hilbert space H, we use the notation $T \cdot S := T_1 S_1 + \cdots + T_n S_n$. We define the *commutator* of the tuples T and S by $[T, S] := T \cdot S - S \cdot T$. With these definitions, if

$\tau I = (\tau_1 I, \ldots, \tau_n I)$ with $\tau_j \in \mathbb{C}$, $j = 1, \ldots, n$, and σI is similar, then by a straightforward calculation,

(5) $$[T - \tau I, S - \sigma I] = [T, S].$$

3 Weighted symmetric Fock spaces

In [Ka], large families of weighted symmetric (bosonic) Fock spaces of holomorphic functions on $\mathbb{B}$ are studied following [A]. They are the spaces in which we develop uncertainty principles. The material in this section is taken from [Ka].

Definition 3.1. Let $b := (b_k)_k$ be a weight sequence satisfying $b_0 = 1$, $b_k > 0$ for all $k = 0, 1, 2, \ldots$, and

(6) $$\limsup_{k\to\infty} b_k^{1/k} \le 1.$$

We define positive-definite kernels by

(7) $$K_b(z, w) := \sum_{k=0}^{\infty} b_k \langle z, w\rangle^k = \sum_{k=0}^{\infty} b_k \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} z^\alpha \overline{w}^\alpha \qquad (z, w \in \mathbb{B})$$

and spaces $\mathcal{F}_b$ as the reproducing kernel Hilbert spaces generated by these kernels.

Condition (6) causes the series in (7) to converge absolutely and uniformly for (z, w) in compact subsets of $\mathbb{B} \times \mathbb{B}$, thereby defining K_b as a holomorphic function of $z \in \mathbb{B}$ and a conjugate holomorphic function of $w \in \mathbb{B}$.

Theorem 3.2. *The space $\mathcal{F}_b$ consists of all $f \in H(\mathbb{B})$ with Taylor expansions*

(8) $$f(z) = \sum_{|\alpha|=0}^{\infty} f_\alpha z^\alpha$$

converging absolutely and uniformly on compact subsets of $\mathbb{B}$ for which

(9) $$\|f\|_b^2 := \sum_{|\alpha|=0}^{\infty} \frac{1}{b_{|\alpha|}} \frac{\alpha!}{|\alpha|!} |f_\alpha|^2 < \infty$$

and is equipped with the inner product

$$\langle f, g\rangle_b := \sum_{|\alpha|=0}^{\infty} \frac{1}{b_{|\alpha|}} \frac{\alpha!}{|\alpha|!} f_\alpha \overline{g}_\alpha.$$

Further,

(10) $$\mathcal{B}_b := \left\{ e_\alpha^b(z) := \sqrt{b_{|\alpha|} \frac{|\alpha|!}{\alpha!}}\, z^\alpha : \alpha \in \mathbb{N}^n \right\}$$

is an orthonormal basis for $\mathcal{F}_b$. Moreover, holomorphic polynomials in the n variables $z_1, \ldots, z_n$ are dense in each $\mathcal{F}_b$.

In particular, for each $\alpha \in \mathbb{N}^n$,

$$\|z^\alpha\|_b^2 = \frac{1}{b_{|\alpha|}} \frac{\alpha!}{|\alpha|!}.$$

We now describe a particular family of holomorphic kernels and associated Hilbert function spaces that include many well-known spaces as special cases. They are included in the family of Bergman–Besov spaces on $\mathbb{B}$.

Definition 3.3. For $q \in \mathbb{R}$ and $k = 0, 1, 2, \dots$, we set

$$b_k(q) := \begin{cases} \dfrac{(1+n+q)_k}{k!}, & \text{if } q > -(1+n), \\ \dfrac{k!}{(1-(n+q))_k}, & \text{if } q \le -(1+n), \end{cases}$$

denote by $K_q(z, w) := \sum_{k=0}^{\infty} b_k(q)\langle z, w\rangle^k$ the reproducing kernel with coefficient sequence $(b_k(q))_k$ and by $\mathcal{F}_q$ the Hilbert space generated by the kernel K_q.

By (2),

$$b_k(q) \sim k^{n+q} \qquad (k \to \infty) \tag{11}$$

for any $q \in \mathbb{R}$ assuring that (6) is satisfied. So an $f \in H(\mathbb{B})$ given by (8) belongs to $\mathcal{F}_q$ if and only if

$$\sum_{|\alpha|=1}^{\infty} \frac{1}{|\alpha|^{n+q}} \frac{\alpha!}{|\alpha|!} |f_\alpha|^2 < \infty. \tag{12}$$

Note that

$$K_q(z, w) = \begin{cases} \dfrac{1}{(1-\langle z, w\rangle)^{1+n+q}} = {}_2F_1(1+n+q, 1; 1; \langle z, w\rangle), & \text{if } q > -(1+n), \\ {}_2F_1(1, 1; 1-(n+q); \langle z, w\rangle), & \text{if } q \le -(1+n), \end{cases}$$

where ${}_2F_1$ is the Gauss hypergeometric function. In particular,

$$K_{-(1+n)}(z, w) = \frac{1}{\langle z, w\rangle} \log \frac{1}{1-\langle z, w\rangle}.$$

Thus $\mathcal{F}_q$ is the weighted Bergman space A_q^2 for $q > -1$, the Hardy space H^2 for $q = -1$, the Drury–Arveson space $\mathcal{A}$ for $q = -n$, and the Dirichlet space $\mathcal{D}$ for $q = -(1+n)$. We simply write A^2 for the unweighted Bergman space when $q = 0$. If $q < -(1+n)$, then the functions in $\mathcal{F}_q$ are bounded on $\mathbb{B}$. The inner products and hence the norms of all the spaces in the $\mathcal{F}_q$ family can be expressed as integrals on $\mathbb{B}$ of either the functions or their sufficiently high-order derivatives (see [Ka] for details).

4 Uncertainty principles in spaces on the disk

We start with the case of the function spaces on the unit disk, that is, $n = 1$. Many of the formulas in Section 3 are simplified mainly because now $|\alpha| = \alpha = k$. So the terms

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of the orthonormal basis $\mathcal{B}_b$ of $\mathcal{F}_b$ are $e_k^b(z) = \sqrt{b_k}\, z^k$ for $k = 0, 1, 2, \ldots$; in particular, $e_0^b = 1$. Equivalently,

$$\|z^k\|_b^2 = \frac{1}{b_k} \qquad (k = 0, 1, 2, \ldots). \tag{13}$$

The homogeneous expansion of a function $f \in H(\mathbb{D})$ is now its Taylor expansion which can also be written in terms of the orthonormal basis of $\mathcal{F}_b$ as

$$f(z) = \sum_{k=0}^{\infty} f_k z^k = \sum_{k=0}^{\infty} \frac{f_k}{\sqrt{b_k}}\, e_k^b(z) \qquad (z \in \mathbb{D}). \tag{14}$$

Definition 4.1. We define the operator C_b on $\mathcal{F}_b$ by first letting

$$C_b e_k^b := \sqrt{k+1}\, e_{k+1}^b$$

on $\mathcal{B}_b$ and then by extending it linearly to its span.

Thus, the operator C_b on $\mathcal{F}_b$ is nothing but the weighted shift operator with the weight sequence $(\sqrt{k+1})_k$. Such operators are investigated in detail in [Sh]. Since the weight sequence is unbounded, C_b is unbounded on $\mathcal{F}_b$. However, it is densely defined since polynomials are dense in every $\mathcal{F}_b$.

By [Sh] or in a similar way to the unweighted shifts on $\mathcal{F}_b$ studied in [Ka], the adjoint $A_b := C_b^*$ of C_b is given by

$$A_b e_k^b = \sqrt{k}\, e_{k-1}^b \qquad (k \geq 1)$$

and $A_b e_0^b = 0$. Similarly, A_b is densely defined and unbounded on $\mathcal{F}_b$, and $A_b^* = C_b$. The choice of the letters C and A is no coincidence, because these operators are exactly the creation and annihilation operators on many-body quantum systems (see [T, p. 106] or [MR, p. 92]). If $f \in \mathcal{F}_b$ is given by (14), then

$$\begin{aligned} C_b f(z) &= \sum_{k=0}^{\infty} \sqrt{k+1} \sqrt{\frac{b_{k+1}}{b_k}} f_k z^{k+1}, \\ A_b f(z) &= \sum_{k=1}^{\infty} \sqrt{k} \sqrt{\frac{b_{k-1}}{b_k}} f_k z^{k-1}. \end{aligned} \tag{15}$$

Continuing,

$$A_b C_b e_k^b = (k+1) e_k^b \qquad \text{and} \qquad C_b A_b e_k^b = k e_k^b \quad (k \geq 1).$$

If $f \in \mathcal{F}_b$, then

$$\begin{aligned} C_b A_b f(z) &= \sum_{k=1}^{\infty} k f_k z^k =: N f(z), \\ A_b C_b f(z) &= \sum_{k=0}^{\infty} (k+1) f_k z^k = N f(z) + I f(z), \end{aligned} \tag{16}$$

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where N denotes the *number operator* of physics which is the same as the *radial derivative* of mathematics. Thus,

(17) $$(A_bC_b - C_bA_b)f = f, \qquad \text{that is,} \qquad [A_b, C_b] = I$$

on the subspace of $\mathcal{F}_b$ that is the intersection of the domains if A_bC_b and C_bA_b. In fact, $(\sqrt{k+1})_k$ is the only positive weight sequence with initial term 1 such that this exact commutation relation holds. This commutation relation is well known, but in general not in reference to an uncertainty relation (see [T, p. 104]). Here, we use it to formulate uncertainty relations on spaces not generally considered in quantum physics and also identify clearly the domains of the operators.

Let us describe the domains of the operators. The domain $D(C_b)$ of C_b is the subspace of all $f \in \mathcal{F}_b$ for which also $C_bf \in \mathcal{F}_b$. So, by Theorem 3.2, (13), (15), and (16),

(18) $$\mathcal{E}_b := \left\{ f \in H(\mathbb{D}) : \sum_{k=1}^{\infty} \frac{k}{b_k} |f_k|^2 < \infty \right\} = D(C_b) = D(A_b),$$
$$\mathcal{G}_b := \left\{ f \in H(\mathbb{D}) : \sum_{k=1}^{\infty} \frac{k^2}{b_k} |f_k|^2 < \infty \right\} = D(A_bC_b) = D(C_bA_b).$$

It is clear that $\mathcal{G}_b \subset \mathcal{E}_b \subset \mathcal{F}_b$ and each inclusion is dense since all three sets contain polynomials. By [R, Theorem 13.9], we conclude the following.

Proposition 4.2. *C_b, A_b, A_bC_b, and C_bA_b are closed operators.*

Let $L_b = C_b + A_b$ and $M_b = i(C_b - A_b)$ be the self-adjoint operators as in Section 2. Then

$$L_bf(z) = \sqrt{b_1}\, f_0 z + \sum_{k=1}^{\infty} \left(\sqrt{k+1} \sqrt{\frac{b_{k+1}}{b_k}}\, z^2 + \sqrt{k} \sqrt{\frac{b_{k-1}}{b_k}} \right) f_k z^{k-1}$$
$$= \frac{1}{\sqrt{b_1}} f_1 + \sum_{k=1}^{\infty} \left(\sqrt{k} \sqrt{\frac{b_k}{b_{k-1}}}\, f_{k-1} + \sqrt{k+1} \sqrt{\frac{b_k}{b_{k+1}}}\, f_{k+1} \right) z^k,$$

where we change variables from $k+1$ to k in the first sum and from $k-1$ to k in the second sum to write the second expression. We do not show $M_bf(z)$ since it is so similar to $L_bf(z)$. Both L_b and M_b are closed operators with domain $\mathcal{E}_b$. By (3) and (17),

(19) $$[L_b, M_b] = 2iI$$

on $\mathcal{G}_b = D([L_b, M_b])$. The uncertainty principle in $\mathcal{F}_b$ implied by Corollary 2.2(i) is the following.

Theorem 4.3. *For $f \in \mathcal{G}_b \subset \mathcal{F}_b$ and $\lambda, \mu \in \mathbb{R}$,*

$$\|(C_b + A_b - \lambda I)f\|_b \,\|(C_b - A_b - i\mu I)f\|_b \ge \|f\|_b^2.$$

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For $\lambda = \mu = 0$, equality holds for a function in $\mathcal{G}_b$ if and only if it is a complex scalar multiple of

$$f^b(z) = \sum_{l=0}^{\infty} \left(\frac{\gamma+1}{\gamma-1} \right)^l \sqrt{\frac{1 \cdot 3 \cdot 5 \cdots (2l-1)}{2 \cdot 4 \cdot 6 \cdots 2l}} \sqrt{b_{2l}}\, z^{2l}$$

for some $\gamma < 0$. For $(\lambda, \mu) \neq (0,0)$, Taylor series coefficients of the functions that give equality can be obtained from a three-term recurrence relation.

Proof By (19) and (4), for $\lambda, \mu \in \mathbb{R}$,

$$\begin{aligned} 2i\langle f, f\rangle_b &= \big\langle [L_b, M_b] f, f \big\rangle_b = \big\langle [L_b - \lambda I, M_b - \mu I] f, f \big\rangle_b \\ &= \big\langle (L_b - \lambda I)(M_b - \mu I) f, f \big\rangle_b - \big\langle (M_b - \mu I)(L_b - \lambda I) f, f \big\rangle_b \\ &= \big\langle (M_b - \mu I) f, (L_b - \lambda I) f \big\rangle_b - \overline{\big\langle (M_b - \mu I) f, (L_b - \lambda I) f \big\rangle_b} \\ &= 2i \operatorname{Im} \big\langle (M_b - \mu I) f, (L_b - \lambda I) f \big\rangle_b . \end{aligned}$$

Hence,

$$\|f\|_b \leq \left| \big\langle (M_b - \mu I) f, (L_b - \lambda I) f \big\rangle_b \right| \leq \|(M_b - \mu I) f\|_b \|(L_b - \lambda I) f\|_b$$

on $\mathcal{G}_b$, which is the desired uncertainty inequality.

Equality holds in the first inequality if and only if $\langle M_b f, L_b f\rangle_b$ is pure imaginary with positive imaginary part. Equality holds in the second inequality if and only if $L_b f = \beta M_b f$ for some $\beta \in \mathbb{C}$. Applying the two conditions together, we deduce that equality holds in the uncertainty inequality for an $f^b(z) = \sum_{k=0}^{\infty} f_k^b z^k \in \mathcal{G}_b$ if and only if $L_b f^b - \lambda f^b = i\gamma(M_b f^b - \mu f^b)$ for some $\gamma < 0$. Equivalently,

$$\begin{aligned} \frac{f_1^b}{\sqrt{b_1}} - \lambda f_0^b + \sum_{k=1}^{\infty} \left(\sqrt{k} \sqrt{\frac{b_k}{b_{k-1}}}\, f_{k-1}^b + \sqrt{k+1} \sqrt{\frac{b_k}{b_{k+1}}}\, f_{k+1}^b - \lambda f_k^b \right) z^k \\ = \frac{\gamma f_1^b}{\sqrt{b_1}} - i\gamma\mu f_0^b + \gamma \sum_{k=1}^{\infty} \left(\sqrt{k+1} \sqrt{\frac{b_k}{b_{k+1}}}\, f_{k+1}^b - \sqrt{k} \sqrt{\frac{b_k}{b_{k-1}}}\, f_{k-1}^b - i\mu f_k^b \right) z^k . \end{aligned} \tag{20}$$

Setting the coefficients of z^k on both sides of (20) equal to each other, letting $f_{-1}^b = 0$ for convenience, and using $\gamma < 0$, we obtain

$$f_{k+1}^b = \frac{\gamma+1}{\gamma-1} \sqrt{\frac{k}{k+1}} \sqrt{\frac{b_{k+1}}{b_{k-1}}}\, f_{k-1}^b - \frac{\lambda - i\gamma\mu}{\gamma - 1} \frac{1}{\sqrt{k+1}} \sqrt{\frac{b_{k+1}}{b_k}}\, f_k^b \tag{21}$$

for $k = 0, 1, 2, \ldots$. From this three-term recurrence relation, it is possible to determine the Taylor series coefficients of f^b in terms of f_0^b and f_1^b and check for what $\gamma < 0$ this f^b belongs to $\mathcal{G}_b$. But the computations are cumbersome and we only work out the details of the representative case $\lambda = \mu = 0$. As a side note, if $\gamma = 1$, then the only function satisfying (20) is the zero function.

For $\lambda = \mu = 0$, the constant terms of (20) give

$$(1 - \gamma) f_1^b = 0 \tag{22}$$

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and the recurrence relation (21) takes the form

$$f^b_{k+1} = \frac{\gamma+1}{\gamma-1}\sqrt{\frac{k}{k+1}}\sqrt{\frac{b_{k+1}}{b_{k-1}}}\, f^b_{k-1}. \tag{23}$$

Since $\gamma < 0$, (22) gives $f^b_1 = 0$. Then, from (23), $f^b_3 = f^b_5 = \cdots = 0$ as well. Letting now $k = 2l - 1$, the coefficients with even indices satisfy

$$f^b_{2l} = \left(\frac{\gamma+1}{\gamma-1}\right)^l \sqrt{\frac{1\cdot 3\cdot 5\cdots(2l-1)}{2\cdot 4\cdot 6\cdots 2l}}\sqrt{b_{2l}}\, f^b_0 \qquad (l = 1, 2, \ldots).$$

Denoting the first square root by d_{2l} and setting $d_0 = 1$, we have

$$f^b(z) = f^b_0 \sum_{l=0}^{\infty}\left(\frac{\gamma+1}{\gamma-1}\right)^l d_{2l}\sqrt{b_{2l}}\, z^{2l} = f^b_0 \sum_{l=0}^{\infty} c^b_{2l} z^{2l}. \tag{24}$$

Simply $d_{2l} \le 1/\sqrt{2}$ and by (18), $f^b \in \mathcal{G}_b$ if and only if

$$\sum_{l=1}^{\infty}\frac{l^2}{b_{2l}}|c^b_{2l}|^2 = \sum_{l=1}^{\infty} l^2\left|\frac{\gamma+1}{\gamma-1}\right|^{2l} d^2_{2l} < \infty. \tag{25}$$

Since $(d_{2l})_l$ is bounded, the finiteness in (25) is implied by the finiteness of

$$\sum_{l=1}^{\infty} l^2\left|\frac{\gamma+1}{\gamma-1}\right|^{2l},$$

and this holds when $|\gamma+1| < |\gamma-1|$, that is, $\gamma < 0$.

Thus, the precise range of values of γ for having a nontrivial $f^b \in \mathcal{G}_b$ given in (24) for which equality holds in the uncertainty inequality for $\lambda = \mu = 0$ is $\gamma < 0$. ■

We next specialize to the $\mathcal{F}_q$ spaces of Definition 3.3 in which the $b_k(q)$ have specific values for $q \in \mathbb{R}$, and use the subscript q as in C_q to denote an operator on $\mathcal{F}_q$. We have

$$C_q z^k = c^q_{k+1} z^{k+1} = \begin{cases} \sqrt{2+q+k}\, z^{k+1}, & \text{if } q > -2, \\ \dfrac{k+1}{\sqrt{-q+k}}\, z^{k+1}, & \text{if } q \le -2, \end{cases}$$

and

$$A_q z^k = a^q_{k-1} z^{k-1} = \begin{cases} \dfrac{k}{\sqrt{1+q+k}}\, z^{k-1} = \dfrac{1}{\sqrt{1+q+k}}(z^k)', & \text{if } q > -2, \\ \sqrt{-q+k-1}\, z^{k-1} = \dfrac{\sqrt{-q+k-1}}{k}(z^k)', & \text{if } q \le -2, \end{cases}$$

where primes denote differentiation. These explicit formulas show that both C_q and A_q are fractional differential operators of order 1/2 since $c^q_k \sim a^q_k \sim k^{1/2}$ for each $q \in \mathbb{R}$. For comparison, by (16), $C_b A_b$ and $A_b C_b$ are differential operators of order 1.

Three values of q give three important spaces; $\mathcal{F}_0$ is the Bergman space A^2, $\mathcal{F}_{-1}$ is the Hardy space H^2, and $\mathcal{F}_{-2}$ is the Dirichlet space $\mathcal{D}$. There is no distinction between the Hardy space and the Drury–Arveson space when $n = 1$. The precise forms of C_q and A_q on these spaces can be read off from the above formulas.

Further, for $f \in \mathcal{F}_q$,

$$L_q f(z) = \begin{cases} \dfrac{1}{\sqrt{2+q}} f_1 + \sum_{k=1}^{\infty} \left(\sqrt{1+q+k}\, f_{k-1} + \dfrac{k+1}{\sqrt{2+q+k}} f_{k+1} \right) z^k, & \text{if } q > -2, \\ \sqrt{-q}\, f_1 + \sum_{k=1}^{\infty} \left(\dfrac{k}{\sqrt{-1-q+k}} f_{k-1} + \sqrt{-q+k}\, f_{k+1} \right) z^k, & \text{if } q \le -2, \end{cases}$$

and $M_q f(z)$ is similar. So L_q and M_q are also fractional differential operators of order $1/2$ for each $q \in \mathbb{R}$.

The domains of C_q, A_q, L_q, and M_q are all the same, $\mathcal{E}_q$. A comparison of (18), (12), and (11) reveals that $\mathcal{E}_q = \mathcal{F}_{q-1}$. Similarly, the domain of $A_q C_q$, $C_q A_q$, and $[L_q, M_q]$ is $\mathcal{G}_q = \mathcal{F}_{q-2}$. So, for example, the domain of C_0 acting on the Bergman space A^2 is the Hardy space H^2 and the domain of $[L_0, M_0]$ again acting on A^2 is the Dirichlet space $\mathcal{D}$.

The formulas are especially simple for the Hardy space $H^2 = \mathcal{F}_{-1}$ in which all the $b_k(-1) = 1$. An $f \in H(\mathbb{D})$ belongs to H^2 if and only if $\sum_{k=0}^{\infty} |f_k|^2 < \infty$. Further, $C_{-1} z^k = \sqrt{k+1}\, z^{k+1}$, $A_{-1} z^k = \sqrt{k}\, z^{k-1}$,

$$L_{-1} f(z) = f_1 + \sum_{k=1}^{\infty} \left(\sqrt{k}\, f_{k-1} + \sqrt{k+1}\, f_{k+1} \right) z^k,$$

and M_{-1} is similar. The functions that give equality in the uncertainty relation for $\lambda = \mu = 0$ are complex scalar multiples of

$$f^{(-1)}(z) = \sum_{l=0}^{\infty} \left(\frac{\gamma+1}{\gamma-1} \right)^l \sqrt{\frac{1 \cdot 3 \cdot 5 \cdots (2l-1)}{2 \cdot 4 \cdot 6 \cdots 2l}}\, z^{2l}$$

with $\gamma < 0$.

5 Uncertainty principles in spaces on the ball

We now let $n > 1$, use the full multivariable orthonormal basis $\mathcal{B}_b$ of $\mathcal{F}_b$ on $\mathbb{B}$ given in (10), and define creation and annihilation operator tuples.

Definition 5.1. For $j = 1, \dots, n$, we define the operators C_{b_j} and A_{b_j} on $\mathcal{F}_b$ by

$$C_{b_j} e_\alpha^b := \sqrt{\alpha_j + 1}\, e_{\alpha+\varepsilon_j}^b \qquad \text{and} \qquad A_{b_j} e_\alpha^b := \sqrt{\alpha_j}\, e_{\alpha-\varepsilon_j}^b \quad (\alpha_j \ge 1),$$

and by $A_{b_j} e_\alpha^b = 0$ if $\alpha_j = 0$. We also define the operator tuples $C_b := (C_{b_1}, \dots, C_{b_n})$ and $A_b := (A_{b_1}, \dots, A_{b_n})$.

The C_{b_j} and A_{b_j} can be called weighted shift operators, but they shift to basis elements that are not immediate neighbors. They are densely defined operators since they are defined at least on polynomials and they are unbounded. For example, C_{b_j} is unbounded at least on a subsequence of $\mathcal{B}_b$ starting with any e_α^b and goes by increasing α_j by 1 each time.

As before, $A_{b_j} = C_{b_j}^*$ and $C_{b_j} = A_{b_j}^*$ for $j = 1, \dots, n$, so they are all closed operators. Also,

$$(26)\qquad C_{b_j} z^\alpha = \sqrt{|\alpha|+1}\sqrt{\frac{b_{|\alpha|+1}}{b_{|\alpha|}}}\, z^{\alpha+\varepsilon_j} \qquad \text{and} \qquad A_{b_j} z^\alpha = \frac{\alpha_j}{\sqrt{|\alpha|}}\sqrt{\frac{b_{|\alpha|-1}}{b_{|\alpha|}}}\, z^{\alpha-\varepsilon_j},$$

the latter for $\alpha_j \geq 1$. Hence,

$$C_{b_j} A_{b_j} z^\alpha = \alpha_j z^\alpha \qquad \text{and} \qquad A_{b_j} C_{b_j} z^\alpha = (\alpha_j + 1) z^\alpha.$$

Thus,

$$(27)\qquad \begin{aligned} (C_b \cdot A_b) f(z) &= \sum_{|\alpha|=1}^{\infty} |\alpha| f_\alpha z^\alpha = N f(z), \\ (A_b \cdot C_b) f(z) &= \sum_{|\alpha|=0}^{\infty} (|\alpha| + n) f_\alpha z^\alpha = N f(z) + n I f(z), \end{aligned}$$

where N is the number operator as before, resulting in

$$(A_b \cdot C_b - C_b \cdot A_b) f = n f, \qquad \text{that is,} \qquad [A_b, C_b] = nI$$

on the intersection of the domains of $A_b \cdot C_b$ and $C_b \cdot A_b$.

The domains of C_{b_j} and A_{b_j} are obtained using (26) and (9) as

$$D(C_{b_j}) = D(A_{b_j}) = \left\{ f \in H(\mathbb{B}) : \sum_{|\alpha|=1}^{\infty} \frac{1}{b_{|\alpha|}} \frac{\alpha!}{|\alpha|!} \alpha_j |f_\alpha|^2 < \infty \right\}.$$

Then the domain of the tuples C_b and A_b is

$$\mathcal{E}_b := \left\{ f \in H(\mathbb{B}) : \sum_{|\alpha|=1}^{\infty} \frac{1}{b_{|\alpha|}} \frac{\alpha!}{|\alpha|!} |\alpha| |f_\alpha|^2 < \infty \right\}.$$

Similarly, the domain of $A_b \cdot C_b$ and $C_b \cdot A_b$ is

$$(28)\qquad \mathcal{G}_b := \left\{ f \in H(\mathbb{B}) : \sum_{|\alpha|=1}^{\infty} \frac{1}{b_{|\alpha|}} \frac{\alpha!}{|\alpha|!} |\alpha|^2 |f_\alpha|^2 < \infty \right\}.$$

We define further operator tuples by $L_b := C_b + A_b = (C_{b_1} + A_{b_1}, \dots, C_{b_n} + A_{b_n})$ and by $M_b := i(C_b - A_b)$ similarly. Explicitly,

$$L_{b_j} f(z) = \frac{1}{\sqrt{b_1}} f_{\varepsilon_j} + \sum_{|\alpha|=1}^{\infty} \left(\sqrt{|\alpha|}\sqrt{\frac{b_{|\alpha|}}{b_{|\alpha|-1}}}\, f_{\alpha-\varepsilon_j} + \frac{\alpha_j + 1}{\sqrt{|\alpha|+1}}\sqrt{\frac{b_{|\alpha|}}{b_{|\alpha|+1}}}\, f_{\alpha+\varepsilon_j} \right) z^\alpha,$$

and $M_{b_j} f(z)$ is similar. The C_{b_j} commute with each other and so do the A_{b_j}. Also, C_{b_j} and A_{b_k} commute if $j \neq k$. Then the L_{b_j} commute among themselves and so do the M_{b_j}. Further, $D(L_b) = D(M_b) = \mathcal{E}_b$.

We call L_b and M_b self-adjoint due to $L_{b_j}^* = L_{b_j}$ and $M_{b_j}^* = M_{b_j}$ for each $j = 1, \dots, n$. Since the sums in L_b and M_b and the products in $L_b \cdot M_b$ and $M_b \cdot L_b$ are

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applied componentwise, we readily obtain

$$[L_b, M_b] = 2i[A_b, C_b] = 2inI$$

on $\mathcal{G}_b = D([L_b, M_b])$. We then obtain the following theorem, which introduces a joint average uncertainty inequality for operator tuples.

Theorem 5.2. *For $f \in \mathcal{G}_b \subset \mathcal{F}_b$ and $\lambda_j, \mu_j \in \mathbb{R}$, for $j = 1, \dots, n$,*

$$\frac{1}{n}\sum_{j=1}^{n} \|(C_{b_j} + A_{b_j} - \lambda_j I)f\|_b \, \|(C_{b_j} - A_{b_j} - i\mu_j I)f\|_b \geq \|f\|_b^2.$$

For $n = 2$ and $\lambda_j = \mu_j = 0$, for $j = 1, 2$, equality holds for a function in $\mathcal{G}_b$ if and only if it is a linear combination of f^b and g^b given in (32) and (33) in the proof.

Proof Using (5) with λ and μ in place of τ and σ, for $f \in \mathcal{G}_b$,

$$\begin{aligned}
2in\langle f, f\rangle_b &= \big\langle [L_b, M_b]f, f\big\rangle_b = \big\langle [L_b - \lambda I, M_b - \mu I]f, f\big\rangle_b \\
&= \big\langle (L_b - \lambda I)\cdot(M_b - \mu I)f, f\big\rangle_b - \big\langle (M_b - \mu I)\cdot(L_b - \lambda I)f, f\big\rangle_b \\
&= \sum_{j=1}^{n}\big(\langle M_{b_j}f - \mu_j f, L_{b_j}f - \lambda_j f\rangle_b - \langle L_{b_j}f - \lambda_j f, M_{b_j}f - \mu_j f\rangle_b\big) \\
&= 2i\sum_{j=1}^{n} \operatorname{Im}\langle M_{b_j}f - \mu_j f, L_{b_j}f - \lambda_j f\rangle_b.
\end{aligned}$$

Hence, on $\mathcal{G}_b$,

$$\|f\|_b^2 \leq \frac{1}{n}\sum_{j=1}^{n}\big|\langle M_{b_j}f - \mu_j f, L_{b_j}f - \lambda_j f\rangle_b\big| \leq \frac{1}{n}\sum_{j=1}^{n}\|M_{b_j}f - \mu_j f\|_b \|L_{b_j}f - \lambda_j f\|_b.$$

For equality, we only work out the case indicated in the statement of the theorem, because computations in the general case are too tedious. But we can let $n > 1$ be arbitrary in the initial steps. As in the proof of Theorem 4.3, equality holds in the first inequality if and only if each $\langle M_{b_j}f, L_{b_j}f\rangle_b$ is pure imaginary with positive imaginary part. Equality holds in the second inequality if and only if $L_{b_j}f = \beta_j M_{b_j}f$ for some $\beta_j \in \mathbb{C}$ for each j. The two conditions together imply that equality holds in the uncertainty inequality for an $f^b(z) = \sum_{|\alpha|=0}^{\infty} f_\alpha^b z^\alpha \in \mathcal{G}_b$ if and only if $L_{b_j}f^b = i\gamma_j M_{b_j}f^b$ for some $\gamma_j < 0$. Equivalently,

$$\begin{aligned}
(29) \qquad &\frac{1}{\sqrt{b_1}}f_{\varepsilon_j}^b + \sum_{|\alpha|=1}^{\infty}\left(\sqrt{|\alpha|}\sqrt{\frac{b_{|\alpha|}}{b_{|\alpha|-1}}}f_{\alpha-\varepsilon_j}^b + \frac{\alpha_j + 1}{\sqrt{|\alpha + 1}}\sqrt{\frac{b_{|\alpha|}}{b_{|\alpha|+1}}}f_{\alpha+\varepsilon_j}^b\right)z^\alpha \\
&= \frac{\gamma_j}{\sqrt{b_1}}f_{\varepsilon_j}^b + \gamma_j\sum_{|\alpha|=1}^{\infty}\left(\frac{\alpha_j + 1}{\sqrt{|\alpha + 1}}\sqrt{\frac{b_{|\alpha|}}{b_{|\alpha|+1}}}f_{\alpha+\varepsilon_j}^b - \sqrt{|\alpha|}\sqrt{\frac{b_{|\alpha|}}{b_{|\alpha|-1}}}f_{\alpha-\varepsilon_j}^b\right)z^\alpha
\end{aligned}$$

for each j.

For each j, the constant terms of (29) give $(1-\gamma_j)f^b_{\varepsilon_j} = 0$, which yields

$$f^b_{\varepsilon_j} = 0 \tag{30}$$

since $\gamma_j < 0$. Setting the coefficients of z^α on both sides of (29) equal to each other and using $\gamma_j < 0$, we obtain

$$f^b_{\alpha+\varepsilon_j} = \frac{\gamma_j+1}{\gamma_j-1}\,\frac{\sqrt{|\alpha|}\sqrt{|\alpha|+1}}{\alpha_j+1}\sqrt{\frac{b_{|\alpha|+1}}{b_{|\alpha|-1}}}\,f^b_{\alpha-\varepsilon_j}. \tag{31}$$

From now on, we consider only the case $n = 2$. So $\alpha = (\alpha_1, \alpha_2)$, $j = 1, 2$, and we have only $\varepsilon_1 = (1, 0)$ and $\varepsilon_2 = (0, 1)$. By (30) and (31), $f^b_{(\text{odd},\text{even})} = 0$ and $f^b_{(\text{even},\text{odd})} = 0$. Further, by (31), every $f^b_{(\text{even},\text{even})}$ depends on $f^b_{(0,0)}$. Similar to the proof of Theorem 4.3, with $\alpha = (2l, 2m)$, we obtain

$$f^b_{(2l,2m)} = \left(\frac{\gamma_1+1}{\gamma_1-1}\right)^l \left(\frac{\gamma_2+1}{\gamma_2-1}\right)^m \frac{\sqrt{(2l+2m)!}}{2^{l+m}\,l!\,m!}\sqrt{b_{2l+2m}}\,f^b_{(0,0)}$$

for $l, m = 0, 1, 2, \ldots$. Denoting the third factor by p_{lm}, these coefficients define

$$f^b(z_1, z_2) = \sum_{l,m=0}^{\infty} \left(\frac{\gamma_1+1}{\gamma_1-1}\right)^l \left(\frac{\gamma_2+1}{\gamma_2-1}\right)^m p_{lm}\sqrt{b_{2l+2m}}\,z_1^{2l} z_2^{2m}. \tag{32}$$

Again, by (31), every $f^b_{(\text{odd},\text{odd})}$ depends on $f^b_{(1,1)}$. With $\alpha = (2l+1, 2m+1)$, similar to above, we obtain

$$f^b_{(2l+1,2m+1)} = \left(\frac{\gamma_1+1}{\gamma_1-1}\right)^l \left(\frac{\gamma_2+1}{\gamma_2-1}\right)^m \frac{\sqrt{(2l+2m+2)!}}{1\cdot3\cdots(2l+1)\cdot1\cdot3\cdots(2m+1)}\sqrt{\frac{b_{2l+2m+2}}{2b_2}}\,f^b_{(1,1)}$$

for $l, m = 0, 1, 2, \ldots$. Denoting the third factor by q_{lm}, these coefficients define

$$g^b(z_1, z_2) = \sum_{l,m=0}^{\infty} \left(\frac{\gamma_1+1}{\gamma_1-1}\right)^l \left(\frac{\gamma_2+1}{\gamma_2-1}\right)^m q_{lm}\sqrt{\frac{b_{2l+2m+2}}{2b_2}}\,z_1^{2l+1} z_2^{2m+1}. \tag{33}$$

We must also check if f^b and g^b belong to $\mathcal{G}_b$. It is a routine calculation that

$$\frac{\alpha!}{|\alpha|!}\,p_{lm}^2 \le 1 \qquad \text{and} \qquad \frac{\alpha!}{|\alpha|!}\,q_{lm}^2 \le 1.$$

Using the first, it is straightforward to see that the sum in (28) is

$$\le 8 \sum_{l,m=1}^{\infty} \left(\frac{\gamma_1+1}{\gamma_1-1}\right)^{2l} \left(\frac{\gamma_2+1}{\gamma_2-1}\right)^{2m} (l^2+m^2),$$

which is finite if and only if $\gamma_1 < 0$ and $\gamma_2 < 0$. This shows $f^b \in \mathcal{G}_b$ for all $\gamma_1 < 0$ and $\gamma_2 < 0$. Similarly, $g^b \in \mathcal{G}_b$ for all $\gamma_1 < 0$ and $\gamma_2 < 0$. ∎

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Specializing to $\mathcal{F}_q$, by Definition 3.3,

$$L_{q_j} f(z) = \frac{f_{\varepsilon_j}}{\sqrt{1+n+q}} + \sum_{|\alpha|=1}^{\infty} \left(\sqrt{n+q+|\alpha|}\, f_{\alpha-\varepsilon_j} + \frac{(1+\alpha_j) f_{\alpha+\varepsilon_j}}{\sqrt{1+n+q+|\alpha|}} \right) z^\alpha$$

if $q > -(1+n)$ and it equals

$$\sqrt{1-n-q}\, f_{\varepsilon_j} + \sum_{|\alpha|=1}^{\infty} \left(\frac{|\alpha|\, f_{\alpha-\varepsilon_j}}{\sqrt{-n-q+|\alpha|}} + \sqrt{1-n-q+|\alpha|}\, \frac{1+\alpha_j}{\sqrt{1+|\alpha|}}\, f_{\alpha+\varepsilon_j} \right) z^\alpha$$

if $q \le -(1+n)$. The expressions for $C_{q_j} f(z)$, $A_{q_j} f(z)$, and $M_{q_j} f(z)$ are similar. By (11), domains of operators behave as in $n = 1$: $\mathcal{E}_q = \mathcal{F}_{q-1}$ and $\mathcal{G}_q = \mathcal{F}_{q-2}$.

For $n > 1$, there are four important values of q. Still $q = 0$ and $q = -1$ give the Bergman and Hardy spaces A^2 and H^2. But $q = -n$ gives the Drury–Arveson space $\mathcal{A}$ and $q = -(1+n)$ gives the Dirichlet space $\mathcal{D}$. The formulas are simplest for $\mathcal{A}$ since the Drury–Arveson space is special for multivariable operator theory due to $b_k(-n) = 1$ for all $k = 0, 1, 2, \ldots$. So

$$L_{(-n)_j} f(z) = f_{\varepsilon_j} + \sum_{|\alpha|=1}^{\infty} \left(\sqrt{|\alpha|}\, f_{\alpha-\varepsilon_j} + \frac{1+\alpha_j}{\sqrt{1+|\alpha|}}\, f_{\alpha+\varepsilon_j} \right) z^\alpha,$$

and with $q = -n = -2$,

$$f^{(-2)}(z_1, z_2) = \sum_{l,m=0}^{\infty} \left(\frac{\gamma_1+1}{\gamma_1-1} \right)^l \left(\frac{\gamma_2+1}{\gamma_2-1} \right)^m p_{lm}\, z_1^{2l} z_2^{2m},$$

which have almost the same forms as the corresponding quantities in the Hardy space when $n = 1$.

References

[A] W. Arveson, *Subalgebras of* C*-*algebras III: multivariable operator theory*. Acta Math. **181**(1998), 159–228.

[CZ] Y. Chen and K. Zhu, *Uncertainty principles for the Fock space*. Sci. Sin. Math. **45**(2015), 1847–1853.

[CD] J. G. Christensen and C. B. Deng, *Uncertainty principles and differential operators on the weighted Bergman space*. Complex Anal. Oper. Theory **17**(2023), Article no. 47, 14 pp.

[F] G. B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, 122, Princeton University Press, Princeton, NJ, 1989.

[FS] G. B. Folland and A. Sitaram, *The uncertainty principle: a mathematical survey*. J. Fourier Anal. Appl. **3**(1997), 207–238.

[G] K. Gröchenig, *Foundations of time–frequency analysis*, Applied and Numerical Harmonic Analysis, Birkhüser, Boston, MA, 2001.

[Ka] H. T. Kaptanoğlu, *Aspects of multivariable operator theory on weighted symmetric Fock spaces*. Commun. Contemp. Math. **16**(2014), Article no. 1350034, 49 pp.

[Kr] E. Kreyszig, *Introductory functional analysis with applications*, Wiley, New York, 1978.

[L] P. Lian, *Heisenberg's uncertainty principle associated with the Caputo fractional derivative*. J. Math. Phys. **62**(2021), Article no. 042102, 6 pp.

[MR] P. A. Martin and F. Rothen, *Many-body problems and quantum field theory: an introduction*, 2nd ed., Texts and Monographs in Physics, Springer, Berlin, 2004.

[R] W. Rudin, *Functional analysis*, 2nd ed., McGraw-Hill, New York, 1991.

[Sh] A. L. Shields, *Weighted shift operators and analytic function theory*. In: C. Pearcy (ed.), Topics in operator theory, Mathematical Surveys, 13, American Mathematical Society, Providence, RI, 1974, pp. 49–128.

[So1] F. Soltani, *Uncertainty principles of Heisenberg type on Dirichlet space*. Ann. Univ. Ferrara Sez. VII Sci. Mat. **67**(2021), 191–202.

[So2] F. Soltani, *Uncertainty principles of Heisenberg type for the Bargmann transform*. Afrika Mat. **327**(2021), 1629–1643.

[T] L. A. Takhtajan, *Quantum mechanics for mathematicians*, Graduate Studies in Mathematics, 95, American Mathematical Society, Providence, RI, 2008.

[UT] I. Urizar-Lanz and G. Tóth, *Number-operator–annihilation-operator uncertainty as an alternative for the number-phase uncertainty relation*. Phys. Rev. A. **81**(2010), Article no. 052108, 7 pp.

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