Besov spaces and Bergman projections on the ball

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Abstract
A class of radial differential operators are investigated yielding a natural classification of diagonal Besov spaces on the unit ball of $\mathbb{C}^N$. Precise conditions are given for the boundedness of Bergman projections from certain $L^p$ spaces onto Besov spaces. Right inverses for these projections are also provided. Applications to complex interpolation are presented.

Les espaces de Besov et les projections de Bergman dans la boule

Résumé
Nous étudions une classe d’opérateurs différentiels radiaux conduisant à une classification naturelle des espaces de Besov diagonaux dans la boule unité de $\mathbb{C}^N$. Nous donnons les conditions précises pour la bornitude des projections de Bergman de certains espaces $L^p$ sur des espaces de Besov. Nous déterminons aussi des inverses à droite pour ces projections. Nous présentons des applications à l’interpolation complexe.

1. Introduction

Let $\mathbb{B}$ denote the unit ball of $\mathbb{C}^N$ and $H(\mathbb{B})$ the space of holomorphic functions on $\mathbb{B}$. Let $\nu$ be the normalized volume measure on $\mathbb{B}$ and let $d\nu_q(z) = (1 - |z|^2)^{\nu} \, d\nu(z)$. Diagonal Besov spaces $B^{p,q}_v$ are defined in one of two ways. One definition requires an $f \in H(\mathbb{B})$ to satisfy $(1 - |z|^2)^{m-v} R^m f(z) \in L^p(\nu^{-1})$ for some integer $m > v$, where $v$ is real (see [6]). Here $R$ is the radial derivative; see below. Another requires an $f \in H(\mathbb{B})$ to satisfy $R^{1+v} f(z) \in L^p(\nu^{-1})$, where $v > 0$, $R^v$ is a $v$-th-order radial derivative, and $p > 1$ (see [12]).

The purpose of this Note is to give a new definition that characterizes these spaces using exactly the same parameters as those of the weighted Bergman spaces $A^p_q(\nu) \cap H(\mathbb{B})$: $p > 0$ and $q \in \mathbb{R}$. The function whose $p$-th power is considered is not $f$ any more, but a product of a $t$-th order radial derivative of $f$ and the $t$-th power of $1 - |z|^2$. We call the spaces thus defined $B^p_q$, and it turns out that they naturally extend the $A^p_q$ spaces to all $q \in \mathbb{R}$ while satisfying $A^p_q = B^p_q$ for $q > -1$. The value of $t$ turns out to be irrelevant as long as $(\Re t)p + q > -1$. Eq. (2) summarizes the new classification.

Detailed proofs and further results will be presented elsewhere.

2. Radial differential operators

The radial derivative at $z$ of a holomorphic function $f$ is $Rf(z) = \sum_{k=1}^{\infty} k f_k(z)$, where $f_k$ is the $k$-th term in the homogeneous expansion of $f$. Following [7], we define some more general linear operators.

DEFINITION. – Let \( f \in H(\mathbb{B}) \). We define
\[
D^t_q f = \sum_{k=0}^{\infty} \frac{(s+t)_k}{(s)_k} f_k \quad (s \in \mathbb{C} \setminus \mathbb{N}, \ s + t \in \mathbb{C} \setminus \mathbb{N}),
\]
where \((a)_0 = 1\) and \((a)_m = a(a+1) \cdots (a+m-1)\) for positive integer \(m\). If, say, \(s = -m\) with \(m \in \mathbb{N}\), we write the corresponding factor as \((\lambda)_k\), divide it by \((-1)^m(\lambda + m)\), and then let \(\lambda \to -m\). So
\[
D^t_q f = f_0 + \cdots + f_m + \sum_{k=m+1}^{\infty} \frac{(t-m)_k}{m!(1)_k-m-1} f_k \quad (s = -m \in \mathbb{N}, \ s + t \in \mathbb{C} \setminus \mathbb{N}).
\]

The definition of \(D^t_q\) when \(s + t\) or both \(s\) and \(s + t\) are nonpositive integers is similar.

It is shown in [5] that \(D^t_q\) is a continuous operator on \(H(\mathbb{B})\). If \(d_{s,t,k}\) is the coefficient of \(f_k\) in \(D^t_q f\), Stirling’s formula gives \(d_{s,t,k} \sim k!\) as \(k \to \infty\). Thus \(D^t_q\) is a radial differential operator of order \(t \in \mathbb{C}\). Every \(D^t_q\) is a bijection on \(H(\mathbb{B})\) and thus invertible. A case by case checking reveals that \(D^{t+s}_q D^{-r}_q = D^{t-r}_q\) for all \(r, s, t \in \mathbb{C}\).

3. \(B^q\) spaces

DEFINITION. – Let \(q \in \mathbb{R}\) and \(0 < p \leq \infty\) with \(-qp + q > -1\) (read \(-q > 0\) when \(p = \infty\)). We define \(B^q_p(\mathbb{B}) = B^q_p(\mathbb{B})\) as the space of \(f \in H(\mathbb{B})\) for which the function \((1 - |z|^2)^{-q} D^{-q}_q f(z)\) belongs to \(L^p(\mathbb{B})\).

The \(B^q_p\) norm of \(f\) is defined as the \(L^p(\mathbb{B})\) norm of \((1 - |z|^2)^{-q} D^{-q}_q f(z)\).

We use the term norm even when \(0 < p < 1\). More explicitly,
\[
B^q_p(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \|f\|_{B^q_p}^p = \int_{\mathbb{B}} |D^{-q}_q f(z)|^p du_q(z) < \infty \right\} \quad (0 < p < \infty),
\]
\[
B^q_\infty(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \|f\|_{B^q_\infty} = \sup_{z \in \mathbb{B}} (1 - |z|^2)^{-q} |D^{-q}_q f(z)| < \infty \right\}.
\]

Let us also define \(B^q_0(\mathbb{B})\) as that subspace of \(B^q_\infty(\mathbb{B})\) with the same norm consisting of functions \(f\) for which \((1 - |z|^2)^{-q} D^{-q}_q f(z)\) belongs to \(C_0(\mathbb{B})\), continuous functions on \(\mathbb{B}\) with 0 boundary values. Since every \(D^t_q\) is invertible, there is no nonzero \(f \in B^q_0\) for which \(\|f\|_{B^q_0}\) is zero.

THEOREM 1. – An \(f \in H(\mathbb{B})\) belongs to \(B^q_p(\mathbb{B})\) if and only if for some \(s\) and \(t\) satisfying \((\Re t) p + q > -1\) (read \(\Re t > 0\) when \(p = \infty\)) the function \((1 - |z|^2)^t D^t_q f(z)\) belongs to \(L^p(\mathbb{B})\). The \(L^p(\mathbb{B})\) norm of \((1 - |z|^2)^t D^t_q f(z)\) is equivalent to the \(B^q_p\) norm of \(f\).

The case \(q = - (N + 1)\) is handled in [15,16]. Also, the \(B^q_p\) spaces are the spaces \(A^{-qp+q+1}_{-q}\) of [9], which considers only \(-qp + q + 1 > 0\).

DEFINITION. – Let \(q \in \mathbb{R}\) and \(0 < p \leq \infty\). We define \(B^q_\infty(\mathbb{B}) = B^q_\infty(\mathbb{B})\) as the space of \(f \in H(\mathbb{B})\) for which the function \((1 - |z|^2)^t D^t_q f(z)\) for some \(s\) and \(t\) satisfying \((\Re t) p + q > -1\) (read \(\Re t > 0\) for \(p = \infty\)) belongs to \(L^p(\mathbb{B})\).

COROLLARY. – The spaces \(B^q_\infty\) for \(q > -1\) and the Bergman spaces \(A^q_\infty\) coincide.

By Theorem 1, \(B^q_\infty\) spaces are the same for all \(q \in \mathbb{R}\). This space is the Bloch space \(\mathbb{B}\). Similarly, the spaces \(B^q_\infty\) are all the same for \(q \in \mathbb{R}\) and are the little Bloch space \(\mathbb{B}_0\). The next result is in [16] for \(s > N\).

COROLLARY. – A function \(f \in H(\mathbb{B})\) belongs to \(\mathbb{B}\) (resp. \(\mathbb{B}_0\)) if and only if for some \(s\) and \(t\) with \(\Re t > 0\) the function \((1 - |z|^2)^t D^t_q f(z)\) is uniformly bounded on \(\mathbb{B}\) (resp. belongs to \(C_0(\mathbb{B})\)).

COROLLARY. – Our space \(B^q_p\) and the diagonal Besov space \(BS^p_{q_1} B^q_{p_1}\) coincide.
Thus $B^p_q$ spaces for $p \geq 1$ are Banach spaces. Each $B^2_q$ space has several equivalent inner products

$$ (f, g)_{s,t} = \int_{\mathbb{B}} (1 - |z|^2)^s D^t_{z}f(z)(1 - |z|^2)^t D^s_{\overline{z}}g(z) \, dv_q(z), \quad (1) $$

one for each $s$ and each $t$ satisfying $2 \Re t + q > -1$. The monomials $\{z^n\}$ form an orthogonal set with respect to each of these inner products.

**Theorem 2.** Each $B^2_q$ is a Hilbert space with reproducing kernel $K_q(z, w) = (1 - \langle z, w \rangle)^{-(N+1+q)}$ for $q > -(N+1)$ and

$$ K_q(z, w) = 1 + \sum_{k=1}^{\infty} \frac{(k-1)!}{(-N-q)^k} \langle z, w \rangle^k $$

for $q \leq -(N+1)$; in particular, $K_{-(N+1)}(z, w) = 1 - \log(1 - \langle z, w \rangle)$.

Similar descriptions in [13] and [2] led the author to this research. In fact, the spaces in Theorem 2 are known as Dirichlet-type spaces, $B^2_{(N+1)}$ being the Dirichlet space $D$ and $B^2_{-1}$ being the Hardy space $H^2$. The space $B^2_{-N}$ attracts a lot of attention in operator theory (see [3] and [8]) due to the universal property of its kernel in Nevanlinna–Pick interpolation (see [1]) and is denoted $\mathbb{P}$ here. Another description of $B^2_q$ spaces for $q \geq -(N+1)$ without any derivatives is given in [4]. The same reference contains an example that shows the inclusion $B^2_{q_1} \subset B^2_{q_2}$ for $q_1 < q_2$ is proper for $q \geq -(N+1)$, which actually works for all real $q$.

Let $0 < p_1 < 1 < p_2 < 2 < p_3$ and $q_1 < -(N+1) < -N < q_2 < -1 < 0 < q_3$. Then

$$ \begin{align*}
\cdots & A_{N+1}^{p_1} \ni A_{N+1}^{q_1} \ni A_{N+1}^{q_2} \ni A_{N+1}^{q_3} \ni \mathbb{P} & q_3 = q \\
\cup & \cup \cup \cup \cup \cup & 0 = q \\
\cdots & B^{p_1}_{-N} \ni B^{1}_{-N} \ni B^{q_1}_{-N} \ni \mathbb{P} & -N = q \\
\cup & \cup \cup \cup \cup & 1 = q \\
\cdots & B^{p_2}_{-N} \ni B^{p_3}_{-N} \ni \mathbb{P} & -1 = q \\
\cup & \cup \cup \cup \cup & 0 = q \\
\end{align*} $$

(2)

The inclusions of the level $q = -(N+1)$ follow from their Möbius invariance and are in [10] for $p > 1$. Further, $B^1_{-(N+1)}$ is the minimal Möbius-invariant space $\mathcal{M}$; see [11]. Thus also $\mathcal{M} \subset B^{p_2}_{-(N+1)}$.

**4. Bergman projections**

We restrict ourselves to $1 \leq p \leq \infty$ from now on. **Bergman projections** are the linear operators $P_s$ with $s \in \mathbb{C}$ defined by

$$ P_s f(z) = C \int_{\mathbb{B}} \frac{(1 - |w|^2)^s}{(1 - \langle z, w \rangle)^{N+1+s}} f(w) \, dv(w) \quad (z \in \mathbb{B}) $$
for $f \in L^1(v_x)$. It is clear that $P_s f \in H(B)$. The coefficient $C$ is a normalization constant. The following theorem appears in [11] for real $s > -1$.

**THEOREM 3.** Let $1 \leq p < \infty$ and $q \leq -1$. The Bergman projection $P_s$ maps $L^p(v_x)$ boundedly onto $B^q_p$ if and only if $q + 1 < p(\text{Re} s + 1)$ and $N + 1 + s$ is not a nonpositive integer. Given such an $s$, if $t$ satisfies $(\text{Re} t) p + q > -1$, then $P_s((1 - |z|^2) D_{N+1} f(z)) = C' f(z)$ for all $f \in B^q_p$.

**COROLLARY.** The Bergman projection $P_s$ maps $L^\infty(v_x) = L^\infty(v)$ boundedly onto $\mathcal{B}$ and each of $C(\overline{B})$ and $C_0(\overline{B})$ onto $\mathcal{B}_0$ if and only if $\text{Re} s > -1$. Given such an $s$, if $t$ satisfies $\text{Re} t > 0$, then $P_s((1 - |z|^2) D_{N+1} f(z)) = C'' f(z)$ for all $f \in \mathcal{B}$ and hence for all $f \in \mathcal{B}_0$ too.

### 5. Duality and interpolation

**THEOREM 4.** Let $1 \leq p < \infty$ and $q \leq -1$. The dual space $(B^q_p)^*$ can be identified with $B^{1 - q}_p$ under the pairing $(\cdot, \cdot)_{N+1, -q}$ of $1$, where $1/p + 1/p' = 1$. In particular, the Bloch space $\mathcal{B}$ is the dual space of all $B^1_1$. The dual space $B^1_0$ can be identified with each of $B^1_q$ under the pairings $(\cdot, \cdot)_{N+1, -q}$ of $1$.

**THEOREM 5.** Suppose $1 \leq p_0 < p < p_1 \leq \infty$ with $1/p = (1-\theta)/p_0 + \theta/p_1$ for some $\theta \in (0,1)$. Then the complex interpolation space $[B^{p_0}_q, B^{p_1}_q]_\theta$ is $B^{p_0}_q$.

The case $q = -(N + 1)$ of Theorem 4 is in [15]. For the definitions on complex interpolation, see [14].

Let $\psi$ be a holomorphic automorphism of $\mathcal{B}$. It is shown in [9] using (1) that the Bergman spaces $A^p_q$ for $0 < p \leq \infty$ and $q > -1$ are invariant under each of the isometries

$$U^p,q f(z) = f(\psi(z)) \left( J\psi(z) \right)^{(2/p)(1 + q/(N + 1))}.$$

**THEOREM 6.** Suppose $2 \leq p \leq \infty$, $-(N + 1) < q \leq -1$, and $\psi$ as above. Then $U^{p,q}$ is a bounded linear transformation on $B^q_p$.

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**References**


